NONUNIFORM CENTER BUNCHING AND THE GENERICITY OF ERGODICITY AMONG C^1 PARTIALLY HYPERBOLIC SYMPLECTOMORPHISMS

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Abstract. We introduce the notion of nonuniform center bunching for partially hyperbolic diffeomorphims, and extend previous results by Burns–Wilkinson and Avila–Santamaria–Viana. Combining this new technique with other constructions we prove that C^1-generic partially hyperbolic symplectomorphisms are ergodic. We also construct new examples of partially hyperbolic diffeomorphisms that are stably ergodic.

1. Introduction

1.1. **Abundance of Ergodicity.** Let (M, ω) be a closed (ie, compact without boundary) symplectic C^∞ manifold of dimension $2N$. Let $\mathrm{Diff}^1_\omega(M)$ be the space of ω-preserving C^1 diffeomorphisms, endowed with the C^1 topology. Let m be the measure induced by the volume form $\omega^{\wedge N}$, normalized so that $m(M) = 1$.

Let $PH^1_\omega(M)$ be the set of diffeomorphisms $f \in \mathrm{Diff}^1_\omega(M)$ that are partially hyperbolic[1], i.e., there exists an invariant splitting $T_xM = E^u(x) \oplus E^c(x) \oplus E^s(x)$, into nonzero bundles such that, for some Riemannian metric[2] $\|\cdot\|$ on M, we have

$$\|(Df|E^u(x))^{-1}\|^{-1} > \|Df|E^c(x)\| \geq \|(Df|E^c(x))^{-1}\|^{-1} > \|Df|E^s(x)\|$$

for every $x \in M$. Such a splitting is automatically continuous.

Theorem A. *The set of ergodic diffeomorphisms is residual in $PH^1_\omega(M)$.*

Our result is motivated by the following well-known conjecture of Pugh and Shub [PS]: *There is a C^2 open and dense subset of the space of C^2 volume-preserving partially hyperbolic diffeomorphisms formed by ergodic maps*. Among the known results in this direction, we have:

- F. and M. A. Rodriguez-Hertz, and Ures [RRU] proved that C^r-stable ergodicity is dense among C^r volume-preserving partially hyperbolic diffeomorphisms with one-dimensional center bundle, for all $r \geq 2$. (See also [BMVW] for an earlier result.)
- F. and M. A. Rodriguez-Hertz, Tahzibi, and Ures [RRTU] proved that ergodicity holds on a C^1 open and dense subset of the C^2 volume-preserving partially hyperbolic diffeomorphisms with two-dimensional center bundle.

Together with the result from Avila [A], it follows that ergodicity is C^1 generic among volume-preserving partially hyperbolic diffeomorphisms with center dimension at most 2. On the other hand, the techniques yielding the results above seem less effective for the understanding of the case of symplectic maps, and indeed Theorem A is the first result on denseness of ergodicity for non-Anosov partially

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[1] There are two definitions of partial hyperbolicity: the *absolute* and the *relative*; see [AV]. We will use the latter, which is the weakest.

[2] Such a metric is called *adapted*; see [G].

hyperbolic symplectomorphisms, even allowing for constraints on the center dimension. Our approach develops some new tools of independent interest, as we explain next.

1.2. **Center Bunching Properties.** To support their conjecture, Pugh and Shub [PS] provided a criterion for a volume-preserving partially hyperbolic map to be ergodic, based on the property of *accessibility*, together with some technical hypotheses. A significantly improved version of this criterion was obtained by Burns and Wilkinson [BW]: accessibility and *center bunching* imply ergodicity. Dolgopyat and Wilkinson [DW] showed that accessibility is open and dense in the C^1 topology, but center bunching is not a dense condition unless the center dimension is 1 (which cannot happen for symplectic maps). In this paper we introduce and exploit a weaker condition, called *nonuniform center bunching*.

In the context of general (not necessarily volume-preserving) partially hyperbolic diffeomorphisms, the center bunching hypothesis in [BW] is a global, uniform property, requiring that at every point in the manifold, the nonconformality of the action on the center bundle be dominated by the hyperbolicity in both the stable and unstable bundles. By contrast, the nonuniform center bunching property introduced here is a property of asymptotic nature about the orbit of a single point; it is the intersection of a forward bunching property of the forward orbit and a backward bunching property of the backward orbit. The precise definitions are slightly technical (see Section 2). However, for Lyapunov regular points (which by Oseledets' theorem have full probability), forward (resp. backwards) center bunching means that the biggest difference between the Lyapunov exponents in the center bundle is smaller than the absolute value of the exponents in the stable (resp. unstable) bundle. The set CB^+ of forward center bunched points for a partially hyperbolic diffeomorphism f has the useful property of being *$\mathcal{W}^s$-saturated*, meaning that it is a union of entire stable manifolds of f; similarly the set CB^- of backward center bunched points is $\mathcal{W}^u$-saturated, i.e. a union of unstable manifolds.

Our next main result, Theorem B, generalizes the core result of [BW] (Theorem 5.1 of that paper). It states that for any C^2 partially hyperbolic diffeomorphism, *the set of Lebesgue density points of any bi essentially saturated set meets CB^+ in a $\mathcal{W}^s$-saturated set and CB^- in a $\mathcal{W}^u$-saturated set.* (A bi essentially saturated set is one that coincides mod 0 with a $\mathcal{W}^s$-saturated set and mod 0 with a $\mathcal{W}^u$-saturated set.)

Burns and Wilkinson [BW] obtain their ergodicity criterion as a simple consequence of their technical core result. Indeed, assuming accessibility (or even essential accessibility), ergodicity in [BW] follows in one step from the core result, using a Hopf argument; it is not necessary to establish local ergodicity first (as one does in proving ergodicity for hyperbolic systems). It is unclear to us whether the the Burns–Wilkinson criterion for ergodicity can be improved by replacing uniform center bunching by almost everywhere nonuniform center bunching, in part because the uniform version in [BW] is by nature *not* a "local ergodicity" result. In reality, it is possible to deduce a new ergodicity criterion (Corollary C) from Theorem B. Namely, ergodicity follows from almost everywhere nonuniform center bunching together with a stronger form of essential accessibility, where we only allow *su*-paths whose corners are center-bunched points. While this accessibility condition is far from automatic, it can be verified in some interesting classes of examples: see §1.4 below.

1.3. **Outline of the Proof of Theorem A.** Let us explain how nonuniform center bunching combines with other ingredients to yield Theorem A. Take a symplectomorphism with the following C^1 generic properties:

(a) it is stably accessible, by Dolgopyat and Wilkinson [DW];
(b) all central Lyapunov exponents vanish at almost every point, by Bochi [B].

Notice that property (b) implies almost every point is center bunched. But Theorem B requires C^2 regularity. This is achieved by taking a perturbation, which still has property (a), but loses property (b). What happens is that each point in some set of measure close to 1 has small center Lyapunov exponents and thus is center bunched.

Before getting useful consequences from Theorem B, we need to provide a local source of ergodicity. This is achieved through a novel application of the Anosov–Katok [AK] examples. (By comparison, [RRTU] uses Bonatti–Díaz blenders.) We proceed as follows. By perturbing, we find a periodic point whose center eigenvalues have unit modulus. Perturbing again, we create a disk tangent to the center direction that is invariant by a power of the map. We can choose any dynamics close to the identity on this disk, so we select an ergodic Anosov–Katok map. Ergodicity is spread from the center disk to a ball around the periodic point using Theorem B, and then to the whole manifold by accessibility. (In fact, since the set of center bunched points is not of full measure, a G_δ argument is necessary to conclude ergodicity – see Section 3 for the precise procedure.)

1.4. **Further Applications of Nonuniform Center Bunching.** By means of our ergodicity criterion (Corollary C) we construct an example of a stably ergodic partially hyperbolic diffeomorphism that is almost everywhere nonuniformly center bunched (but not center bunched in the sense of [BW]) in a robust way.

We also prove in this paper an extension of Theorem B to sections of bundles over partially hyperbolic diffeomorphisms. This result, Theorem D, brings into the nonuniform setting a recent result of Avila, Santamaria and Viana [ASV], which they use to show that the generic bunched $SL(n, \mathbb{R})$ cocycle over an accessible, center bunched, volume-preserving partially hyperbolic diffeomorphism has a nonvanishing exponent. The result from [ASV] has also been used in establishing measurable rigidity of solutions to the cohomological equation over center-bunched systems; see [W]. Theorem D has similar applications in the setting where nonuniform center bunching holds, and we detail some of them in Section 6.

We conceive that our methods may be further extended to apply in certain "singular partially hyperbolic" contexts where partial hyperbolicity holds on an open, noncompact subset of the manifold M but decays in strength near the boundary. Such conditions hold, for example, for geodesic flows on certain nonpositively curved manifolds. Under suitable accessibility hypotheses, these systems should be ergodic with respect to volume.

1.5. **Questions.** Combining results of [DW] and Brin [Br], one obtains that topological transitivity holds for a C^1 open and dense set of partially hyperbolic symplectomorphisms. On the other hand, the C^1-interior of the ergodic symplectomorphisms is contained in the partially hyperbolic diffeomorphisms [HT, SX]. This suggests the following natural question.

Question 1. Can Theorem A be improved to an open (and dense) instead of residual set?

Notice that it is not known even whether the set of C^1 Anosov ergodic maps has non-empty interior.

Dropping partial hyperbolicity, recall that C^1 generic symplectic and volume-preserving diffeomorphisms are transitive by [ABC] and [BC], while ergodicity is known to be C^0-generic among volume-preserving homeomorphisms by [OU]. So the following well-known question arises:

Question 2. Is ergodicity generic among C^1 symplectic and volume-preserving diffeomorphisms?

1.6. **Organization of the Paper.** In Section 2 we define nonuniform center bunching, state Theorem B, and derive Corollary C from it.

In Section 3 we prove Theorem A following the outline given in §1.3. As we have explained, the proof uses the existence (after perturbation) of a periodic point with elliptic central behavior. Such a result goes along the lines of [BDP, HT, SX], but we have not been able to find a precise reference. In Section 4, which can be read independently from the rest of the paper, we provide a proof of this result by reducing it to its ergodic counterpart and applying the Ergodic Closing Lemma. This approach is different from the one taken in the literature. For this reason, we included an appendix explaining how to use it to reobtain some results from [BDP].

The proof of Theorem B, despite having much in common with [BW], is given here in full detail in Section 5. In Section 6 we formulate and prove the more general Theorem D. The new examples of stably ergodic maps are constructed in Section 7.

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2. Nonuniform Center Bunching and Consequences

Throughout this section, f denotes a fixed C^2 partially hyperbolic of a closed manifold M of dimension d. (We do not require f to be symplectic or even volume-preserving.)

2.1. **Saturated Sets.** If $\mathcal{F}$ is a foliation with smooth leaves, a set $X \subseteq M$ is said to be *$\mathcal{F}$-saturated* if it is a union of entire leaves of $\mathcal{F}$. We say that a measurable set X is *essentially $\mathcal{F}$-saturated* if it coincides Lebesgue mod 0 with a $\mathcal{F}$-saturated set.

We also say that a set X is *$\mathcal{F}$-saturated at a point x* if there exist $0 < \delta_0 < \delta_1$ such that for any $z \in X \cap B(x, \delta_0)$, we have $\mathcal{F}(z, \delta_1) \subset X$. (Here $\mathcal{F}(z, \delta_1)$ denotes the connected component of $\mathcal{F}(z) \cap B(z, \delta_1)$ containing z.)

A measurable set X is called *bi essentially saturated* if it is both essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated. (Here $\mathcal{W}^u$ and $\mathcal{W}^s$ are the unstable and stable foliations of the partially hyperbolic diffeomorphism f.)

2.2. **Nonuniform Center Bunching.** If $A : V \to W$ is a linear transformation between Banach spaces, we denote by $\mathbf{m}(A)$ the *conorm* of A, defined by

$$\mathbf{m}(A) = \inf_{v \in V, \|v\|=1} \|A(v)\|.$$

If A is invertible, then $\mathbf{m}(A) = \|A^{-1}\|^{-1}$.

We say that a point $p \in M$ is *forward center bunched* if there exist $\theta > 1$ and a sequence $0 = i_0 < i_1 < \cdots$ such that $i_{k+1}/i_k \to 1$ and for every $k \geq 0$,

$$\|D_{f^{i_k}(p)} f^{i_{k+1}-i_k} | E^s\|^{-1} \geq \theta^{i_{k+1}-i_k} \cdot \frac{\|D_{f^{i_k}(p)} f^{i_{k+1}-i_k} | E^c\|}{\mathbf{m}\Big(D_{f^{i_k}(p)} f^{i_{k+1}-i_k} | E^c\Big)}.$$

The point p is called *backwards center bunched* if it is forward center bunched with respect to f^{-1}. The set of forward, resp. backwards, center bunched points is denoted by CB^+, resp. CB^-. Also denote $CB = CB^+ \cap CB^-$. It is easy to see that these sets are f-invariant. Moreover, in Section 5 we show:

Proposition 2.1. *CB^+ is $\mathcal{W}^s$-saturated and CB^- is $\mathcal{W}^u$-saturated.*

A much deeper property is:

Theorem B. *Let f be a C^2 partially hyperbolic diffeomorphism. Let X be a bi essentially saturated set, and let $\hat{X}$ denote the set of Lebesgue density points of X. Then $\hat{X} \cap CB^+$ is $\mathcal{W}^s$-saturated and $\hat{X} \cap CB^-$ is $\mathcal{W}^u$-saturated.*

We remark that the hypotheses of Theorem B are weaker than the center bunching hypothesis in [BW]. In the setting of [BW], $CB^+ = CB^- = M$ and one takes $i_k = k$ in the definition of forward center bunching (In fact, the center bunching hypothesis in [BW] is equivalent to the condition $CB^+ = CB^- = M$ see Remark 2.4 below.)

Another remark is that, as in [BW], it is essential that X is *both* essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated in order to conclude anything.

2.3. **Relation with Lyapunov Spectrum.** Let us formulate sufficient conditions for center bunching in terms of Lyapunov exponents.

Oseledets' Theorem asserts that there exists a set of full probability (that is, a Borel set of full measure with respect to any f-invariant probability) where Lyapunov exponents and Oseledets' splitting are defined (see for example [Arno, Theorem 3.4.11 and Remark 4.2.8]). The elements of this set are called *Lyapunov regular points*.

If $p \in M$ is a Lyapunov regular point, we write the Lyapunov exponents (with multiplicity) of f at p as:

$$\underbrace{\lambda_1 \geq \cdots \geq \lambda_k}_{E^u} > \underbrace{\lambda_{k+1} \geq \cdots \geq \lambda_\ell}_{E^c} > \underbrace{\lambda_{\ell+1} \geq \cdots \geq \lambda_d}_{E^s} .$$

(The braces are shorthands meaning that $\dim E^u = k$, $\dim E^c = \ell - k$, $\dim E^s = d - \ell$.) We say the Lyapunov spectrum of f at p satisfies the *forward center bunched condition* if

$$\lambda_{k+1} - \lambda_\ell < -\lambda_{\ell+1} ,$$

and the *backwards center bunched condition* in the case that

$$\lambda_{k+1} - \lambda_\ell < \lambda_k .$$

Notice that if f is symplectic then, by the symmetry between the exponents, the forward and the backwards center bunching conditions are equivalent to:

$$2\lambda_{k+1} < \lambda_k .$$

Proposition 2.2. *A Lyapunov regular point is forward (resp. backwards) center bunched if and only if its spectrum satisfies the forward (resp. backwards) center bunched condition.*

Proof. We only need to prove the forward part of the proposition, and the backwards part will follow by symmetry.

Fix a point p and let

$$\Theta(j, n) = \|D_{f^j(p)} f^n | E^s\|^{-1} \cdot \mathbf{m}\Big(D_{f^j(p)} f^n | E^c\Big) \cdot \|D_{f^j(p)} f^n | E^c\|^{-1} \quad j, n \geq 0. \tag{2.1}$$

Assume that p is forward center bunched, and let θ and i_k be as in the definition of forward center bunching. Since

$$\Theta(0, i_k) \geq \Theta(0, i_1)\Theta(i_1, i_2 - i_1) \cdots \Theta(i_{k-1}, i_k - i_{k-1}) \geq \theta^{i_k} ,$$

we have

$$\limsup_{n\to+\infty} \frac{1}{n} \log \Theta(0,n) > 0. \tag{2.2}$$

If p is Lyapunov regular then the lim sup above equals $-\lambda_{\ell+1} + \lambda_\ell - \lambda_{k+1}$. Thus p has center bunched Lyapunov spectrum.

Conversely, assume that the point p is Lyapunov regular and has center bunched Lyapunov spectrum. Fix some τ with $0 < \tau < -\lambda_{\ell+1} - \lambda_{k+1} + \lambda_\ell$. We claim that

(2.3) for every $\delta > 0$ there exists $c_\delta > 0$ such that $\Theta(j,n) > c_\delta e^{-\delta j} e^{\tau n}$ for all $j, n \geq 0$.

Before giving the proof, let us see how to conclude from here. Let $i_0 = 0$. Inductively define i_{k+1} as the least $i > i_k$ such that $\Theta(i_k, i - i_k) > e^{(\tau/2)(i-i_k)}$. Let us see that this sequence of times satisfies the requirements of the definition of forward center bunching, with $\theta = \tau/2$. For any $\delta > 0$, we have

$$c_\delta e^{-\delta i_k} e^{\tau(i_{k+1}-i_k-1)} < \Theta(i_k, i_{k+1} - i_k - 1) \leq e^{(\tau/2)(i_{k+1}-i_k-1)}\,.$$

It follows that if i_k is sufficiently large (depending on δ) then $(i_{k+1} - i_k)/i_k < 3\delta/\tau$. This proves that $i_{k+1}/i_k \to 1$ and hence that $p \in CB^+$.

We are left to prove (2.3). For $1 \leq i \leq d = \dim M$, let $E^i(p)$ be the Oseledets space corresponding to the Lyapunov exponent $\lambda_i(p)$. (This notation is not standard because those spaces are not necessarily different.) A consequence of the Lyapunov regularity of p is that, for each $i = 1, \dots, d$, the quotient $n^{-1} \log \|D_p f^n(v)\|$ converges to λ_i uniformly over unit vectors $v \in E^i(p)$. Thus for every $\delta > 0$ there exists $K_\delta > 1$ such that

$$K_\delta^{-1} e^{(\lambda_i - \delta)n} \leq \|D_p f^n(v)\| \leq K_\delta e^{(\lambda_i+\delta)n}\,, \text{for all unit vectors } v \in E^i(p) \text{ and } n \geq 0.$$

Hence, for each n, $j \geq 0$, we have

$$\|D_{f^j(p)} f^n|_{E^i}\| \leq \|D_p f^{n+j}|_{E^i}\| / \mathbf{m}(D_p f^j|_{E^i}) \leq K_\delta^2 e^{2\delta j} e^{(\lambda_i+\delta)n}\,. \tag{2.4}$$

Another consequence of Lyapunov regularity (see [Arno, Corollary 5.3.10]) is that the angles between (sums of different) Oseledets spaces along the orbit of p are subexponential. In particular, for each $\delta > 0$ we can find $K'_\delta > 1$ such that

$$(K'_\delta)^{-1} e^{-\delta(j+n)} \leq \frac{\|D_{f^j(p)} f^n|_{E^s}\|}{\max_{i\in[\ell+1,d]} \|D_{f^j(p)} f^n|_{E^i}\|} \leq K'_\delta e^{\delta(j+n)}\,, \quad \text{for each } n,\, j \geq 0.$$

It follows from (2.4) that there exists $K''_\delta > 1$ such that

$$\|D_{f^j(p)} f^n|_{E^s}\| \leq K''_\delta e^{3\delta j} e^{(\lambda_{\ell+1}+2\delta)n}\,, \quad \text{for each } n,\, j \geq 0.$$

This controls the first term in (2.1). The other two are dealt with in an analogous way, and (2.3) follows. ◻

Remark 2.3. If $p \in CB^+$ then we have seen that (2.2) holds, where Θ is defined by (2.1). However, condition (2.2) alone does not imply forward center bunching, as the following abstract example shows: Let

$$A'_s = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad A''_s = \begin{pmatrix} e^{-1/2} & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad A_c = \begin{pmatrix} 1 & 0 \\ 0 & e^{3/4} \end{pmatrix}.$$

Assume that $D_{f^j(p)} f|_{E^s}$ equals A'_s if $2^m - 2 \leq j < 2^m + 2^{m-1} - 2$ for some integer $m \geq 1$, and equals A''_s otherwise. Also assume that $D_{f^j(p)} f|_{E^c}$ equals A_c for every $j \geq 0$. Then for every $n \geq 0$, we have $\|D_p f^n|_{E^s}\| = e^{-n}$, $\mathbf{m}(D_p f^n|_{E^c}) = 1$, and $\|D_p f^n|_{E^c}\| = e^{(3/4)n}$, so condition (2.2) is satisfied. On the other hand, if $j = 2^m + 2^{m-1} - 2$ and $n = 2^{m-1}$ then $\|D_{f^j(p)} f^n|_{E^s}\| = e^{(-1/2)n}$ and therefore $\Theta(j,n) = e^{(-1/4)n}$. It follows that p cannot be forward center bunched.

Remark 2.4. If $CB^+ = CB^- = M$ then f is center bunched in the sense of [BW]. Indeed, let $\Theta_p(j,n)$ be as in (2.1), with a subscript to indicate dependence on the point. Assuming $CB^+ = M$, compactness implies that there exists $\theta > 1$ and m such that for every $p \in M$ there exists i with $1 \leq i \leq m$ such that $\Theta_p(0,i) > \theta$. It follows that there is $c > 0$ such that $\Theta_p(0,n) > c\theta^{n/m}$. We reason analogously for f^{-1}. The conclusion follows from an adapted metric argument along the lines of [G].

2.4. **An Ergodicity Criterion.** Let us extract a criterion for ergodicity from Theorem B. (It is not used in the proof of Theorem A, so the reader can skip the rest of this section.)

Corollary C. *Let f be a C^2 partially hyperbolic volume-preserving diffeomorphism. Let $CB = CB^+ \cap CB^-$ be the set of center bunched points. Assume that almost every pair of points $x, y \in CB$ can be connected by an su-path whose corners are in CB.*

Let X be a bi essentially saturated set such that $X \cap CB$ has positive measure. Then X has full measure in CB. If CB has full measure, then f is ergodic, and in fact a K-system.

In Section 7 we give applications of Corollary C to prove stable ergodicity of certain partially hyperbolic diffeomorphisms that are *not* center bunched.

Proof of Corollary C. Let f and X satisfy the hypotheses of Corollary C and let $\hat{X}$ be the set of Lebesgue density points of X. Then for almost every $x \in \hat{X} \cap CB$ and almost every $y \in CB$, there is an *su*-path from x to y with corners $x_0 = x, x_1, \ldots, x_k = y$ all lying in CB (that is, so that x_i lies in $CB \cap (\mathcal{W}^s(x_{i+1}) \cup \mathcal{W}^u(x_{i+1}))$, for $i = 0, \ldots, k-1$). Fix such an x and y and such an *su*-path. Applying Theorem B inductively to each pair x_i, x_{i+1}, we obtain that x_i lies in $\hat{X}$, for $i = 1, \ldots k$, and so $y \in \hat{X}$. This implies that almost every $y \in CB$ lies in $\hat{X}$, and hence X has full measure in CB.

A standard argument shows that a volume-preserving partially hyperbolic diffeomorphism is ergodic if and only if every bi essentially saturated, invariant set has measure 0 or 1. Moreover, f is a K-system if every bi essentially saturated set, invariant or not, has measure 0 or 1 (see [BW], Section 5). If CB has full measure, then any bi essentially saturated set has 0 or full measure in CB, and hence has measure 0 or 1. It follows that f is ergodic, and in fact a K-system. ◻

3. Proof of Theorem A

For $\varepsilon > 0$, let us call a diffeomorphism $f \in PH^1_\omega(M)$ *ε-nearly ergodic* if for any bi essentially saturated and mod 0 invariant set X, either $m(X) < \varepsilon$ or $m(X) > 1 - \varepsilon$. The bulk of the proof of Theorem A consists in showing the following:

Proposition 3.1. *For any $\varepsilon > 0$, the ε-nearly ergodic diffeomorphisms form a dense subset of $PH^1_\omega(M)$.*

In §§3.1, 3.2, and 3.3 we review some results from the literature, which are used to prove the proposition in §3.4. Then in §3.5 we explain how Proposition 3.1 implies Theorem A.

3.1. **Zero Center Exponents.** Given $f \in PH^1_\omega(M)$, the partially hyperbolic splitting $TM = E^u \oplus E^c \oplus E^s$ is not necessarily unique. *We consider from now on only the unique splitting of minimal center dimension.* If this center dimension is constant on a C^1-neighborhood of f, we say that f has *unbreakable* center bundle. Such f's form an open dense subset of $PH^1_\omega(M)$ (by upper-semicontinuity of the center dimension).

To get center bunching, we will use the following:

Theorem 3.2 (Bochi [B], Theorem C). *There is a residual set $\mathcal{R} \subset PH^1_\omega(M)$ such that if $f \in \mathcal{R}$ then all Lyapunov exponents in the center bundle vanish for a.e. point.*

In other words, $\lambda^c(f) = 0$ for generic f, where

$$\lambda^c(f) = \lim_{n\to+\infty} \frac{1}{n} \int_M \log \|Df^n|E^c_f\| \, dm = \inf_n \frac{1}{n} \int_M \log \|Df^n|E^c_f\| \, dm.$$

Notice that $\lambda^c(f)$ is an upper semicontinuous function of f. Therefore, for any $\delta > 0$, the set of $f \in PH^1_\omega(M)$ with $\lambda^c(f) < \delta$ is open and dense (and thus, by [Z], it contains C^2 maps).

3.2. **Accessibility.** There are two results about accessibility that we will need: one says that it is frequent, and the other gives a useful consequence.

Theorem 3.3 (Dolgopyat and Wilkinson [DW]). *There is an open and dense subset of $PH^1_\omega(M)$ formed by accessible symplectomorphisms.*

Theorem 3.4 (Brin [Br][3]). *If f is a C^2 volume-preserving partially hyperbolic diffeomorphism with the accessibility property then almost every point has a dense orbit.*

3.3. **Creating an Ergodic Center Disk.** The last ingredient we will need in the proof of Proposition 3.1 is Lemma 3.8 below, whose proof needs its own preparations. We begin finding a suitable periodic point:

Theorem 3.5. *Let f have unbreakable center. There exists a C^1-perturbation $\tilde{f}$ that has a periodic point with* $\dim E^c$ *eigenvalues of modulus* 1.

This result can be obtained along the lines of [SX] or [HT] (which prove symplectic versions of the results of [BDP]). In Section 4 we give a different proof, relying on [B] and the Ergodic Closing Lemma [M].

The following symplectic pasting lemma is established using generating functions, see [AM]:

Lemma 3.6. *Let $f \in \mathrm{Diff}^r_\omega(M)$. Given $\varepsilon > 0$ there is $\delta > 0$ such that if $U \subset M$ is an open set of diameter less than δ, and $g : U \to M$ is a C^r-symplectic map that is δ-C^1-close to $f|U$, then g can be extended to some $\hat{g} \in \mathrm{Diff}^r_\omega(M)$ that is ε-C^1-close to f.*

The Anosov–Katok constructions enter here:

Theorem 3.7. *Let $L : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be a symplectic linear map with all eigenvalues of modulus* 1*. Then there exist an arbitrarily small neighborhood U of* 0 *in $\mathbb{R}^{2N}$ and an* ergodic *symplectic diffeomorphism $g : U \to U$ that is C^∞-close to $L|U$.*

Proof. We may assume that L has only simple eigenvalues $\lambda_1^{\pm1}, \dots, \lambda_N^{\pm1}$, all in the unit circle. For $1 \le i \le N$, let E^i be the L-invariant two-dimensional subspaces associated to the eigenvalues λ_i, λ_i^{-1}. Let $A_i : E^i \to \mathbb{R}^2$ be linear maps conjugating $L|E^i$ to rigid rotations R_i. Fix $\varepsilon > 0$. If $g_i : \mathbb{D} \to \mathbb{D}$ are area-preserving maps of the unit disk $\mathbb{D} \subset \mathbb{R}^2$ that are C^∞-close to $R_i|\mathbb{D}$, then the formula

$$g(x) = \varepsilon A_1^{-1} g_1(\varepsilon^{-1} A_1 x_1) + \cdots + \varepsilon A_N^{-1} g_N(\varepsilon^{-1} A_N x_N), \text{ where } x = x_1 + \cdots + x_N \text{ with } x_i \in E^i,$$

defines, on a small neighborhood U of 0 in $\mathbb{R}^{2N}$, a symplectic map $g : U \to U$ that is C^∞-close to $L|U$. Now using a well known result of Anosov and Katok [AK], we choose maps g_i as above that are weakly mixing. It follows (see [P], Theorem 2.6.1) that g is weakly mixing (and hence ergodic) as well. □

[3] In fact, Brin proved the result for *absolute* partially hyperbolic maps. Another proof was given by Burns, Dolgopyat, and Pesin, see [BuDP, Lemma 5]. Their proof applies to *relative* partially hyperbolic maps (the weaker definition taken in this paper): the only necessary modification is to use the property of absolute continuity of stable and unstable foliations in the relative case, which is proven by Abdenur and Viana in [AV].

Lemma 3.8. *For all f in a C^1 dense subset of $PH^1_\omega(M)$, the following properties hold: The map f is C^2, and there is a immersed closed disk D^c such that:*

1) *the tangent space T_xD^c coincides with $E^c(x)$ at each $x \in D^c$;*
2) *there is some ℓ such that D^c is f^ℓ-invariant, and moreover $D^c \cap f^i(D^c) = \varnothing$ for $0 < i < \ell$;*
3) *the restriction of f^ℓ to D^c is ergodic (with respect to the Riemannian volume m_c);*
4) *the disk is center bunched in the sense that*
$$\frac{\|Df|E^c(x)\|}{\mathbf{m}(Df|E^c(x))} < \min\left(\mathbf{m}(Df|E^u(x)), \|Df|E^s(x)\|^{-1}\right) \quad \textit{for all } x \in D^c;$$
5) *f is dynamically coherent in a box neighborhood B of D^c (that is, there are foliations $\mathcal{W}^c$, $\mathcal{W}^{uc}$, $\mathcal{W}^{cs}$ in the box B that integrate the distributions E^c, $E^u \oplus E^c$, $E^c \oplus E^s$).*

Proof. We will explain how to perturb a given f in order to obtain the desired properties.

First use Theorem 3.5 to perturb f and find a periodic point p of period ℓ such that all eigenvalues of $Df^\ell|E^c(p)$ have modulus 1. Also assume that these eigenvalues are distinct and their arguments are rational mod 2π, so that $Df^\ell|E^c(p)$ is diagonalizable and a power of it is the identity.

Take a neighborhood U of p that is disjoint from $f^i(U)$ for $1 \le i \le \ell - 1$, and such that there is a symplectic chart $\phi : U \to \mathbb{R}^{2N}$, (that is, the form $\phi_*\omega$ coincides with $\sum_{i=1}^N dp_i \wedge dq_i$, where $p_1, \dots, p_N, q_1, \dots, q_N$ are coordinates in $\mathbb{R}^{2N}$.) We can also assume that $\phi(p) = 0$ and $D\phi(p)$ sends the spaces $E^u(p)$, $E^c(p)$, and $E^s(p)$ to the planes $p_1 \cdots p_u$, $p_{u+1} \cdots p_N q_{u+1} \cdots q_N$, and $q_1 \cdots q_u$, respectively (where $u = \dim E^u$.)

Using Lemma 3.6, we can perturb f so that $\phi \circ f^\ell \circ \phi^{-1}$ coincides with the linear map $D\phi(p) \circ Df^\ell(p) \circ D\phi^{-1}(0)$ on a neighborhood of p. For simplicity, we omit the chart in the writing, thus
$$f^\ell(x_u, x_c, x_s) = (L_u(x_u), L_c(x_c), L_s(x_s)).$$
Recall that there is a power of L_c that is the identity. So, if necessary changing the point p and the period ℓ, we can assume L_c is the identity.

Next we use Theorem 3.7. Let $g : D^c \to D^c$ be an ergodic symplectic diffeomorphism, where the disk $D^c \subset \mathbb{R}^{\dim E^c}$ is contained in the chart domain. Consider the (symplectic) map
$$G(x_u, x_c, x_s) = (L_u(x_u), g(x_c), L_s(x_s)), \text{ defined in a neighborhood of } (0, 0, 0).$$
Now use Lemma 3.6 again to find a global $\tilde{f} : M \to M$ close to f, such that (still in charts)
$$f^\ell(x_u, x_c, x_s) = G(x_u, x_c, x_s) \text{ in a neighborhood of } (0, 0, 0).$$
Rename $\tilde{f}$ to f. Then f has all the desired properties. □

3.4. **Getting Near-Ergodicity.**

Proof of Proposition 3.1. Fix an open set $\mathcal{U} \subseteq PH^1_\omega(M)$ and $\varepsilon > 0$. Let $\delta > 0$ be small. Using Theorems 3.3 and 3.2, we can assume that the set $\mathcal{U}$ is composed of maps f that are accessible and satisfy $\lambda^c(f) < \delta$. With a good choice of δ, the latter property implies that for any $f \in \mathcal{U}$, the measure of the set of Lyapunov regular points whose Lyapunov spectrum satisfies the center bunching condition is at least $1 - \varepsilon$. Thus, by Proposition 2.2,
$$m(CB^+) > 1 - \varepsilon.$$

Now take $f \in \mathcal{U}$ given by Lemma 3.8. Thus we have a center bunched disk D^c that is ergodic (w.r.t. the measure m_c) by a power f^ℓ, disjoint from its first $\ell - 1$ iterates, and has a dynamically coherent box neighborhood B.

We will prove that f is ε-nearly ergodic. So take any bi essentially saturated set mod 0 invariant set X. Let X_1 be the (invariant) set of its Lebesgue density points, and $X_0 = M \setminus X_1$. By Proposition 2.1 and Theorem B, $X_j \cap CB^+$ is $\mathcal{W}^s$-saturated and $X_j \cap CB^-$ is $\mathcal{W}^u$-saturated for both $j = 0, 1$.

The map f^ℓ has the invariant ergodic measure m_c, supported on D^c. Thus, for some $i \in \{0, 1\}$ (that will be kept fixed in the sequel),

$$m_c(X_i) = 0.$$

By Oseledets' Theorem, m_c-almost every point is Lyapunov regular for f^ℓ, and hence for f as well. By Property 4 in Lemma 3.8, all these points have center bunched Lyapunov spectrum, and thus are (forward and backwards) center bunched, by Proposition 2.2. Hence for m_c-almost every $x \in D^c$, the unstable manifold $\mathcal{W}^u(x)$ is contained in X_{1-i}. Dynamical coherence gives a foliation $\mathcal{W}^{uc}$ in the box B (which integrates $E^u \oplus E^c$); let D^{uc} be the leaf that contains D^c, with an induced Riemannian volume measure m_{uc}. It follows from the absolute continuity of the $\mathcal{W}^u$ foliation that

$$m_{uc}(X_i \cap D^{uc}) = 0.$$

Since the set $Y_i = X_i \cap CB^+$ is $\mathcal{W}^s$-saturated and $m_{uc}(Y_i \cap D^{uc}) = 0$, absolute continuity gives $m(Y_i \cap B) = 0$. It follows that $m(Y_i) = 0$; indeed if the invariant set Y_i had positive measure then, by Theorem 3.4, it would have a positive measure intersection with every set of nonempty interior, for example the box B. Recalling that $m(CB^+) > 1 - \varepsilon$, we get that $m(X_i) < \varepsilon$. This means that either $m(X) < \varepsilon$ or $m(X) > 1 - \varepsilon$, as we wanted to prove. □

3.5. **The G_δ Argument.** We now explain how Proposition 3.1 implies Theorem A.

Given $f \in \mathrm{Diff}^1_\omega(M)$ and a continuous function $\varphi : M \to \mathbb{R}$, define functions:

$$\varphi_{f,n}(x) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^j(x)), \qquad \varphi_f(x) = \lim_{n \to +\infty} \varphi_{f,n}(x) \quad \text{(defined a.e.)}.$$

For $\varphi \in C^0(M, \mathbb{R})$, $a \in \mathbb{R}$, and $\varepsilon > 0$, let $\mathcal{G}(\varphi, a, \varepsilon)$ be the set of f such that $m[\varphi_f \geq a] \geq 1 - \varepsilon$ or $m[\varphi_f \leq a] \geq 1 - \varepsilon$. (Here $[\varphi_f \geq a]$ is a shorthand for the set of $x \in M$ where $\varphi_f(x)$ exists and is greater than or equal to a.)

Lemma 3.9. *$\mathcal{G}(\varphi, a, \varepsilon)$ is a G_δ subset of* $\mathrm{Diff}^1_\omega(M)$.

Proof. Define

$$\mathcal{F}(\varphi, a, \alpha) = \Big\{ f;\ m[\varphi_f \geq a] \geq \alpha \Big\}.$$

So we have

$$\mathcal{G}(\varphi, a, \varepsilon) = \mathcal{F}(\varphi, a, 1 - \varepsilon) \cup \mathcal{F}(-\varphi, -a, 1 - \varepsilon),$$

We are going to prove that $\mathcal{F}(\varphi, a, \alpha)$ is a G_δ. Since the finite union of G_δ's is a G_δ, [4] the lemma will follow.

Let φ, a, α be fixed. Given $b < a$, $\beta < \alpha$, and $n_0, n_1 \in \mathbb{N}$ with $n_0 \leq n_1$, let $\mathcal{U}(b, \beta, n_0, n_1)$ be the set of f such that the set $\big[\max_{n \in [n_0, n_1]} \varphi_{f,n} > b\big]$ has measure $> \beta$. Then $\mathcal{U}(b, \beta, n_0, n_1)$ is open.

We will check that:

$$\mathcal{F}(\varphi, a, \alpha) = \bigcap_{b<a} \bigcap_{\beta<\alpha} \bigcap_{n_0} \bigcup_{n_1 > n_0} \mathcal{U}(b, \beta, n_0, n_1)\,, \tag{3.1}$$

where b and β are take rational values. First, we have:

$$[\varphi_f \geq a] = [\limsup \varphi_{f,n} \geq a] = \bigcap_{b<a} \bigcap_{n_0} \bigcup_{n \geq n_0} [\varphi_{f,n} > b] \quad \bmod 0.$$

---

[4] Proof: $\bigcap A_n \cup \bigcap B_n = \bigcap (A_1 \cap \cdots \cap A_n) \cup (B_1 \cap \cdots \cap B_n)$.

Then we have the following equivalences:

$$m[\varphi_f \geq a] \geq \alpha \iff \forall b < a,\ \forall n_0,\ m\left(\bigcup_{n \geq n_0} [\varphi_{f,n} > b]\right) \geq \alpha$$

$$\iff \forall b < a,\ \forall n_0,\ \forall \beta < \alpha,\ \exists n_1 > n_0 \text{ s.t. } m\left(\bigcup_{n=n_0}^{n_1} [\varphi_{f,n} > b]\right) \geq \alpha\,.$$

This proves (3.1), and hence that $\mathcal{F}(\varphi, a, \alpha)$ is a G_δ. □

Proof of Theorem A. First, we claim that if f is a ε-nearly ergodic map, then $f \in \mathcal{G}(\varphi, a, \varepsilon)$ for any $\varphi \in C^0(M, \mathbb{R})$ and $a \in \mathbb{R}$. Indeed, let X be the (invariant) set of points $x \in M$ where $\limsup \varphi_{f,n}(x) \geq a$. This is a bi essentially saturated set, because it is $\mathcal{W}^s$-saturated and it coincides mod 0 with the $\mathcal{W}^u$-saturated set $[\limsup \varphi_{f^{-1},n} \geq a]$. Since f is ε-nearly ergodic, $m(X)$ is either less than ε or greater than $1 - \varepsilon$. So either $m[\varphi_f > a]$ or $m[\varphi_f \leq a]$ is greater than $1 - \varepsilon$, showing that $f \in \mathcal{G}(\varphi, a, \varepsilon)$.

It follows from Proposition 3.1 that the sets $\mathcal{G}(\varphi, a, \varepsilon)$ are dense in $PH^1_\omega(M)$, while Lemma 3.9 says they are G_δ. Thus to complete the proof of the theorem, we need only to see that the set of ergodic diffeomorphisms is precisely

$$\bigcap_{\varphi, a, \varepsilon} \mathcal{G}(\varphi, a, \varepsilon)\,,$$

where φ varies on a dense subset $\mathcal{D}$ of $C^0(M, \mathbb{R})$, and a and ε take rational values. Indeed, if f is not ergodic then we can find $\varphi \in \mathcal{D}$, $a < b$ and $0 < \varepsilon < 1/2$ such that $[\varphi_f < a]$ and $[\varphi_f > b]$ both have measure greater than ε. Then f cannot belong to $\mathcal{G}(\varphi, a, \varepsilon) \cap \mathcal{G}(\varphi, b, \varepsilon)$. □

4. A Proof of Theorem 3.5

The following is the symplectic version of Mañé's Ergodic Closing Lemma [M], proved by Arnaud [Arna]. If $f \in \mathrm{Diff}^1_\omega(M)$ and $x \in M$, we say that x is *f-closable* if for every $\varepsilon > 0$ there exists a ε-perturbation $\tilde{f} \in \mathrm{Diff}^1_\omega(M)$ such that x is periodic for $\tilde{f}$ and moreover $d(\tilde{f}^i x, f^i x) < \varepsilon$ for every i between 0 and the $\tilde{f}$-period of x.

Theorem 4.1 ([Arna]). *For every $f \in \mathrm{Diff}^1_\omega(M)$, m-almost every point is f-closable.*

The first step to obtain Theorem 3.5 is to find an "almost elliptic" periodic point, that is a periodic point whose center eigenvalues are close to the unit circle:

Lemma 4.2. *Let $f \in PH^1_\omega(M)$ have unbreakable center. Then for every $\varepsilon > 0$ there exists an ε-perturbation $\tilde{f}$ and a periodic point x of period p for $\tilde{f}$ such that all eigenvalues μ_i of $D\tilde{f}^p|E^c(x)$ satisfy $\big|\log|\mu_i|\big| \leq \varepsilon p$.*

Proof. Let f and ε be given. Write $D^c f = Df|E^c$. Since the eigenvalues of a symplectic map are symmetric, to prove the lemma it suffices to find an ε-perturbation $\tilde{f}$ with a periodic point x of period p such that:

$$\|D^c \tilde{f}^{mp}(x)\| < e^{\varepsilon mp} \text{ for some } m \geq 1. \tag{4.1}$$

Due to Theorem 3.2, we can assume that $\lambda^c(f) = 0$. Therefore there exists k such that $\frac{1}{k}\int_M \log\|D^c f^k\|\, dm < \varepsilon$. Hence, for all x in a set of positive measure,

$$\lim_{m \to \infty} \frac{1}{km} \sum_{i=0}^{m-1} \log\|D^c f^k(f^{ik}(x))\| < \varepsilon, \tag{4.2}$$

By Theorem 4.1, we can take an f-closable point x such that (4.2) holds. If x is periodic then (4.2) follows with $\tilde{f} = f$. Otherwise, let $f_j \in PH^1_\omega(M)$ be a sequence converging to f in the C^1 topology such that x is periodic (of period p_j) for f_j. Then $p_j \to \infty$. Let $m_j = \lfloor p_j/k \rfloor$. We estimate:

$$\frac{1}{p_j} \log \|D^c f_j^{p_j}(x)\| \leq \frac{1}{km_j} \sum_{i=0}^{m_j-1} \log \|D^c f_j^k(f_j^{ik}(x))\| + \frac{1}{m_j} \log \|D^c f_j\|_\infty$$

As $j \to \infty$, the right hand side converges to the left hand side of (4.2). Thus the result follows with $\tilde{f} = f_j$ for j sufficiently large. □

Next we see how the eigenvalues can be adjusted:

Lemma 4.3. *Let $\varepsilon > 0$. Let $A_1, \ldots, A_n$ be symplectic matrices and let $2d$ be the number of eigenvalues μ_i of $A_n \cdots A_1$ (counted with multiplicity) such that $\frac{1}{n}\log|\mu_i| \leq \varepsilon$. Then there exist symplectic matrices $B_1, \ldots, B_n$ such that $\|B_i - Id\| \leq e^\varepsilon - 1$ and $A_n B_n \cdots A_1 B_1$ has exactly $2d$ eigenvalues (counted with multiplicity) in the unit circle.*

Proof. Assume the matrices have size $2N \times 2N$. Let $\{p_1, \ldots, p_N, q_1, \ldots, q_N\}$ be the canonical symplectic and orthonormal basis of $\mathbb{R}^{2N}$.

Write $A^i = A_i \cdots A_1$. Let $\lambda_1 > \cdots > \lambda_t$ be the Lyapunov exponents of A^n and let $\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_t = \mathbb{R}^{2N}$ be the Lyapunov filtration of A^n, that is, $A^n(F_i) = F_i$ and the action of A^n on F_i/F_{i-1} has eigenvalues of modulus e^{λ_i}. Let $r(i)$ be the dimension of F_i. Let $m \geq 0$ be maximal with $\lambda_m > 0$, and let $0 \leq u \leq m$ be maximal with $\lambda_u \geq \varepsilon$. Notice that $\dim F_{t-u}/F_u = 2d$.

Let $F_i^k = A^k(F_i)$, $0 \leq k \leq n-1$. There exists symplectic orthogonal matrices C_0, $\ldots$, C_{n-1} such that if $i \leq m$ then $C_k F_i^k$ is spanned by p_1, $\ldots$, $p_{r(i)}$. It follows that if $i > m$ then $C_k F_i^k$ is spanned by $p_1, \ldots, p_d, q_d, \ldots, q_{r(i)-d}$.

Let us consider a symplectic matrix Λ such that $\Lambda p_k = p_k$ and $\Lambda q_k = q_k$, unless $r(i-1) < k \leq r(i)$ for some $u < i \leq m$, in which case we let $\Lambda p_k = e^{-\lambda_i/n} p_k$, $\Lambda q_k = e^{\lambda_i/n} q_k$.

Let $B_k = C_k^{-1} \Lambda C_k$. Then $T = A_n B_n \cdots A_1 B_1$ preserves the spaces F_i, $0 \leq i \leq t$, and T acts on F_i/F_{i-1} as A^n, unless $u < i \leq m$ or $k - m < i \leq k - u$, in which case T acts as $e^{-\lambda_i} A^n$. It follows that the action of T on F_{k-u}/F_u has only zero Lyapunov exponents. The result follows. □

Proof of Theorem 3.5. By Lemma 4.2 we can perturb f and create an "almost elliptic" periodic point. Lemma 4.3 says that the derivatives along this orbit can be perturbed to become completely elliptic. Using Lemma 3.6 we can realize this by a further perturbation of the diffeomorphism. □

The argument above could have been carried out by appealing to the easier cocycle version of [B] obtained in [BV]: see the Appendix of this paper.

5. Proof of Theorem B

We adopt as much as possible notation that is consistent with the notation in [BW], as the proof of Theorem B has many parallels with the proof of Theorem 3.1 there. A few statements are also adapted bearing in mind the needs of the proof of Theorem D given in the appendix.

5.1. **Density.** If ν is a measure and A and B are ν-measurable sets with $\nu(B) > 0$, we define the *density of A in B* by:

$$\nu(A : B) = \frac{\nu(A \cap B)}{\nu(B)}.$$

A point $x \in M$ is a *Lebesgue density point* of a measurable set $X \subseteq M$ if

$$\lim_{r\to 0} m(X : B_r(x)) = 1.$$

The Lebesgue Density Theorem implies that if X is a measurable set and $\widehat{X}$ is the set of Lebesgue density points of X, then $m(X \,\triangle\, \widehat{X}) = 0$.

Lebesgue density points can be characterized using nested sequences of measurable sets. We say that a sequence of measurable sets Y_n *nests* at point x if $Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset \{x\}$, and

$$\bigcap_n Y_n = \{x\}.$$

A nested sequence of measurable sets Y_n is *regular* if there exists $\delta > 0$ such that, for all $n \geq 0$, we have $m(Y_n) > 0$, and

$$m(Y_{n+1}) \geq \delta m(Y_n).$$

Two nested sequences of sets Y_n and Z_n are *internested* if there exists a $k \geq 1$ such that, for all $n \geq 0$, we have

$$Y_{n+k} \subseteq Z_n, \qquad \text{and} \qquad Z_{n+k} \subseteq Y_n.$$

The following lemma is a straightforward consequence of the definitions.

Lemma 5.1 ([BW], Lemma 2.1)**.** *Let Y_n and Z_n be internested sequences of measurable sets, with Y_n regular. Then Z_n is also regular. If the sets Y_n have positive measure, then so do the Z_n, and, for any measurable set X,*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} m(X : Z_n) = 1.$$

5.2. **Foliations and Absolute Continuity.** Let $\mathcal{F}$ be a foliation with smooth d-dimensional leaves. An open set $U \subset M$ is a *foliation box for $\mathcal{F}$* if it is the image of $\mathbb{R}^{n-d} \times \mathbb{R}^d$ under a homeomorphism that sends each vertical $\mathbb{R}^d$-slice into a leaf of $\mathcal{F}$. The images of the vertical $\mathbb{R}^d$-slices are called *local leaves of $\mathcal{F}$ in U*.

A *smooth transversal* to $\mathcal{F}$ in U is a smooth codimension-d disk in U that intersects each local leaf in U exactly once and whose tangent bundle is uniformly transverse to $T\mathcal{F}$. If τ_1 and τ_2 are two smooth transversals to $\mathcal{F}$ in U, we have the *holonomy map* $h_{\mathcal{F}} : \tau_1 \to \tau_2$, which takes a point in τ_1 to the intersection of its local leaf in U with τ_2.

If $S \subseteq M$ is a smooth submanifold, we denote by m_S the volume of the induced Riemannian metric on S. If $\mathcal{F}$ is a foliation with smooth leaves, and A is contained in a single leaf of $\mathcal{F}$ and is measurable in that leaf, then we denote by $m_{\mathcal{F}}(A)$ the induced Riemannian volume of A in that leaf.

A foliation $\mathcal{F}$ with smooth leaves is *transversely absolutely continuous with bounded Jacobians* if for every angle $\alpha \in (0, \pi/2]$, there exists $C \geq 1$ and $R_0 > 0$ such that, for every foliation box U of diameter less than R_0, any two smooth transversals τ_1, τ_2 to $\mathcal{F}$ in U of angle at least α with $\mathcal{F}$, and any m_{τ_1}–measurable set A contained in τ_1:

$$C^{-1} m_{\tau_1}(A) \leq m_{\tau_2}(h_{\mathcal{F}}(A)) \leq C m_{\tau_1}(A). \tag{5.1}$$

The foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ for a partially hyperbolic diffeomorphism are transversely absolutely continuous with bounded Jacobians (see [AV]).

Let $\mathcal{F}$ be an absolutely continuous foliation and let U be a foliation box for $\mathcal{F}$. Let τ be a smooth transversal to $\mathcal{F}$ in U. Let $Y \subseteq U$ be a measurable set. For a point $q \in \tau$, we define the *fiber $Y(q)$ of Y over q* to be the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base τ_Y of Y* is the set of all $q \in \tau$ such that the fiber $Y(q)$ is $m_{\mathcal{F}}$-measurable and $m_{\mathcal{F}}(Y(q)) > 0$. The absolute continuity of $\mathcal{F}$ implies that τ_Y is m_τ-measurable. We say "Y fibers over Z" to indicate that $Z = \tau_Y$.

If, for some $c \geq 1$, the inequalities

$$c^{-1} \leq \frac{m_{\mathcal{F}}(Y(q))}{m_{\mathcal{F}}(Y(q'))} \leq c$$

hold for all $q, q' \in \tau_Y$, then we say that Y *has c-uniform fibers*. A sequence of measurable sets Y_n contained in U has *c-uniform fibers* if each set in the sequence has c-uniform fibers, with c independent of n.

Proposition 5.2 ([BW], § 2.3). *Suppose that the foliation $\mathcal{F}$ is absolutely continuous with bounded Jacobians. Let U be a foliation box for $\mathcal{F}$, and let τ be a smooth transversal to $\mathcal{F}$ in U. Let Y_n and Z_n be sequences of measurable subsets of U with c-uniform fibers.*

1) *Suppose that there exists $\delta > 0$ such that:*
   (a) *for all $n \geq 0$,*
   $$m_\tau(\tau_{Y_{n+1}}) \geq \delta m_\tau(\tau_{Y_n});$$
   (b) *for all $n \geq 0$, there are points $z \in \tau_{Y_{n+1}}, z' \in \tau_{Y_n}$ with*
   $$m_{\mathcal{F}}(Y_{n+1}(z)) \geq \delta m_{\mathcal{F}}(Y_n(z')).$$
   Then Y_n is regular.
2) *Suppose that $\tau_{Y_n} = \tau_{Z_n}$, for all n and that Y_n and Z_n both nest at a common point x. Then, for any set $X \subseteq U$ that is essentially $\mathcal{F}$-saturated at x, we have the equivalence:*
   $$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m(X : Z_n) = 1.$$
3) *For every measurable set X that is $\mathcal{F}$-saturated at x, we have the equivalence:*
   $$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m_\tau(\tau_X : \tau_{Y_n}) = 1.$$

5.3. **Construction of an Adapted Metric.** We begin with some notation. Again fix the diffeomorphism $f \colon M \to M$. For $x \in M$ and $j \in \mathbb{Z}$ we denote by x_j the jth iterate $f^j(x)$. If α, β are positive functions defined on the forward orbit $\mathcal{O}^+(p) = \{p_j;\ j \geq 0\}$ of some $p \in M$, we write $\alpha \prec \beta$ if there exists a positive constant $\lambda < 1$ such that for all $y \in \mathcal{O}^+(p)$:

$$\frac{\alpha(y)}{\beta(y)} < \lambda.$$

Notice that if α, β happen to extend from $\mathcal{O}^+(p)$ to continuous functions on M satisfying the pointwise inequality $\alpha < \beta$, then compactness of M implies that $\alpha \prec \beta$.

If α is a positive function, and $j \geq 1$ is an integer, let

$$\alpha_j(x) = \alpha(x)\alpha(x_1)\cdots\alpha(x_{j-1}),$$

and

$$\alpha_{-j}(x) = \alpha(x_{-j})^{-1}\alpha(x_{-j+1})^{-1}\cdots\alpha(x_{-1})^{-1}.$$

We set $\alpha_0(x) = 1$. Observe that α_j is a multiplicative cocycle; in particular, we have $\alpha_{-j}(x)^{-1} = \alpha_j(x_{-j})$.

Lemma 5.3. *Let $f \colon M \to M$ be C^1 and partially hyperbolic, and let $p \in CB^+$. Then there exist functions $B, \nu, \hat{\nu}, \gamma, \hat{\gamma} \colon \mathcal{O}^+(p) \to \mathbb{R}_+$, bounded from below, and a Riemannian metric $\|\cdot\|_\star$ defined on $T_{\mathcal{O}^+(p)}M$ with the following properties:*

1) *$\nu \prec \gamma\hat{\gamma} \leq 1$ and $\hat{\nu} \prec 1$;*
2) *for y in $\mathcal{O}^+(p)$,*

$$\|D_y f|_{E^s}\|_\star \prec \nu(y) \prec \gamma(y) \prec \mathbf{m}_\star(D_y f|_{E^c}) \leq \|D_y f|_{E^c}\|_\star \prec \hat{\gamma}(y)^{-1} \prec \hat{\nu}(y)^{-1} \prec \mathbf{m}_\star(D_y f|_{E^u});$$

3) $\limsup_{j\to\infty} B(p_j)^{1/j} = 1$;

4) *for all $v \in T_{p_j}M$ and $j \geq 0$:*

$$\|v\| \leq \|v\|_\star \leq B(p_j)\|v\|. \tag{5.2}$$

Proof. Let $\|\cdot\|_1$ be a Riemannian metric on $T_{\mathcal{O}^+(p)}M$ that coincides with $\|\cdot\|$ on each of the three spaces $E^s(p_j)$, $E^c(p_j)$, and $E^u(p_j)$, but with respect to which those three spaces are orthogonal. Notice that there exists a constant $C \geq 1$ such that $C^{-1}\|v\| \leq \|v\|_1 \leq C\|v\|$ for every $v \in T_{\mathcal{O}^+(p)}M$.

Let us define another Riemannian metric $\|\cdot\|_2$ on $T_{\mathcal{O}^+(p)}M$ as follows. Let i_k be as in the definition of forward center bunching. With respect to the inner product induced by $\|\cdot\|_1$, the linear map $D_{p_{i_k}}f^{i_{k+1}-i_k}$ can be written in a unique way as $O_k P_k^{i_{k+1}-i_k}$ where $P_k : T_{p_{i_k}}M \to T_{p_{i_k}}M$ is selfadjoint positive and $O_k : T_{p_{i_k}}M \to T_{p_{i_{k+1}}}M$ is an isometry: indeed $P_k^{2(i_{k+1}-i_k)} = (D_{p_{i_k}}f^{i_{k+1}-i_k})^* \cdot D_{p_{i_k}}f^{i_{k+1}-i_k}$. Notice that P_k preserves the spaces $E^s(p_{i_k})$, $E^c(p_{i_k})$, and $E^u(p_{i_k})$. Define $\|\cdot\|_2$ on $T_{\mathcal{O}^+(p)}M$ so that for $i_k \leq j < i_{k+1}$, the map

$$D_{p_{i_k}}f^{j-i_k} \cdot P_k^{-(j-i_k)} : (T_{p_{i_k}}M, \|\cdot\|_1) \to (T_{p_j}M, \|\cdot\|_2)$$

is an isometry. By construction, for each $i_k \leq j < i_{k+1}$, and for each subbundle $F = E^u$, E^c, E^s, we have $\|D_{p_j}f|_F\|_2^{i_{k+1}-i_k} = \|D_{p_{i_k}}f^{i_{k+1}-i_k}|_F\|$ and $\mathbf{m}_2(D_{p_j}f|_F)^{i_{k+1}-i_k} = \mathbf{m}(D_{p_{i_k}}f^{i_{k+1}-i_k}|_F)$. The definitions of partial hyperbolicity and forward center bunching then immediately imply that there exists $\rho < 1$ such that

$$\|D_y f|_{E^s}\|_2 \leq \rho^2 \mathbf{m}_2(D_y f|_{E^c}) \min\{1, \|D_y f|_{E^c}\|_2^{-1}\}, \quad \text{and}$$
$$\max\{1, \|D_y f|_{E^c}\|_2\} \leq \rho^2 \mathbf{m}_2(D_y f|_{E^u})$$

for every $y \in \mathcal{O}^+(p)$.

Notice that $\|\cdot\|_2$ and $\|\cdot\|_1$ coincide for $T_{p_{i_k}}M$ for each k. Let $C_j \geq 1$ be minimal such that $C_j^{-1}\|v\| \leq \|v\|_2 \leq C_j\|v\|$ for every $v \in T_{p_j}M$. The condition $i_{k+1}/i_k \to 1$ then implies that $C_j^{1/j} \to 1$. Let $D_j \geq C_j$ be a sequence such that $D_j \leq D_{j+1} \leq \rho^{-1}D_j$ and $D_j^{1/j} \to 1$. For every $j \geq 0$, let $\|\cdot\|_\star = D_j\|\cdot\|_2$ over $T_{p_j}M$, and $B(p_j) = D_jC_j$. For $y \in \mathcal{O}^+(p)$, we define $\nu(y) = \rho^{-1/4}\|D_y f|_{E^s}\|_\star$, $\gamma(y) = \rho^{1/4}\mathbf{m}_\star(D_y f|_{E^c})$, $\hat{\gamma}(y) = (\rho^{1/4}\|D_y f|_{E^c}\|_\star)^{-1}$, and $\hat{\nu}(y) = (\rho^{-1/4}\mathbf{m}_\star(D_y f|_{E^u}))^{-1}$. All desired properties are straightforward to check. $\square$

We next show that the sets CB^+ and CB^- are respectively $\mathcal{W}^s$ and $\mathcal{W}^u$-saturated.

Proof of Proposition 2.1. We will use the previous lemma and its proof. Let $p \in CB^+$ and $q \in \mathcal{W}^s(p)$, and let $p_j = f^j(p)$, $q_j = f^j(q)$. Choose invertible linear maps $A_j : T_{p_j}M \to T_{q_j}M$, bounded and with bounded inverses with respect to $\|\cdot\|$, that preserve the bundles E^s and E^c, and such that $A_{j+1}^{-1}D_{q_j}fA_j$ is exponentially close to $D_{p_j}f$ (here we use that Df and the bundles E^s and E^c are Hölder). This implies that $A_{j+1}^{-1}D_{q_j}fA_j$ is also exponentially close to $D_{p_j}f$ with respect to $\|\cdot\|_\star$. It follows that there exists $\delta > 0$ such that

$$\|A_{j+1}^{-1}D_{q_j}fA_j|_{E^s}\|_\star \cdot \mathbf{m}_\star(A_{i_{j+1}}^{-1}D_{q_j}fA_j|_{E^c})^{-1} \cdot \|A_{j+1}^{-1}D_{q_j}fA_j|_{E^c}\|_\star \leq 1 - \delta$$

for every sufficiently large j. Let i_k be as in the definition of forward center bunching for p. By the proof of the previous lemma, $\|\cdot\|_\star$ and $\|\cdot\|$ coincide modulo a constant factor over $E^s(p_{i_k})$ and $E^c(p_{i_k})$, so

$$\|A_{i_{k+1}}^{-1}D_{q_{i_k}}f^{i_{k+1}-i_k}A_{i_k}|_{E^s}\| \cdot \mathbf{m}(A_{i_{k+1}}^{-1}D_{q_{i_k}}f^{i_{k+1}-i_k}A_{i_k}|_{E^c})^{-1} \cdot \|A_{i_{k+1}}^{-1}D_{q_{i_k}}f^{i_{k+1}-i_k}A_{i_k}|_{E^c}\| \leq (1-\delta)^{i_{k+1}-i_k}$$

for every k sufficiently large. Since the maps A_j, A_j^{-1} are uniformly bounded with respect to $\|\cdot\|$, and preserve E^s and E^c, we see that there exists $n \geq 1$ such that every

$k \geq 0$,

$$\|D_{q_{i_{nk}}} f^{i_{nk+n}-i_{nk}}|_{E^s}\|^{-1} \geq (1+\delta)^{i_{nk+n}-i_{nk}} \, \frac{\|D_{q_{i_{nk}}} f^{i_{nk+n}-i_{nk}}|_{E^c}\|}{\mathbf{m}(D_{q_{i_{nk}}} f^{i_{nk+n}-i_{nk}}|_{E^c})}.$$

Since $i_{nk+n}/i_{nk} \to 1$, we conclude that $q \in CB^+$.

It follows by symmetry that CB^- is $\mathcal{W}^u$-saturated. □

Fix $R_0 > 0$ less than injectivity radius of M in the original $\|\cdot\|$ metric. Let exp denote the exponential map for the $\|\cdot\|$ metric. Consider the neighborhood $\mathcal{N}_{R_0}$ of $\mathcal{O}^+(p)$ defined by

$$\mathcal{N}_{R_0} = \bigsqcup_{j\geq 0} B(p_j, R_0),$$

where $B(x,r)$ denotes the ball of radius r centered at x in the original Riemannian metric. The manifold $\mathcal{N}_{R_0}$ carries the restriction of the original Riemannian metric. When we speak of volumes and induced Riemannian volumes on submanifolds of $\mathcal{N}_{R_0}$, it will always be with respect to this metric.

We introduce two other metrics on $\mathcal{N}_{R_0}$ that will be used in this proof, one of them closely related to (and comparable to) the original metric. The first metric is the *flat* $\|\cdot\|$ *metric*, denoted $\|\cdot\|_\flat$, which is the (locally) flat Riemannian metric defined as follows. For $x \in B(p_j, R_0)$, and $v, w \in T_{p_j}M$, we set

$$\langle v, w\rangle_\flat = \langle D_x \exp_{p_j}^{-1}(v), D_x \exp_{p_j}^{-1}(w)\rangle_{p_j},$$

where we make the standard identification $T_u(T_pM) \simeq (T_pM)$. In the distance $d_\flat$ induced by this metric, we have, for $q, q' \in B(p_j, R_0)$, $d_\flat(q,q') = \|\exp_{p_j}^{-1}(q) - \exp_{p_j}^{-1}(q')\|_{p_j}$. Compactness of M implies that $\|\cdot\|$ and $\|\cdot\|_\flat$ are comparable.

Next we extend the $\|\cdot\|_\star$ metric, which is defined on $T_{\mathcal{O}^+(p)}M$, to a flat metric $\|\cdot\|_\star$ on $\mathcal{N}_{R_0}$ using the same type of construction. For $x \in B(p_j, R_0)$, and $v, w \in T_{p_j}M$, we set

$$\langle v, w\rangle_\star = \langle D_x \exp_{p_j}^{-1}(v), D_x \exp_{p_j}^{-1}(w)\rangle_{\star, p_j}.$$

Denote by $d_\star$ the distance induced by this Riemannian metric, so that, for $q, q' \in B(p_j, R_0)$, we have $d_\star(q,q') = \|\exp_{p_j}^{-1}(q) - \exp_{p_j}^{-1}(q')\|_\star$.

The results of this section imply that on $B(p_j, R_0)$, we have $Kd_\flat \leq d_\star \leq B(p_j)d_\flat$. Thus on any component $B(p_j, R_0)$, the $\star$ and $\flat$ metrics are uniformly comparable. The degree of comparability decays subexponentially as $j \to \infty$. For $q \in \mathcal{N}_{R_0}$ and $r > 0$ sufficiently small, we denote by $B_\star(q,r)$ the $d_\star$-ball of radius r centered at q.

By uniformly rescaling the $\|\cdot\|_\flat$ and $\|\cdot\|_\star$ metrics by the same constant factor, we may assume that for some $R > 1$, and any $x \in M$, the Riemannian balls $B_\star(x,R)$ and $B(x,R)$ are contained in foliation boxes for both $\mathcal{W}^s$ and $\mathcal{W}^u$. We assume both R and R_0 are large enough so that all the objects considered in the sequel are small compared with R and R_0.

5.4. **Fake Invariant Foliations.** Let $\mathbf{r} : \mathcal{O}^+(p) \to \mathbb{R}_+$ be any positive function such that $\sup_{j\geq 0} \mathbf{r}(p_j) \leq R_0$. Denote by $\mathcal{N}_\mathbf{r}$ the following neighborhood of $\mathcal{O}^+(p)$:

$$\mathcal{N}_\mathbf{r} = \bigsqcup_{j\geq 0} B(p_j, \mathbf{r}(j)).$$

If $\mathcal{F}$ is a foliation of $\mathcal{N}_\mathbf{r}$, and $B_\star(x,r)$ is contained in a foliation box U for $\mathcal{F}$, then we will denote by $\mathcal{F}_\star(x,r)$ the intersection of the local leaf of $\mathcal{F}$ at x with $B_\star(x,r)$. Notice that $\mathcal{F}_\star(x,r) \subseteq \mathcal{F}(x, K^{-1}r)$.

Proposition 5.4. *For every $\varepsilon > 0$, there exist functions $\mathbf{r}, \mathbf{R}\colon \mathcal{O}^+(p) \to \mathbb{R}$ satisfying:*

$$\mathbf{r} < \mathbf{R}, \quad \sup_{y \in \mathcal{O}^+(p)} \mathbf{R}(y) < R_0, \quad \inf_{j\geq 0} \frac{\mathbf{r}(p_{j+1})}{\mathbf{r}(p_j)} > e^{-\varepsilon}, \quad \textit{and} \quad \inf_{j\geq 0} \frac{\mathbf{R}(p_{j+1})}{\mathbf{R}(p_j)} > e^{-\varepsilon},$$

and such that the neighborhood $\mathcal{N}_{\mathbf{R}}$ is foliated by foliations $\widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^c$, $\widehat{\mathcal{W}}^{cu}$ and $\widehat{\mathcal{W}}^{cs}$ with the following properties, for each $\beta \in \{u, s, c, cu, cs\}$:

1) **Almost tangency to invariant distributions:** *For each $q \in \mathcal{N}_{\mathbf{R}}$, the leaf $\widehat{\mathcal{W}}^\beta(q)$ is C^1 and the tangent space $T_q\widehat{\mathcal{W}}^\beta(q)$ lies in a cone of $\|\cdot\|_\star$-angle ε about $E^\beta(q)$ and also within a cone of $\|\cdot\|$-angle ε about $E^\beta(q)$.*
2) **Local invariance:** *for each $y \in \mathcal{O}^+(p)$ and $q \in B(y, \mathbf{r}(y))$,*
$$f(\widehat{\mathcal{W}}^\beta(q, \mathbf{r}(y))) \subset \widehat{\mathcal{W}}^\beta(q_1), \text{ and } f^{-1}(\widehat{\mathcal{W}}^\beta(q_1, \mathbf{r}(y_1))) \subset \widehat{\mathcal{W}}^\beta(q).$$
3) **Exponential growth bounds at local scales:** *The following hold for all $n \geq 0$ and $y \in \mathcal{O}^+(p)$.*
   (a) *Suppose that $q_j \in B_\star(y_j, \mathbf{r}(y_j))$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^s(q, \mathbf{r}(y))$, then $q'_n \in \widehat{\mathcal{W}}^s(q_n, \mathbf{r}(y_n))$, and
   $$d_\star(q_n, q'_n) \leq \nu_n(y) d_\star(q, q').$$
   If $q'_j \in \widehat{\mathcal{W}}^{cs}(q_j, \mathbf{r}(y_j))$ for $0 \leq j \leq n-1$, then $q'_n \in \widehat{\mathcal{W}}^{cs}(q_n)$, and
   $$d_\star(q_n, q'_n) \leq \hat{\gamma}_n(y)^{-1} d_\star(q, q').$$
   (b) *Suppose that $q_{-j} \in B_\star(y_{n-j}, \mathbf{r})$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^u(q, \mathbf{r}(y_n))$, then $q'_{-n} \in \widehat{\mathcal{W}}^u(q_{-n}, \mathbf{r}(y))$, and
   $$d_\star(q_{-n}, q'_{-n}) \leq \hat{\nu}_n(y) d_\star(q, q').$$
   If $q'_{-j} \in \widehat{\mathcal{W}}^{cu}(q_{-j}, \mathbf{r}(y_{n-j}))$ for $0 \leq j \leq n-1$, then $q'_{-n} \in \widehat{\mathcal{W}}^{cu}(q_{-n})$, and
   $$d_\star(q_{-n}, q'_{-n}) \leq \gamma_n(y)^{-1} d_\star(q, q').$$
4) **Coherence:** *$\widehat{\mathcal{W}}^s$ and $\widehat{\mathcal{W}}^c$ subfoliate $\widehat{\mathcal{W}}^{cs}$; $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^c$ subfoliate $\widehat{\mathcal{W}}^{cu}$.*
5) **Uniqueness:** *$\widehat{\mathcal{W}}^s(p) = \mathcal{W}^s(p, \mathbf{R}(p))$, and $\widehat{\mathcal{W}}^u(p) = \mathcal{W}^u(p, \mathbf{R}(p))$.*
6) **Regularity:** *The foliations $\widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^c$, $\widehat{\mathcal{W}}^{cu}$ and $\widehat{\mathcal{W}}^{cs}$ and their tangent distributions are uniformly Hölder continuous, in both the $d_\star$ and d metrics.*
7) **Regularity of the strong foliation inside weak leaves:** *the restriction of the foliation $\widehat{\mathcal{W}}^s$ to each leaf of $\widehat{\mathcal{W}}^{cs}$ is absolutely continuous with bounded jacobians, and the restriction of the foliation $\widehat{\mathcal{W}}^u$ to each leaf of $\widehat{\mathcal{W}}^{cu}$ is absolutely continuous with bounded jacobians (with respect to the standard Riemannian metric and volume).*

   There exists a constant $L > 0$ such that for any $p' \in \mathcal{W}^s(p)$, the $\widehat{\mathcal{W}}^s$-holonomy map $h^s\colon \widehat{\mathcal{W}}^c(p) \to \widehat{\mathcal{W}}^c(p')$ is L-bi-Lipschitz at p. That is, for all $q \in \widehat{\mathcal{W}}^c(p)$, we have:
   $$L^{-1} d_\star(p, q) \leq d_\star(h^s(p), h^s(q)) \leq L d_\star(p, q).$$

Proof of Proposition 5.4. The proof follows closely the proof of Proposition 3.1 in [BW]. Our construction will be performed in two steps. In the first, we construct foliations of each tangent space T_yM, $y \in \mathcal{O}(p)$. In the second step, we use the exponential map $\exp_y$ to project these foliations from a neighborhood of the origin in T_yM to a neighborhood of y.

The argument diverges slightly from the argument in [BW] in that, because we are in the nonuniform setting, the Hölder continuity of Df (in this case Lipschitz continuity) must be used explicitly in the construction of the fake foliations.

Step 1. We extend the $\|\cdot\|_\star$-metric on $T_{O^+(p)}M$ to a metric on $T_{O(p)}M$, which we also denote by $\|\cdot\|_\star$, by setting it equal to $\|\cdot\|$ on $\bigsqcup_{j\leq 0} T_{p_j}M$. Extend the function B to $O(p)$ by setting $B(p_j) = 1$ for $j \leq 0$.

Fix a constant $R_1 < R_0$ such that the diameter of $f(B(x, R_1))$ is less than R_0, for all $x \in M$. For $v \in T_{p_j}M$, $\|v\| \leq R_1$, let $\tilde{f}_j(v) = \exp^{-1}_{p_{j+1}} \circ f \circ \exp_{p_j}(v)$. Then $D_0\tilde{f}_j = D_{p_j}f$ and so, since f is C^2:

$$\tilde{f}_j(v) = D_{p_j}f(v) + O(\|v\|^2), \qquad \text{and} \qquad \|D_v\tilde{f}_j - D_{p_j}f\| \leq O(\|v\|), \tag{5.3}$$

uniformly in j. Fix a family of smooth bump functions $\{\beta_r : \mathbb{R} \to [0, 1], r > 0\}$ with the properties that $|\beta_r'| \leq 3r^{-1}$, $\beta_r(t) = 1$ for $|t| \leq r^2$, and $\beta_r(t) = 0$ for $|t| \geq 4r^2$.

For $r \in (0, R_1)$, define $F_{j,r} : T_{p_j}M \to T_{p_{j+1}}(M)$ by:

$$F_{j,r}(v) = \beta_r(\|v\|^2)\tilde{f}_j(v) + (1 - \beta_r(\|v\|^2))D_{p_j}f(v).$$

One easily checks using (5.3) that $d_{C^1}(F_{j,r}, D_{p_j}f) \leq O(r)$, uniformly in j and that $F_{j,r}(v) = \tilde{f}(v)$ for $\|v\| \leq r$, and $F_{j,r}(v) = D_{p_j}f(v)$ for $\|v\| \geq 2r$.

For any function $\mathbf{r}\colon O(p) \to \mathbb{R}_+$ with $\sup_{y\in O(p)} \mathbf{r}(y) < R_1$, define a C^2 bundle map $F_\mathbf{r}\colon T_{O(p)}M \to T_{O(p)}M$, by setting $F_\mathbf{r} = F_{\mathbf{r}(\mathbf{p_j}),j}$ on $T_{p_j}M$. Then $F_\mathbf{r}$ covers $f : O(p) \to O(p)$, and has the following properties:

1) $F_\mathbf{r}$ coincides with $\exp^{-1}_{p_{j+1}} \circ f \circ \exp_{p_j}$ on the $\|\cdot\|$-ball of radius $\mathbf{r}(p_j)$ in $T_{p_j}M$ and with $D_{p_j}f$ outside the ball of radius $2\mathbf{r}(p_j)$
2) The C^1 distance from $F_\mathbf{r}$ to Df on approaches 0 uniformly as $|\mathbf{r}|_\infty \to 0$. In particular, on $T_{p_j}M$, we have $d_{C^1}(F_\mathbf{r}, D_{p_j}f) \leq O(\mathbf{r}(p_j))$.

When measured in the $\|\cdot\|_\star$-metric, the C^1 distance between two functions on $T_{p_j}(M)$ is multiplied by $B(p_j)$. It follows that:

3) On $T_{p_j}M$, we have $d_{C^1}(F_\mathbf{r}, D_{p_j}f)_\star \leq O(B(p_j)\mathbf{r}(p_j))$, uniformly in j; that is, $d_{C^1}(F_\mathbf{r}, Df)_\star \leq O(B\mathbf{r})$.

Let $\varepsilon > 0$ be given. Fix $\varepsilon_1 < \varepsilon$ such that

$$e^{-2\varepsilon} > \sup_{y\in O^+(p)} \max\left\{\nu(y),\, \hat{\nu}(y),\, \frac{\nu(y)}{\gamma(y)},\, \frac{\nu(y)}{\gamma(y)},\, \frac{\hat{\nu}(y)}{\hat{\gamma}(y)},\, \frac{\nu(y)}{\gamma\hat{\gamma}(y)}\right\}. \tag{5.4}$$

For $c > 0$, define a function $\mathbf{R}_c\colon O(p) \to \mathbb{R}_+$ by

$$\mathbf{R}_c(p_j) = \begin{cases} c, & \text{if } j \leq 0 \\ ce^{-j\varepsilon'}, & \text{if } j > 0. \end{cases}$$

Since $\limsup_{j\to\infty} B(p_j)\mathbf{R}_c(p_j) = 0$, the argument above shows that $d_{C^1}(F_{\mathbf{r}_c}, Df)_\star$ tends to 0 uniformly as $c \to 0$. This also implies that the C^1 distance in the original Riemannian metric $\|\cdot\|$ tends to 0 uniformly in c.

Since Df is uniformly partially hyperbolic in both metrics, we may choose c sufficiently small so that $F = F_{\mathbf{R}_c}$ is uniformly partially hyperbolic in both $\|\cdot\|$ and $\|\cdot\|_\star$ metrics. Note that F is C^{1+Lip} in the $\|\cdot\|_\star$ metric, with Lipschitz constant of DF, DF^{-1} on $T_{p_j}M$ bounded by a constant $L(p_j) > 0$ with the property $\limsup_{j\to\infty} L(p_j)^{1/j} = 1$. F is uniformly C^2 in $\|\cdot\|$ metric. Note also that F is $C^{1-\varepsilon}$ in the $\|\cdot\|_\star$ metric, with Hölder constant of DF, DF on $T_{p_j}M$ bounded by a constant. If c is small enough, the equivalents of inequalities (3)–(6) will hold for TF.

If c is sufficiently small, standard graph transform arguments give stable, unstable, center-stable, and center-unstable foliations for F_r inside each T_pM. These foliations are uniquely determined by the extension F and the requirement that their leaves be graphs of bounded functions. We obtain a center foliation by intersecting the leaves of the center-unstable and center-stable foliations. While TM

is not compact, all of the relevant estimates for F are uniform, and it is this, not compactness, that counts.

The uniqueness of the stable and unstable foliations imply, via a standard argument (see, e.g. [HPS], Theorem 6.1 (e)), that the stable foliation subfoliates the center-stable, and the unstable subfoliates the center-unstable.

We now discuss the regularity properties of these foliations of TM. Our foliations of TM have been constructed as the unique fixed points of graph transform maps. We can apply the above results to the F-invariant splittings of TTM as the sum of the stable and center-unstable bundles for F and as the sum of the center-stable and unstable bundles for F. It follows from the pointwise Hölder section theorem (see [PSW], Theorem A) that both the center-unstable and unstable bundles and the corresponding foliations are Hölder continuous as long as F is $C^{1+\delta}$ for some $\delta > 0$. Since F is $C^{1+\delta}$ uniformly in both $\|\cdot\|$ and $\|\cdot\|_\star$ metrics, it follows that the bundles are uniformly Hölder in both metrics.

We obtain the Hölder continuity of the center-stable and stable bundles for F_r and the corresponding foliations by thinking of the same splittings as F_r^{-1}-invariant. Hölder regularity of the center bundle and foliation is obtained by noticing the the center is the intersection of the center-stable and center-unstable.

The absolute continuity with bounded Jacobians of the unstable foliation inside of the center-unstable foliation is a standard result, using only partial hyperbolicity, dynamical coherence, the fact that F is uniformly $C^{1+\delta}$, and the Hölder continuity of the bundles in the partially hyperbolic splitting. Similarly, the stable foliation for F is absolutely continuous with bounded Jacobians when considered as a subfoliation of the center-stable.

The Lipschitz continuity of the stable inside of the center-stable is proved in Lemma 5.5 below.

Step 2. We now have foliations of T_yM, for each $y \in \mathcal{O}(p)$. We obtain the foliations $\widehat{\mathcal{W}}^u, \widehat{\mathcal{W}}^c, \widehat{\mathcal{W}}^s, \widehat{\mathcal{W}}^{cu}$, and $\widehat{\mathcal{W}}^{cs}$ by applying the exponential map $\exp_y$ to the corresponding foliations of T_yM inside the ball around the origin of radius $\mathbf{R}_c(y)$.

If c is sufficiently small, then the distribution E^β_q lies within the angular $\varepsilon/2$-cone about the parallel translate of E^β_y, for every $\beta \in \{u, s, c, cu, cs\}$, $y \in \mathcal{O}^+(p)$, and all $q \in B(y, \mathbf{R}_c(y))$. Combining this fact with the preceding discussion, we obtain that property (1) holds if c is sufficiently small.

Property (2) — local invariance — follows from invariance under F_r of the foliations of TM and the fact that $\exp_{f(y)}(F(y, v)) = f(\exp_y(y, v))$ provided $\|v\| \leq \mathbf{R}_c(y)$.

Having chosen c, we now choose c_1 small enough so that, for all $y \in \mathcal{O}^+(p)$, $f(B(y, 2\mathbf{R}_{c_1}(y))) \subset B(f(y), \mathbf{R}_c(y))$ and $f^{-1}(B(y, 2\mathbf{R}_{c_1}(y))) \subset B(f^{-1}(y), \mathbf{R}_c(y))$, and so that, for all $q \in f(B(y, \mathbf{R}_{c_1}(y)))$,

$$\begin{aligned}
q' \in \widehat{\mathcal{W}}^s_p(q, \mathbf{R}_{c_1}(y))) &\implies d_\star(f(q), f(q')) \leq \nu(y)\, d_\star(q, q'),\\
q' \in \widehat{\mathcal{W}}^u_p(q, \mathbf{R}_{c_1}(y))) &\implies d_\star(f^{-1}(q), f^{-1}(q')) \leq \hat{\nu}(f^{-1}(y))\, d_\star(q, q'),\\
q' \in \widehat{\mathcal{W}}^{cs}_p(q, \mathbf{R}_{c_1}(y))) &\implies d_\star(f(q), f(q')) \leq \hat{\gamma}(y)^{-1}\, d_\star(q, q'),\\
q' \in \widehat{\mathcal{W}}^{cu}(q, \mathbf{R}_{c_1}(y))) &\implies d_\star(f^{-1}(q), f^{-1}(q')) \leq \gamma(f^{-1}(y))^{-1}\, d_\star(q, q').
\end{aligned}$$

We set $\mathbf{R} = \mathbf{R}_c$ and $\mathbf{r} = \mathbf{R}_{c_1}$.

Property (3) — exponential growth bounds at local scales — is now proved by a simple inductive argument.

Properties (4)–(7) — coherence, uniqueness, regularity and regularity of the strong foliation inside weak leaves — follow immediately from the corresponding properties of the foliations of TM discussed above, except for the Lipschitz continuity statement, which we now prove:

Lemma 5.5. *The $\widehat{\mathcal{W}}^s$ holonomy maps between $\widehat{\mathcal{W}}^c$ manifolds are Lipschitz at p.*

Proof of Lemma 5.5. Fix a function ρ satisfying $\nu\gamma^{-1} \prec \rho \prec \min\{1, \hat{\gamma}\}$, and such that $\kappa < e^{-\varepsilon_1}$, where

$$\kappa = \sup_{y \in O^+(p)} \max\{(\nu\gamma^{-1}\rho^{-1})(y), (\rho\hat{\gamma}^{-1})(y)\}.$$

Note that this is possible because (5.4) implies that $\sup_{y\in O^+(p)} \max\{\nu\gamma^{-1}(y), \nu\gamma^{-1}\hat{\gamma}^{-1}(y)\} < e^{-2\varepsilon_1}$. Fix a constant $\lambda \in (\kappa, e^{-\varepsilon_1})$. Observe that

$$\sup_{y\in O^+(p)} (\nu\gamma^{-1}\hat{\gamma}^{-1}(y)) < \kappa, \tag{5.5}$$

since $\rho \prec \min\{1, \hat{\gamma}\}$.

Since $B(p_j)^{1/j} \to 1$ as $j \to \infty$, there exists a constant $C > 0$ such that

$$\sup_{j\geq 0} B(p_j)(\kappa\lambda^{-1})^j < C.$$

Let θ be the Hölder exponent of the partially hyperbolic splitting, in the $\star$-metric, and let H be the θ-Hölder norm. Choose $\delta > 0$ and $N > 0$ such that:

- $H\Big((\delta\nu_j(p))^\theta) + (\rho_n\hat{\gamma}_j(p)^{-1})^\theta\Big) < 1/2 - \varepsilon$ for all $n \geq N$ and $j = 0, \dots, n$,
- $\rho_N(p) < \delta/3$, and
- $1 - \lambda - 4\delta C \sup_{y\in O^+(p)} \gamma(y) > 0$.

Finally, choose $K > 2\delta$ satisfying:

$$K > \sup_{j\in\mathbb{N}} \frac{8\delta B(p_{j+1})(\kappa\lambda^{-1})^{j+1}}{1 - \lambda - 4\delta B(p_{j+1})\kappa^{j+1}\gamma(p_{j+1})},$$

and let $L = 3 + 2K$.

We will show that for each $p' \in \widehat{\mathcal{W}}^s_\star(p, \delta/3)$, and for every $q \in \widehat{\mathcal{W}}^c_{loc}(p)$:

$$d_\star(p,q) \leq \rho_N(p) \implies L^{-1}d_\star(p,q) \leq d_\star(h^s(p), h^s(q)) \leq L^{-1}d_\star(p,q),$$

where $h^s \colon \widehat{\mathcal{W}}^c_{loc}(p) \to \widehat{\mathcal{W}}^c(p')$ is the $\widehat{\mathcal{W}}^s$-holonomy map. We prove the righthand inequality; the proof of the lefthand inequality is given by switching the roles of p and p'.

Let $p' \in \widehat{\mathcal{W}}^s_\star(p, \delta/3)$ be given, and let $q \in \widehat{\mathcal{W}}^c_\star(p, \rho_N(p))$. Denote by q' the image of q under h^s (by definition $h^s(p) = p'$). Fix $n \geq N$ such that $\rho_n(p) \leq d_\star(p,q) < \rho_{n-1}(p)$. Note that $d_\star(p,p') < \delta/3 < \delta$ and $d_\star(q,q') < \delta$, by the triangle inequality.

Lemma 5.6. *For $j = 0, \dots, n$, we have $\{p_j, p'_j, q_j, q'_j\} \subset \mathcal{N}_{\mathbf{r}}$. Moreover:*

1) $\rho_n(p)\gamma_j(p) \leq d_\star(p_j, q_j) \leq \rho_{n-1}(p)\hat{\gamma}_j(p)^{-1}$, *and*
2) $\max\{d_\star(p_j, p'_j), d_\star(q_j, q'_j)\} < \delta\nu_j(p)$.

Proof. The proof is a simple inductive argument using Part 3 of Proposition 5.4. □

We will work in $\|\cdot\|$-exponential coordinates in $\mathcal{N}_{\mathbf{r}}$. For $j \in \mathbb{N}$ and $x \in B_\star((p_j), \mathbf{r})$, denote by $\tilde{x}$ the point $\exp^{-1}_{p_j}(x)$. Note that $\tilde{p}_j = 0$. Let $v_j = \tilde{q}_j - \tilde{p}_j$, let $v'_j = \tilde{q}'_j - \tilde{p}'_j$, and let $w_j = v'_j - v_j$. Lemma 5.6 implies that for $j = 0, \dots, n$, we have $(\rho_n\gamma_j)(p) \leq \|v_j\|_\star \leq (\rho_{n-1}\hat{\gamma}_j^{-1})(p)$ and $\|w_j\|_\star \leq d_\star(p_j, p'_j) + d_\star(q_j, q'_j) \leq 2\delta\nu_j(p)$. Let $\pi^c_j : T_{p_j}M \to E^c_{p_j}$ be the linear projection with kernel $(E^u \oplus E^s)_{p_j}$, and let $\pi^{us}_j : T_{p_j}M \to (E^u \oplus E^s)_{p_j}$ be the linear projection with kernel $E^c_{p_j}$.

The vectors v_j and v'_j lie in uniform cones about $E^c_{p_j}$ with respect to the splitting $T_{p_j}M = E^c_{p_j} \oplus (E^u \oplus E^s)_{p_j}$:

Lemma 5.7. *For $j = 0, \dots n$, we have $\|\pi^{us}_j(v_j)\|_\star \le \frac{1}{2}\|v_j\|_\star$, $\|\pi^{us}_j(v'_j)\|_\star \le \frac{1}{2}\|v'_j\|_\star$, $\|v_j\|_\star \le \frac{3}{2}\|\pi^c_j(v_j)\|_\star$ and $\|v'_j\|_\star \le \frac{3}{2}\|\pi^c_j(v'_j)\|_\star$.*

Proof. $T_{p_j}\widehat{\mathcal{W}}^c$ and $T_{q_j}\widehat{\mathcal{W}}^c$ both lie in the ε-cone about $E^c(p_j)$, and the tangent distribution to $\widehat{\mathcal{W}}^c$ is Hölder continuous. Hence $\tan \measuredangle_\star(T_{p_j}\widehat{\mathcal{W}}^c, T_{p'_j}\widehat{\mathcal{W}}^c) \le H d_\star(p_j, p'_j)^\theta \le H(\delta\nu_i(p))^\theta$, and $\tan \measuredangle_\star(T_{q_j}\widehat{\mathcal{W}}^c, T_{q'_j}\widehat{\mathcal{W}}^c) \le H d_\star(q_j, q'_j)^\theta \le H(\delta\nu_i(p))^\theta$. Furthermore $\tan \measuredangle_\star(T_{p_j}\widehat{\mathcal{W}}^c, T_{q_j}\widehat{\mathcal{W}}^c) \le H d_\star(p_j, q_j)^\theta \le H(\rho_n(p)\hat{\gamma}_j(p)^{-1})^\theta$. This implies that

$$\tan \measuredangle_\star(T_{p'_j}\widehat{\mathcal{W}}^c, T_{q'_j}\widehat{\mathcal{W}}^c) \le H\left((\delta\nu_j(p))^\theta) + (\rho_n(p)\hat{\gamma}_j(p)^{-1})^\theta\right) < 1/2 - \varepsilon$$

for $j = 0, \dots, n$, by our choice of δ. □

Since the points $\{p_j, p'_j, q_j, q'_j\}$ all lie in $\mathcal{N}_{\mathbf{r}}$, in which F coincides with $\tilde{f} = \exp^{-1} \circ f \circ \exp$, we have that $\tilde{x}_j = F^j(\tilde{x})$, for $x \in \{p, p', q, q'\}$. The Mean Value Theorem implies that $v_{j-1} = \int_0^1 D_{\tilde{p}_j + tv_j}F(v_j)\,dt$ and $v'_{j-1} = \int_0^1 D_{\tilde{p}'_j + tv'_j}F(v'_j)\,dt$; subtracting these expressions, we obtain:

$$w_{j-1} = \int_0^1 \left(D_{\tilde{p}_j + tv_j}F^{-1}(v_j) - D_{\tilde{p}'_j + tv'_j}F^{-1}(v'_j)\right) dt.$$

and

$$\pi^c_j(w_{j-1}) = \int_0^1 \pi^c_j\left(D_{\tilde{p}_j + tv_j}F^{-1}(v_j) - D_{\tilde{p}'_j + tv'_j}F^{-1}(v'_j)\right) dt.$$

Then $\|\pi^c_{j-1}(w_{j-1})\|_\star \le \text{(I)} + \text{(II)}$ where

$$\text{(I)} = \int_0^1 \left\|\pi^c_{j-1}D_{\tilde{p}_j + tv_j}F^{-1}\left(v_j - v'_j\right)\right\|_\star dt,$$

$$\text{(II)} = \int_0^1 \left\|\left(\pi^c_{j-1}D_{\tilde{p}_j + tv_j}F^{-1} - \pi^c_{j-1}D_{\tilde{p}'_j + tv'_j}F^{-1}\right)(v'_j)\right\|_\star dt.$$

We have

$$\text{(II)} \le \int_0^1 B(p_j)\|v_j - v'_j\|_\star\|v'_j\|_\star\,dt \le B(p_j)\|w_j\|_\star\|v'_j\|_\star,$$

since DF^{-1} is Lipschitz with norm $B(p_j)$ on $T_{p_j}M$.

We next estimate the expression (I). Since $D_{p_j}F^{-1} = D_{p_j}f^{-1}$, which sends the splitting $(E^u \oplus E^c \oplus E^s)_{p_j}$ to $(E^u \oplus E^c \oplus E^s)_{p_{j-1}}$ and has norm on E^c bounded by $\gamma(p_j)^{-1}$, we have that:

$$\int_0^1 \left\|\pi^c_{j-1}D_{\tilde{p}_j}F^{-1}\left(w_j\right)\right\|_\star dt \le \gamma(p_j)^{-1}\|\pi^c_j w_j\|_\star.$$

Hence

$$\begin{aligned}
\text{(I)} &= \int_0^1 \left\|\pi^c_{j-1}D_{\tilde{p}_j + tv_j}F^{-1}\left(w_j\right)\right\|_\star dt \\
&\le \int_0^1 \left\|\pi^c_{j-1}\left(D_{\tilde{p}_j}F^{-1} - D_{\tilde{p}_j + tv_j}F^{-1}\right)\left(w_j\right)\right\|_\star dt + \int_0^1 \left\|\pi^c_{j-1}D_{\tilde{p}_j}F^{-1}\left(w_j\right)\right\|_\star dt \\
&\le \int_0^1 \left\|\pi^c_{j-1}\left(D_{\tilde{p}_j}F^{-1} - D_{\tilde{p}_j + tv_j}F^{-1}\right)\left(w_j\right)\right\|_\star dt + \gamma(p_j)^{-1}\|\pi^c_j w_j\|_\star \\
&\le B(p_j)\|v_j\|_\star\|w_j\|_\star + \gamma(p_j)^{-1}\|\pi^c_j w_j\|_\star,
\end{aligned}$$

again using the Lipschitz continuity of DF^{-1}. We conclude that

$$\begin{aligned} \|\pi^c_{j-1}(w_{j-1})\|_\star &\le \gamma(p_j)^{-1}\|\pi^c_j w_j\|_\star + B(p_j)\left(\|v_j\|_\star\|w_j\|_\star + \|w_j\|_\star\|v'_j\|_\star\right) \\ &\le \gamma(p_j)^{-1}\|\pi^c_j w_j\|_\star + 2\delta B(p_j)\nu_j(p)\left(\|v_j\|_\star + \|v'_j\|_\star\right), \end{aligned} \tag{5.6}$$

using the bound $\|w_j\|_\star \le 2\delta\nu_j(p)$.

Claim. For $j = 0, \dots, n$, we have $\|\pi^c_j w_j\|_\star \le K\lambda^j\gamma_j(p)\|v_0\|_\star$ and $\|v'_j\|_\star \le (3+2K\lambda^j)\|v_j\|_\star$

Proof. We prove it by backwards induction on n. The base case is $j = n$. Observe that:

$$\|\pi^c_n w_n\|_\star \le \|w_n\|_\star \le 2\delta\nu_n(p) = 2\delta\frac{\nu_n(p)}{(\rho\gamma)_n(p)}\gamma_n(p)\rho_n(p) < 2\delta\lambda^n\gamma_n(p)\|v_0\|_\star < K\lambda^n\gamma_n(p)\|v_0\|_\star.$$

Since $\|w_n\|* \le 2\delta\nu_n(p) \le 2\delta\lambda^n(\rho\gamma)_n(p) \le 2\delta\lambda^n\|v_n\|_\star$, we also obtain that

$$\|v'_n\|_\star \le \|v_n\|_\star + \|v_n - v'_n\|_\star = \|v_n\|_\star + \|w_n\|_\star \le \|v_n\|_\star(1 + 2\delta\lambda^n) \le \|v_n\|_\star(3 + 2K\lambda^n).$$

Now suppose that the claim holds for some $(j+1) \le n$. Then, by (5.6):

$$\begin{aligned} \|\pi^c_j(w_j)\|_\star &\le \gamma(p_{j+1})^{-1}\|\pi^c_{j+1}w_{j+1}\|_\star + 2\delta B(p_{j+1})\nu_{j+1}(p)\left(\|v_{j+1}\|_\star + \|v'_{j+1}\|_\star\right) \\ &\le \gamma(p_{j+1})^{-1}K\lambda^{j+1}\gamma_{j+1}(p)\|v_0\|_\star + 2\delta B(p_{j+1})\nu_{j+1}(p)(4 + 2K\lambda^{j+1})\|v_{j+1}\|_\star \\ &\le K\lambda^{j+1}\gamma_j(p)\|v_0\|_\star + 2\delta B(p_{j+1})\|v_0\|_\star(\nu\hat{\gamma}^{-1})_{j+1}(p)(4 + 2K\lambda^{j+1}) \\ &\le K\eta\lambda^j\gamma_j(p)\|v_0\|_\star, \end{aligned}$$

where

$$\begin{aligned} \eta &= \lambda + 8\delta B(p_{j+1})\frac{(\nu\gamma^{-1}\hat{\gamma}^{-1})_{j+1}(p)}{K\lambda^{j+1}}\gamma(p_{j+1}) + 4\delta B(p_{j+1})(\nu\gamma^{-1}\hat{\gamma}^{-1})_{j+1}(p)\gamma(p_{j+1}) \\ &\le \lambda + 8\delta B(p_{j+1})\frac{\kappa^{j+1}}{K\lambda^{j+1}}\gamma(p_{j+1}) + 4\delta B(p_{j+1})\kappa^{j+1}\gamma(p_{j+1}), \end{aligned}$$

by (5.5). Then $\eta < 1$, since

$$K > \frac{8\delta B(p_{j+1})(\kappa\lambda^{-1})^{j+1}}{1 - \lambda - 4\delta B(p_{j+1})\kappa^{j+1}\gamma(p_{j+1})}.$$

This implies that $\|\pi^c_j(w_j)\|_\star \le K\lambda^j\gamma_j(p)\|v_0\|_\star$, completing the inductive step for the first assertion of the claim.

Finally, to prove the inductive step for the second part of the claim, we have:

$$\begin{aligned} \|v'_j\|_\star &\le \|v_j\|_\star + \|v_j - v'_j\|_\star \\ &\le \|v_j\|_\star + \|\pi^c_j(v_j - v'_j)\|_\star + \|\pi^{us}_j(v_j - v'_j)\|_\star \\ &\le \|v_j\|_\star + \|\pi^c_j(w_j)\|_\star + \|\pi^{us}_j(v_j)\|_\star + \|\pi^{us}_j(v'_j)\|_\star \\ &\le \|v_j\|_\star + K\lambda^j\gamma_j(p)\|v_0\|_\star + .5\|v_j\|_\star + .5\|v'_j\|_\star \\ &\le \|v_j\|_\star + K\lambda^j\|v_j\|_\star + .5\|v_j\|_\star + .5\|v'_j\|_\star; \end{aligned}$$

Solving for $\|v'_j\|_\star$, we obtain that $\|v'_j\|_\star \le (3 + 2K\lambda^j)\|v_j\|_\star$, as desired. □

The claim finishes the proof of Lemma 5.5; setting, $j = 0$ we see that

$$d_\star(h^s(p), h^s(q)) = \|v'_0\|_\star \le (3 + 2K)\|v_0\|_\star = Ld_\star(p, q).$$

□

□

Given this proposition, the proof now proceeds as the proof of Theorem 5.1 in [BW], with a few modifications, which we will describe in the sequel.

5.5. **Distortion Estimates in Thin Neighborhoods.** Fix $p \in M$ satisfying the bunching hypotheses of Theorem B. Henceforth the entire analysis will take place in a neighborhood of the forward orbit of p.

We choose $\varepsilon > 0$:

- much smaller than $\pi/2$, which is the $\star$-angle between the bundles of the partially hyperbolic splitting over $\mathcal{O}^+(p)$.
- small enough that

$$e^{-\varepsilon} > \sup_{y \in \mathcal{O}^+(p)} \max\left\{\nu(y),\, \hat{\nu}(y),\, \frac{\nu(y)}{\gamma(y)},\, \frac{\hat{\nu}(y)}{\hat{\gamma}(y)},\, \frac{\nu(y)}{\gamma\hat{\gamma}(y)}\right\}. \tag{5.7}$$

Let $\mathbf{r}, \mathbf{R} \colon \mathcal{O}^+(p) \to \mathbb{R}_+$ and foliations $\widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^c$, $\widehat{\mathcal{W}}^{cu}$ and $\widehat{\mathcal{W}}^{cs}$ be given by Proposition 5.4, using this value of ε. By uniformly rescaling the $\|\cdot\|_\star$ metric on $\mathcal{N}_{\mathbf{R}}$, we may assume that

$$\inf_{y \in \mathcal{O}^+(p)} \mathbf{r}(y) \gg 1.$$

We may also assume that if $x, y \in B_\star(p_j, \mathbf{r})$, then $\widehat{\mathcal{W}}^{cs}(x) \cap \widehat{\mathcal{W}}^u(y)$, $\widehat{\mathcal{W}}^{cs}(x) \cap \mathcal{W}^u_{loc}(y)$, $\widehat{\mathcal{W}}^{cu}(x) \cap \widehat{\mathcal{W}}^s(y)$ and $\widehat{\mathcal{W}}^{cu}(x) \cap \mathcal{W}^s_{loc}(y)$ are single points. We denote by $\widehat{m}_a$ the measure $m_{\widehat{\mathcal{W}}^a}$ induced by the volume form $\|\cdot\|$.

We next choose functions $\sigma, \tau \colon \mathcal{O}^+(p) \to \mathbb{R}_+$ satisfying

$$\sigma \prec \min\{1, \hat{\gamma}\}, \quad \text{and} \quad \nu \prec \tau \prec \sigma\gamma, \tag{5.8}$$

and such that $\kappa = \sup_{y \in \mathcal{O}^+(p)} \sigma\hat{\gamma}^{-1}(y) < e^{-\varepsilon}$ (this is possible because of (5.7)). Note that these inequalities also imply that

$$\tau\hat{\nu} \prec \sigma\gamma\hat{\nu} \prec \sigma\gamma\hat{\gamma} \prec \sigma.$$

For the rest of the proof, except where we indicate otherwise, cocycles will be evaluated at the point p. We will also drop the dependence on p from the notation; thus, if α is a cocycle, then $\alpha_n(p)$ will be abbreviated to α_n.

Using these functions and the fake foliations, we next define a sequence of thin neighborhoods T_n of $\mathcal{W}^s_\star(p, 1)$. We first define a neighborhood S_n in $\widehat{\mathcal{W}}^{cs}(p)$ by:

$$S_n = \bigcup_{x \in \mathcal{W}^s_\star(p,1)} \widehat{\mathcal{W}}^c_\star(x, \sigma_n),$$

and then define the neighborhood T_n by:

$$T_n = f^{-n}\left(\bigcup_{z \in f^n(S_n)} \widehat{\mathcal{W}}^u_\star(z, \tau_n) \cup \mathcal{W}^u_\star(z, \tau_n)\right). \tag{5.9}$$

Lemma 5.8 (c.f. [BW], Lemma 4.3)**.** *The set T_n is well-defined. There exists $C > 0$ such that, for every $n \geq 0$,*

$$f^j(T_n) \subset B_\star(p_j, C\kappa^j),$$

for $j = 0, \ldots, n$.

Proof. Suppose first that $x \in \mathcal{W}^s_\star(p, 1)$ and $y \in \widehat{\mathcal{W}}^c(x, \sigma_n)$. By part 3(a) of Proposition 5.4, we then have

$$y_j \in \widehat{\mathcal{W}}^c_\star(x_j, \hat{\gamma}_j^{-1}\sigma_n) \subset \widehat{\mathcal{W}}^c_\star(x_j, 1) \subset B_\star(p_j, 2),$$

for $0 \le j \le n$. In fact, since $\sigma \prec \min\{\hat\gamma, 1\}$, the quantity $\hat\gamma_j^{-1}\sigma_n < \hat\gamma_j^{-1}\sigma_j \le \kappa^j$ is exponentially small in j, as is the $\star$-diameter of $f^j(\mathcal{W}^s(p,1))$. This implies that for some $C > 0$ and for every $n \ge 0$,

$$f^j(S_n) \subset B_\star(p_j, C\kappa^j), \quad \text{for } j = 0, \dots, n. \tag{5.10}$$

For every $x \in S_n$, we have that $B_\star(x_n, \tau_n) \subset B_\star(p_n, \mathbf{r}(p_n))$, and so the set T_n is well-defined by (5.9). Proposition 5.4 implies that the leaves of $\widehat{\mathcal{W}}^u_{p_j}$ and $\mathcal{W}^u_{loc}$ are uniformly contracted by f^{-1} as long as they stay near the orbit of p. Because $\kappa < e^{-\varepsilon}$, the image of $f^n(T_n)$, for n sufficiently large, remains in the neighborhood $\mathcal{N}_\mathbf{r}$ of $\mathcal{O}^+(p)$ in which the fake foliations are defined and the expansion and contraction estimates hold.

Combining these facts with (5.10), we obtain the conclusion. □

Lemma 5.9 (c.f. [BW], Lemma 4.4)**.** *Let $\alpha : M \to \mathbb{R}$ be a positive, uniformly Hölder continuous function. Then there is a constant $C \ge 1$ such that, for all $n \ge 0$ and all $x, y \in T_n$,*

$$C^{-1} \le \frac{\alpha_n(y)}{\alpha_n(x)} \le C.$$

Proof. Since $d \le K^{-1} d_\star$, Lemma 5.8 implies that the diameter of $f^j(T_n)$ remains exponentially small in the d metric, for $j = 0, \dots, n$. Since f is $C^{1+\delta}$, the lemma follows from the following elementary distortion estimate:

Lemma 5.10 ([BW], Lemma 4.1)**.** *Let $\alpha : M \to \mathbb{R}$ be a positive Hölder continuous function, with exponent $\theta > 0$. Then there exists a constant $H > 0$ such that the following holds, for all $p, q \in M$, $B > 0$ and $n \ge 1$:*

$$\sum_{i=0}^{n-1} d(p_i, q_i)^\theta \le B \quad \Longrightarrow \quad e^{-HB} \le \frac{\alpha_n(p)}{\alpha_n(q)} \le e^{HB},$$

and

$$\sum_{i=1}^{n} d(p_{-i}, q_{-i})^\theta \le B \quad \Longrightarrow \quad e^{-HB} \le \frac{\alpha_{-n}(p)}{\alpha_{-n}(q)} \le e^{HB}.$$

□

5.6. **Juliennes.** The next step is to define juliennes. For each $x \in \mathcal{W}^s_\star(p,1)$ one defines a sequence $\{\widehat{J}^{cu}_n(x)\}_{n\ge0}$ of center-unstable juliennes, which lie in the fake center-unstable manifold $\widehat{\mathcal{W}}^{cu}(x)$ and shrink exponentially as $n \to \infty$ while becoming increasingly thin in the $\widehat{\mathcal{W}}^u$-direction.

Define, for all $x \in \mathcal{W}^s(p,1)$,

$$\widehat{B}^c_n(x) = \widehat{\mathcal{W}}^c_\star(x, \sigma_n).$$

Note that

$$S_n = \bigcup_{y \in \mathcal{W}^s(p,1)} \widehat{B}^c_n(x).$$

For $y \in S_n$, we may then define two types of *unstable juliennes*:

$$\widehat{J}^u_n(y) = f^{-n}(\widehat{\mathcal{W}}^u_\star(y_n, \tau_n))$$

and

$$J^u_n(y) = f^{-n}(\mathcal{W}^u_\star(y_n, \tau_n)).$$

Observe that for all $y \in S_n$, the sets $\widehat{J}^u_n(y)$ and $J^u_n(y)$ are contained in T_n.

For each $x \in \mathcal{W}^s(p,1)$ and $n \geq 0$, we then define the *center-unstable julienne* centered at x of order n:

$$\widehat{J}^{cu}_n(x) = \bigcup_{q \in \widehat{B}^c_n(x)} \widehat{J}^u_n(q).$$

Note that, by their construction, the sets $\widehat{J}^{cu}_n(x)$ are contained in T_n, for all $n \geq 0$ and $x \in \mathcal{W}^s(p,1)$.

The crucial properties of center unstable juliennes are summarized in the next three Propositions. We state them in a slightly more general form than we will need for the proof of Theorem B; the more general formulation will be used in the proof of Theorem D.

Proposition 5.11 (c.f. [BW], Proposition 5.3)**.** *Let $x, x' \in \mathcal{W}^s(p,1)$, and let $h^s : \widehat{\mathcal{W}^{cu}}(x) \to \widehat{\mathcal{W}^{cu}}(x')$ be the holonomy map induced by the stable foliation $\mathcal{W}^s$. Then the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested.*

Proposition 5.12 (c.f. [BW], Proposition 5.4)**.** *There exist $\delta > 0$ and $c \geq 1$ such that, for all $x \in \mathcal{W}^s(p,1)$, and all $q, q' \in S_n$, the following hold, for all $n \geq 0$:*

$$c^{-1} \leq \frac{\widehat{m}_u(\widehat{J}^u_n(q))}{\widehat{m}_u(\widehat{J}^u_n(q'))} \leq c,$$

$$c^{-1} \leq \frac{m_u(J^u_n(q))}{m_u(J^u_n(q'))} \leq c,$$

$$\widehat{m}_u(\widehat{J}^u_{n+1}(q)) \geq \delta \widehat{m}_u(\widehat{J}^u_n(q)),$$

and

$$\widehat{m}_{cu}(\widehat{J}^{cu}_{n+1}(x)) \geq \delta \widehat{m}_{cu}(\widehat{J}^{cu}_n(x)).$$

Proposition 5.13 (c.f. [BW], Proposition 5.5)**.** *Let X be a measurable set that is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated at some point $x \in \mathcal{W}^s(p)$. Then x is a Lebesgue density point of X if and only if:*

$$\lim_{n\to\infty} \widehat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) = 1.$$

Assuming these propositions, we conclude the:

Proof of Theorem B. Let X be a bi essentially saturated set, and let X^s be an essential $\mathcal{W}^s$-saturate of X. Since $m(X \vartriangle X^s) = 0$, the Lebesgue density points of X are precisely the same as those of X^s. Suppose that $x \in \mathcal{W}^s(p,1)$ is a Lebesgue density point of X^s. Proposition 5.13 implies that x is a cu-julienne density point of X^s.

To finish the proof, we show that every $x' \in \mathcal{W}^s(p,1)$ is a cu-julienne density point of X^s. Then by Proposition 5.13, every $x' \in \mathcal{W}^s(p,1)$ is a Lebesgue density point of X^s, and so $\mathcal{W}^s(p,1) \subset \hat{X}$. Notice that if p satisfies the hypotheses of Theorem B, thes so does every $p' \in \mathcal{W}^s(p)$. Hence if $\mathcal{W}^s(p) \cap \hat{X} \neq \emptyset$, then $\mathcal{W}^s(p) \subset \hat{X}$, completing the proof.

Let $h^s : \widehat{\mathcal{W}^{cu}}(x) \to \widehat{\mathcal{W}^{cu}}(x')$ be the holonomy map induced by the stable foliation $\mathcal{W}^s$. The sequence $h^s(\widehat{J}^{cu}_n(x)) \subset \widehat{\mathcal{W}^{cu}}(x')$ nests at x'.

Transverse absolute continuity of h^s with bounded Jacobians implies that

$$\lim_{n\to\infty} \widehat{m}_{cu}(X^s : \widehat{J}^{cu}_n(x)) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \widehat{m}_{cu}(h^s(X^s) : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Since X^s is s-saturated, we then have:

$$\lim_{n\to\infty} \widehat{m}_{cu}(X^s : \widehat{J}^{cu}_n(x)) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \widehat{m}_{cu}(X^s : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Since we are assuming that x is a cu-julienne density point of X^s, we thus have

$$\lim_{n\to\infty} \widehat{m}_{cu}(X^s : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Working inside of $\widehat{\mathcal{W}}^{cu}(x')$, we will apply Lemma 5.1 to the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$, which both nest at x'. Proposition 5.11 implies that these sequences are internested. Proposition 5.12 implies that $\widehat{J}^{cu}_n(x')$ is regular with respect to the induced Riemannian measure $\widehat{m}_{cu}$ on $\widehat{\mathcal{W}}^{cu}(x')$. Lemma 5.1 now implies that

$$\lim_{n\to\infty} \widehat{m}_{cu}(X^s : h^s(\widehat{J}^{cu}_n(x))) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \widehat{m}_{cu}(X^s : \widehat{J}^{cu}_n(x')) = 1,$$

and so x' is a cu-julienne density point of X^s. It follows from Proposition 5.13 that x' is a Lebesgue density point of X^s, and thus of X. □

We extract from this proof a proposition that will be used in the proof of Theorem D:

Proposition 5.14. *Let $Y \subset \widehat{\mathcal{W}}^{cu}(p)$ be a measurable subset, let $x \in \mathcal{W}^s(p, 1)$, and let Y' be the image of Y under $\mathcal{W}^s$-holonomy. Then p is a cu-julienne density point of Y if and only if x is a cu-julienne density point of Y'.*

5.7. **Julienne Quasiconformality.** Here we prove Proposition 5.11. The proof is taken *mutatis mutandis* from [BW].

By a simple argument reversing the roles of x and x', it will suffice to show that k can be chosen so that

$$h^s(\widehat{J}^{cu}_n(x)) \subseteq \widehat{J}^{cu}_{n-k}(x'), \tag{5.11}$$

for all $n \geq k$, whenever x and x' satisfy the hypotheses of the proposition.

In order to prove that k can be chosen so that (5.11) holds, we need two lemmas.

Lemma 5.15. *There exists a positive integer k_1 such that, for all $x, x' \in \mathcal{W}^s(p)$,*

$$\hat{h}^s(\widehat{B}^c_n(x)) \subseteq \widehat{B}^c_{n-k_1}(x'),$$

for all $n \geq k_1$, where $\hat{h}^s : \widehat{\mathcal{W}}^{cu}_{loc}(x) \to \widehat{\mathcal{W}}^{cu}(x')$ is the local $\widehat{\mathcal{W}}^s$ holonomy.

Proof. Proposition 5.4 implies that $\hat{h}^s$ is L-Lipschitz at x, for some $L \geq 1$. Therefore the image of $\widehat{\mathcal{W}}^c_\star(x, \sigma_n)$ under $\hat{h}^s$ is contained in $\widehat{\mathcal{W}}^c_\star(x', L\sigma_n) \subseteq \widehat{\mathcal{W}}^c_\star(x', \sigma_{n-k_1})$, for any k_1 large enough so that $\sigma_{-k_1} > L$. □

Lemma 5.16. *There exists a positive integer k_2 such that the following holds for every integer $n \geq k_2$. Suppose $q, q' \in S_n$, with $q' \in \widehat{\mathcal{W}}^s(q)$. Let $y \in \widehat{J}^u_n(q)$, and let y' be the image of y under $\mathcal{W}^s$ holonomy from $\widehat{\mathcal{W}}^{cu}_{loc}(q)$ to $\widehat{\mathcal{W}}^{cu}(q')$. Then*

$$y' \in \widehat{J}^u_{n-k_2}(z'),$$

for some $z' \in \widehat{\mathcal{W}}^c_\star(q', \sigma_{n-k_2})$.

Proof. Let z' be the unique point in $\widehat{\mathcal{W}}^u(y') \cap \widehat{\mathcal{W}}^c(q')$. It is not hard to see that $z'_j \in \mathcal{N}_r$, for $j = 0, \ldots, n-1$ and that z'_n is the unique point in $\widehat{\mathcal{W}}^u(y'_n) \cap \widehat{\mathcal{W}}^c(q'_n)$. It will suffice to prove that $d_\star(y'_n, z'_n) = O(\tau_n)$ and $d_\star(q', z') = O(\sigma_n)$.

We have $d_\star(q_n, y_n) \leq \tau_n$ because $y \in f^{-n}(\mathcal{W}^u_\star(q_n, \tau_n))$. By Proposition 5.4, 3(a), we also have that $d_\star(q_n, q'_n) = O(\nu_n)$ and $d_\star(y_n, y'_n) = O(\nu_n)$, since $d_\star(q, q')$ and $d_\star(y, y')$ are both $O(1)$. Note that q_n and z'_n are, respectively, the images of y_n and y'_n under $\widehat{\mathcal{W}}^u$-hononomy between $\widehat{\mathcal{W}}^{cs}_{loc}(y_n)$ and $\widehat{\mathcal{W}}^{cs}(q_n)$. Uniform transversality of the foliations $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^{cs}$ implies that

$$d_\star(y'_n, z'_n) = O(\max\{d_\star(q_n, y_n), d_\star(y_n, y'_n)\}) = O(\tau_n),$$

since $\nu < \tau$.

We next show that $d_\star(q', z') = O(\sigma_n)$. By the triangle inequality,

$$d_\star(q'_n, z'_n) \le d_\star(q'_n, q_n) + d_\star(q_n, y_n) + d_\star(y_n, y'_n) + d_\star(y'_n, z'_n).$$

All four of the quantities on the right-hand side are easily seen to be $O(\tau_n)$. Since q'_n and z'_n lie in the same $\widehat{\mathcal{W}}^c$-leaf at $d_\star$-distance $O(\tau_n)$, Proposition 5.4 now implies that $d_\star(q', z') = O((\gamma_n)^{-1}\tau_n)$. But τ and σ were chosen so that $\tau \prec \gamma\sigma$. Hence $(\gamma_n)^{-1}\tau_n < \sigma_n$ and $d_\star(q', z') = O(\sigma_n)$, as desired. □

Proof of Proposition 5.11. As noted above, it suffices to prove the inclusion (5.11). For $q \in \hat{B}^c_n(x)$, let $q' = \hat{h}^s(q)$. Then $q' \in \hat{B}^c_{n-k_1}(x')$ by Lemma 5.15. Hence $q, q' \in S_{n-k_1}$ and we can apply Lemma 5.16 to obtain

$$h^s(\hat{J}^{cu}_n(x)) \subseteq \bigcup_{z \in Q} \hat{J}^u_{n-k_2}(z),$$

where

$$Q = \bigcup_{q' \in \hat{B}^c_{n-k_1}(x')} \hat{B}^c_{n-k_2}(q').$$

For $k \ge k_2$, we have:

$$\bigcup_{z \in Q} \hat{J}^u_{n-k_2}(z) \subseteq \bigcup_{z \in Q} \hat{J}^u_{n-k}(z).$$

It therefore suffices to find $k \ge k_2$ such that $Q \subseteq \hat{B}^c_{n-k}(x')$. This latter inclusion holds if:

$$\sigma_{n-k_1} + \sigma_{n-k_2} \le \sigma_{n-k},$$

which is obviously true for all $n \ge k$, if k is sufficiently large. □

5.8. Julienne Measure.

Next we give the:

Proof of Proposition 5.12. Recall that we are using the standard Riemannian volumes and induced Riemannian volumes on submanfiolds (not the $\|\cdot\|_\star$-volumes).

Lemma 5.17 (cf. inequalities (21), [BW]). *There exists a constant $C_1 > 1$ such that, for all $n \ge 0$:*

$$C_1^{-1} \le \frac{\widehat{m}_u(\widehat{\mathcal{W}}^u(q_n, \tau_n(p)))}{\widehat{m}_u(\widehat{\mathcal{W}}^u(q'_n, \tau_n(p)))} \le C_1, \tag{5.12}$$

for all $q, q' \in S_n$, where $\widehat{\mathcal{W}}^u_\star(x, r) = \widehat{\mathcal{W}}^u(x) \cap B_\star(x, r)$, and $\widehat{m}_u$ is the induced Riemannian metric on $\widehat{\mathcal{W}}^u$-leaves.

Proof. Recall that the flat $\|\cdot\|_\flat$ metric on $T_{p_n}M$ and the Riemannian metric in a neighborhood of p_n (viewed in exponential coordinates at p_n) are uniformly comparable. We will estimate the ratio in (5.12) using the volume on $T_{p_n}M$ induced by $\|\cdot\|_\flat$.

On $T_{p_n}M$, the $\|\cdot\|_\star$-metric is also flat: the ball of radius τ_n at q_n is just a translate by $q_n - q'_n$ of a $\|\cdot\|_\star$-ball of radius τ_n at q'_n. Viewed in the $\|\cdot\|_\flat$ metric, a $\star$-ball of radius τ_n in $T_{p_n}M$ is an ellipsoid with eccentricity bounded by $K^{-1}B_n$. The intersection of such a ball centered at q_n of radius $\tau_n(p)$ with $\widehat{\mathcal{W}}^u(q_n)$ gives the set $\widehat{\mathcal{W}}^u(q_n, \tau_n(p))$. Since the leaves of $\widehat{\mathcal{W}}^u(q_n)$ are tangent to a uniformly Hölder continuous distribution, the volumes of these sets are uniformly comparable to the intersection of $T_{q_n}\widehat{\mathcal{W}}^u(q_n)$ with $B_\star(q_n, \tau_n(p))$. This is also a (u-dimensional) ellipsoid, call it $\mathcal{E}(q_n)$. Similarly we have an ellipsoid $\mathcal{E}(q'_n)$ centered at q'_n.

The distance between the spaces $T_{q_n}\widehat{\mathcal{W}}^u(q_n)$ and $T_{q'_n}\widehat{\mathcal{W}}^u(q_n)$ (translated by $q_n - q'_n$) is on the order of $d(q_n, q'_n)^\theta$, for some $\theta \in (0, 1]$, and so is bounded by $c\beta^n$, where

$\beta = \kappa^\theta < 1$. The bound on the eccentricity of $B_\star(q_n, \tau_n)$ then implies that the ratio between the u-dimensional volumes of $\mathcal{E}(q_n)$ and $\mathcal{E}(q_n')$ is bounded above by $D_n = C'(1 + cK^{-1}B_n\beta^n)^u$ and below by D_n^{-1}, for some constant C'. Since $\limsup_{n\to\infty} B_n^{1/n} = 1$, there exists a constant D such that $D_n \leq D$ for all n. We conclude that there exists a constant C satisfying (5.12), for all n and all $q, q' \in S_n$. □

Let $\widehat{E^s}$, $\widehat{E^c}$, and $\widehat{E^u}$ be the tangent distributions to the leaves of $\widehat{\mathcal{W}^s}$, $\widehat{\mathcal{W}^c}$, and $\widehat{\mathcal{W}^u}$, respectively. They are Hölder continuous by Proposition 5.4, part 6. Furthermore, the restrictions of these distributions to T_n are invariant under Df^j, for $j = 1, \ldots n$. We next observe that the Jacobian $\mathrm{Jac}(Df^n|_{\widehat{E^u}})$ is nearly constant when restricted to the set T_n. More precisely, we have:

Lemma 5.18. *There exists $C_2 \geq 1$ such that, for all $n \geq 1$, and all $y, y' \in T_n$,*

$$C_2^{-1} \leq \frac{\mathrm{Jac}(Df^n|_{\widehat{E^u}})(y)}{\mathrm{Jac}(Df^n|_{\widehat{E^u}})(y')} \leq C_2.$$

Proof. By the Chain Rule, these inequalities follow from Lemma 5.9 with $\alpha = \mathrm{Jac}(Df|_{\widehat{E^u}})$. □

Let $q \in S_n$, and let $X \subseteq \widehat{J^u_n}(q)$ be a measurable set (such as $\widehat{J^u_n}(q)$ itself). Then:

$$\widehat{m}_u(f^n(X)) = \int_X \mathrm{Jac}(Tf^n|_{\widehat{E^u}})(x)\, d\widehat{m}_u(x).$$

From this and Lemma 5.18 we then obtain:

Lemma 5.19. *There exists $C_3 > 0$ such that, for all $n \geq 0$, for any $q, q' \in S_n$, and any measurable sets $X \subset \widehat{J^u_n}(q)$, $X' \subset \widehat{J^u_n}(q')$, we have:*

$$C_3^{-1}\frac{\widehat{m}_u(f^n(X))}{\widehat{m}_u(f^n(X'))} \leq \frac{\widehat{m}_u(X)}{\widehat{m}_u(X')} \leq C_3\frac{\widehat{m}_u(f^n(X))}{\widehat{m}_u(f^n(X'))}.$$

Recall that $f^n(\widehat{J^u_n}(q)) = \widehat{\mathcal{W}^u_\star}(q_n, \tau_n)$, for $q \in S_n$. The first conclusion of Proposition 5.12 now follows from (5.12) and Lemma 5.19 with $X = \widehat{J^u_n}(q)$ and $X' = \widehat{J^u_n}(q')$.

The second conclusion is proved similarly.

We next show that there exists $\delta > 0$ such that

$$\frac{\widehat{m}_u(\widehat{J^u_{n+1}}(q))}{\widehat{m}_u(\widehat{J^u_n}(q))} \geq \delta, \tag{5.13}$$

for all $n \geq 0$ and all $q \in S_n$. To obtain (5.13), we will apply Lemma 5.19 with $q = q'$, $X = \widehat{J^u_{n+1}}(q)$, and $X' = \widehat{J^u_n}(q)$. This gives us:

$$\frac{\widehat{m}_u(\widehat{J^u_{n+1}}(q))}{\widehat{m}_u(\widehat{J^u_n}(q))} \geq C_3^{-1}\frac{\widehat{m}_u(f^n(\widehat{J^u_{n+1}}(q)))}{\widehat{m}_u(f^n(\widehat{J^u_n}(q)))}.$$

But $f^n(\widehat{J^u_{n+1}}(q)) = f^{-1}(\widehat{\mathcal{W}^u_\star}(q_{n+1}, \tau_{n+1}))$ and $f^n(\widehat{J^u_n}(q)) = \widehat{\mathcal{W}^u_\star}(q_n, \tau_n)$, and hence:

$$\frac{\widehat{m}_u(f^n(\widehat{J^u_{n+1}}(q)))}{\widehat{m}_u(f^n(\widehat{J^u_n}(q)))} = \frac{\widehat{m}_u(f^{-1}(\widehat{\mathcal{W}^u_\star}(q_{n+1}, \tau_{n+1})))}{\widehat{m}_u(\widehat{\mathcal{W}^u_\star}(q_n, \tau_n))}.$$

We show that this ratio is uniformly bounded below away from 0. Since τ_{n+1}/τ_n is uniformly bounded, and f is uniformly C^1 in the $\star$-metric on $\mathcal{N}_{\mathbf{r}}$, there exists a constant $\mu < 1$ (independent of n) such that $f^{-1}(\widehat{\mathcal{W}^u_\star}(q_{n+1}, \tau_{n+1}))$ contains the set $\widehat{\mathcal{W}^u_\star}(q_n, \mu\tau_n)$. Since $\|\cdot\|_\star$ is a locally flat metric, the set $B_\star(q_n, \mu\tau_n)$ is just the set $B_\star(q_n, \tau_n)$ dilated (in exponential coordinates at p_j) from q_n by a factor of μ. Since the leaves of the $\mathcal{W}^u$ foliation are uniformly smooth, the volumes of $\widehat{\mathcal{W}}^u_\star(q_n, C\tau_n)$ and $\widehat{\mathcal{W}}^u_\star(q_n, \tau_n)$ in the $\|\cdot\|$-metric are therefore uniformly comparable. This implies that their Riemannian volumes are comparable.

To prove the final claim, we begin by observing that, considered as a subset of $\widehat{\mathcal{W}}^{cu}(x)$, the set $\widehat{J}^{cu}_n(x)$ fibers over $\widehat{B}^c_n(x)$ with $\widehat{\mathcal{W}}^u$-fibers $\widehat{J}^u_n(q)$. We have just proved that these fibers are c-uniform. Since $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ is uniformly bounded away from 0, the ratio

$$\frac{\widehat{m}_c(\widehat{B}^c_{n+1}(x))}{\widehat{m}_c(\widehat{B}^c_n(x))} = \frac{\widehat{m}_c(\widehat{\mathcal{W}}^c(x, \sigma_{n+1}))}{\widehat{m}_c(\widehat{\mathcal{W}}^c(x, \sigma_n))}$$

is bounded away from 0, uniformly in x and n. Thus the sequence of bases $\widehat{B}^c_n(x)$ of $\widehat{J}^{cu}_n(x)$ is regular in the induced Riemannian volume $\widehat{m}_c$. Proposition 5.4, part 7. implies that, considered as a subfoliation of $\widehat{\mathcal{W}}^{cu}(x)$, $\widehat{\mathcal{W}}^u$ is absolutely continuous with bounded Jacobians. Proposition 5.2 implies that the sequence $\widehat{J}^{cu}_n(x)$ is regular, with respect to the induced Riemannian measure $\widehat{m}_{cu}$. This proves the final claim of Proposition 5.12. □

5.9. **Julienne Density.** We now come to the:

Proof of Proposition 5.13. We must show that if a measurable set X is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated at a point $x \in \mathcal{W}^s_\star(p, 1)$, then x is a Lebesgue density point of X if and only if

$$\lim_{n\to\infty} \widehat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) = 1.$$

As in [BW], we will establish the following chain of equivalences:

$$\begin{aligned}
x \text{ is a Lebesgue density point of } X \quad &\iff \quad \lim_{n\to\infty} m(X : B_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} m(X : C_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} m(X : D_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} m(X : E_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} m(X : F_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} m(X : G_n(x)) &= 1\\
&\iff \quad \lim_{n\to\infty} \widehat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) &= 1.
\end{aligned}$$

The sets $B_n(x)$ through $G_n(x)$ are defined as follows. The set $B_n(x)$ is a $\star$-Riemannian ball in M:

$$B_n(x) = B_\star(x, \sigma_n).$$

The sets $C_n(x)$, $D_n(x)$ and $E_n(x)$ will fiber over the same base $D^{cs}_n(x)$, where

$$D^{cs}_n(x) = \bigcup_{x' \in \widehat{\mathcal{W}}^s_\star(x, \sigma_n)} \widehat{B}^c_n(x').$$

Proposition 5.4, part (4) implies that $D^{cs}_n(x)$ is contained in the C^1 submanifold $\widehat{\mathcal{W}}^{cs}(x)$; the sequences $D^{cs}_n(x)$ and $\widehat{\mathcal{W}}^{cs}_\star(x, \sigma_n)$ are internested. Let

$$C_n(x) = \bigcup_{q \in D^{cs}_n(x)} \mathcal{W}^u_\star(q, \sigma_n),$$

and let

$$D_n(x) = \bigcup_{q \in D^{cs}_n(x)} J^u_n(q).$$

The set $E_n(x)$ is nearly identical to $D_n(x)$, with the difference that the J^u_n-fibers are replaced with $\widehat{J}^u_n$-fibers:

$$E_n(x) = \bigcup_{q\in D^{cs}_n(x)} \widehat{J}^u_n(q) = \bigcup_{x'\in \widehat{\mathcal{W}}^s_\star(x,\sigma_n)} \widehat{J}^{cu}_n(x') = \bigcup_{x'\in \mathcal{W}^s_\star(x,\sigma_n)} \widehat{J}^{cu}_n(x').$$

The rightmost equality follows from the fact that $\widehat{\mathcal{W}}^s_\star(x,\sigma_n) = \mathcal{W}^s_\star(x,\sigma_n)$, for all $x\in\mathcal{W}^s(p,1)$ (Proposition 5.4, part (5))

We define $F_n(x)$ to be the foliation product of $\widehat{J}^{cu}_n(x)$ and $\mathcal{W}^s_\star(x,\sigma_n)$:

$$F_n(x) = \bigcup_{q\in\widehat{J}^{cu}_n(x),\ q'\in\mathcal{W}^s_\star(x,\sigma_n)} \mathcal{W}^s(q)\cap\widehat{\mathcal{W}}^{cu}(q').$$

This definition makes sense since the foliations $\widehat{\mathcal{W}}^{cu}$ and $\mathcal{W}^s$ are transverse. Finally, let

$$G_n(x) = \bigcup_{q\in\widehat{J}^{cu}_n(x)} \mathcal{W}^s_\star(q,\sigma_n).$$

We now prove these equivalences, following the outline described above.

First, recall that $B_n(x)$ is a round $d_\star$-ball about x of radius σ_n. The forward implication in the first equivalence is obvious from the definition of $B_n(x)$. The backward implication follows from this definition and the fact that the ratio $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ of successive radii is less than 1, and is bounded away from both 0 and 1 independently of n. From this we also see that $B_n(x)$ is regular.

The set $C_n(x)$ fibers over $D^{cs}_n(x)$, with fiber $\mathcal{W}^u_\star(x',\sigma_n)$ over $x'\in D^{cs}_n(x)$. The sequence $D^{cs}_n(x)$ internests with the sequence of disks $\widehat{\mathcal{W}}^{cs}_\star(x,\sigma_n)$, by continuity and transversality of the foliations $\widehat{\mathcal{W}}^c$ and $\widehat{\mathcal{W}}^s$. Continuity and transversality of the foliations $\mathcal{W}^u$ and $\widehat{\mathcal{W}}^{cs}$ then imply that $C_n(x)$ and $B_n(x)$ are internested.

To prove the equivalence

$$\lim_{n\to\infty} m(X : C_n(x)) = 1 \iff \lim_{n\to\infty} m(X : D_n(x)) = 1,$$

we note that $C_n(x)$ and $D_n(x)$ both fiber over $D^{cs}_n(x)$, with $\mathcal{W}^u$-fibers. Since X is essentially $\mathcal{W}^u$-saturated at x, Proposition 5.2 implies that it suffices to show that the fibers of $C_n(x)$ and $D_n(x)$ are both c-uniform. The fibers of of $C_n(x)$ are easily seen to be uniform, because they are all comparable to balls in $\mathcal{W}^u$ of fixed radius σ_n. The fibers of $D_n(x)$ are the unstable juliennes $J^u_n(x')$, for $x'\in D^{cs}_n(x)$. Uniformity of these fibers follows from Proposition 5.12.

We next prove:

Lemma 5.20. *The sequences $D_n(x)$ and $E_n(x)$ are internested.*

Proof. Recall that

$$D_n(x) = \bigcup_{q\in D^{cs}_n(x)} J^u_n(x), \quad \text{and} \quad E_n(x) = \bigcup_{q\in D^{cs}_n(x)} \widehat{J}^u_n(x).$$

Internesting of the sequences $D_n(x)$ and $E_n(x)$ means that there is a $k\geq 0$ such that, for all $n\geq k$,

$$D_n(x)\subseteq E_{n-k}(x) \quad \text{and} \quad E_n(x)\subseteq D_{n-k}(x).$$

We will show that there is a k for which the first inclusion holds. Reversing the roles of of $\mathcal{W}^u$ and $\widehat{\mathcal{W}}^u$ in the proof gives the second inclusion.

Suppose $y\in D_n(x)$. Then $y\in J^u_n(q) = f^{-n}(\mathcal{W}^u_\star(q_n,\tau_n))$, for some $q\in D^{cs}_n(x)$; in particular,

$$d_\star(y_n,q_n) = O(\tau_n). \tag{5.14}$$

Let $\hat{q}$ be the unique point of intersection of $\widehat{\mathcal{W}}^u(y)$ with $\widehat{\mathcal{W}}^{cs}(x)$. We will show that $y \in E_{n-k}(x)$, for some k that is independent of n. In order to do this, it suffices to show that $\hat{q} \in D^{cs}_{n-k}(x)$ and $y \in \widehat{J}^u_{n-k}(\hat{q}) = f^{-(n-k)}(\widehat{\mathcal{W}}^u_\star(\hat{q}_{n-k}, \tau_{n-k}))$.

In order to prove that $\hat{q} \in D^{cs}_{n-k}(x)$ it will suffice to show that

$$d_\star(q, \hat{q}) = o(\sigma_n) \tag{5.15}$$

(in fact, $O(\sigma_n)$ would suffice, but the argument gives $o(\sigma_n)$). In order to prove that $y \in \widehat{J}^u_{n-k}(\hat{q})$ it will suffice to show that

$$d_\star(y_n, \hat{q}_n) = O(\tau_n). \tag{5.16}$$

Equation (5.15) follows easily from (5.16). Since y_n and $\hat{q}_n$ lie in the same $\widehat{\mathcal{W}}^u$ leaf, Proposition 5.4 and (5.16) imply that

$$d_\star(y, \hat{q}) = O(\hat{\nu}_n \tau_n) = o(\sigma_n), \tag{5.17}$$

since $\hat{\nu}\tau \prec \sigma$. Similarly, Proposition 5.4 and (5.14) imply that

$$d_\star(y, q) = o(\sigma_n). \tag{5.18}$$

Applying the triangle inequality to (5.17) and (5.18) gives (5.15).

It remains to prove (5.16). Recall from the construction of the fake foliations in Proposition 5.4 that, at any point z in the neighborhood $\mathcal{N}_{\mathbf{r}}$ of the orbit of p in which the fake foliations are defined, the tangent space $T_z\widehat{\mathcal{W}}^u(z)$ lies in the ε-cone about $T_z\mathcal{W}^u(z) = E^u(z)$. Furthermore, the angle between $T_z\widehat{\mathcal{W}}^{cs}(z)$ and either $T_z\widehat{\mathcal{W}}^u(z)$ or $T_z\mathcal{W}^u(z)$ is uniformly bounded away from 0. Note that $\hat{q}_n$ is the unique point in $\widehat{\mathcal{W}}^u(y_n) \cap \widehat{\mathcal{W}}^{cs}(x_n)$ and q_n is the unique point in $\mathcal{W}^u(y_n) \cap \widehat{\mathcal{W}}^{cs}(x_n)$; combining this with (5.14) gives:

$$d_\star(y_n, \hat{q}_n) = O(d_\star(y_n, q_n)) = O(\tau_n).$$

This completes the proof. □

We next show:

Lemma 5.21. *$E_n(x)$ and $F_n(x)$ are internested, as are $F_n(x)$ and $G_n(x)$.*

Proof. The sets $E_n(x)$ and $F_n(x)$ both fiber over the same base $\widehat{\mathcal{W}}^s_\star(x, \sigma_n)$. The fibers of $E_n(x)$ are the *cu*-juliennes$\widehat{J}^{cu}_n(x')$, for $x' \in \widehat{\mathcal{W}}^s(x, \sigma_n)$. the fibers of $F_n(x)$ are images of $\widehat{J}^{cu}_n(x)$ under $\mathcal{W}^s$-holonomy from $\widehat{\mathcal{W}}^{cu}(x)$ to $\widehat{\mathcal{W}}^{cu}(x')$, for $x' \in \widehat{\mathcal{W}}^s_\star(x, \sigma_n)$. It follows immediately from Proposition 5.11 that the sequences $E_n(x)$ and $F_n(x)$ are internested.

To see that $F_n(x)$ and $G_n(x)$ are internested, suppose that q' lies in the boundary of the fiber of $F_n(x)$ that lies in $\mathcal{W}^s(q)$ for some $q \in \widehat{J}^{cu}_n(x)$. Then $q' \in \widehat{J}^{cu}_n(x')$ for a point x' that lies in the boundary of $\mathcal{W}^s_\star(x, \sigma_n)$. The diameters of $\widehat{J}^{cu}_n(x)$ and $\widehat{J}^{cu}_n(x')$ are both $O(\sigma_n)$, and $d_\star(x, x') = \sigma_n$. Hence, if k is large enough, we will have

$$\sigma_{n+k} \leq d_\star(q, q') \leq \sigma_{n-k}.$$

Thus all points on the boundary of the fiber of $F_n(x)$ in $\mathcal{W}^s_{loc}(q)$ lie outside $\mathcal{W}^s_\star(q, \sigma_{n+k})$ and inside $\mathcal{W}^s_\star(q, \sigma_{n-k})$. □

We now know that any two of $D_n(x), E_n(x), F_n(x)$ and $G_n(x)$ are internested. As discussed above, to prove the fourth through sixth equivalences, it now suffices to show:

Lemma 5.22. *The sequence $G_n(x)$ is regular for each $x \in \mathcal{W}^s(p, 1)$.*

Proof. The set
$$G_n(x) = \bigcup_{q\in \widehat{J_n^{cu}}(x)} \mathcal{W}^s_\star(q,\sigma_n)$$
fibers over $\widehat{J_n^{cu}}(x)$, with $\mathcal{W}^s$-fibers $\mathcal{W}^s_\star(q,\sigma_n)$. Since $\mathcal{W}^s$ is absolutely continuous, Proposition 5.2 implies that regularity of $G_n(x)$ follows from regularity of the base sequence and fiber sequence. Proposition 5.12 implies that the sequence $\widehat{J_n^{cu}}(x)$ is regular in the induced measure $\widehat{m}_{cu}$. As we remarked above, the ratio $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ is uniformly bounded below away from 0. Consequently, the ratio
$$\frac{m_s(\mathcal{W}^s_\star(q,\sigma_{n+1}))}{m_s(\mathcal{W}^s_\star(q,\sigma_n))}$$
is bounded away 0, uniformly in x, q, and n. The regularity of $G_n(x)$ now follows from Proposition 5.2. ◻

To prove the final equivalence, we use the fact that $G_n(x)$ fibers over $\widehat{J_n^{cu}}(x)$ with c-uniform fibers and apply Proposition 5.2. Here we use the fact that X is $\mathcal{W}^s$-saturated. This completes the proof of Proposition 5.13. ◻

6. Cocycle Saturation

We now explain a generalization of Theorem B involving saturation properties of sections. This brings the results of [ASV] into the nonuniform setting. We review the notations from [ASV]. In this discussion M denotes a closed manifold and $f\colon M\to M$ a partially hyperbolic diffeomorphism.

A Hausdorff topological space P is *refinable* if there exists an increasing sequence of countable partitions $\mathcal{Q}_1 \prec \mathcal{Q}_2 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ into measurable sets such that any sequence $(Q_n)_{n\in\mathbb{N}}$ with $Q_n \in \mathcal{Q}_n$ and $\bigcap Q_n \neq \emptyset$ converges to a point $\eta \in P$ in the sense that every neighborhood of η contains all Q_n for n sufficiently large. Every separable metric space is refinable.

We shall consider continuous fiber bundles $\mathcal{X}$ over M with fiber a Hausdorff topological space P. Such a fiber bundle is *refinable* if P is refinable.

A fiber bundle $\pi\colon \mathcal{X}\to M$ *has stable and unstable holonomies* if, for every $x, y\in M$ with $y\in\mathcal{W}^*(x)$ and $*\in\{u,s\}$, there exists a homeomorphism $h^*_{x,y}\colon \pi^{-1}(x)\to\pi^{-1}(y)$ with the following properties:

1) $h^*_{x,x} = Id_{\pi^{-1}(x)}$, and $h^*_{y,z}\circ h^*_{x,y} = h^*_{x,z}$;
2) the map $(x,y,\eta,d_*(x,y))\mapsto h^*_{x,y}(\eta)$ is continuous on its domain (a subset of $M\times M\times\mathcal{X}\times[0,\infty)$), where $d_*(x,y)$ stands for the distance between x and y in $\mathcal{W}^*(x)$.[5]

Our main result concerns the saturation properties of sections of refinable bundles with stable and unstable holonomies. In analogy with the definition of stable saturated set, we say that a section $\Psi\colon M\to\mathcal{X}$ is *h^s-saturated* if, for every $x\in M$ and $y\in\mathcal{W}^s(x)$:
$$\Psi(y) = h^s_{x,y}(\Psi(x)).$$
We similarly define *h^u-saturated* sections (the terms *s-invariant* and *u-invariant* are used in [ASV]). A section is *bisaturated* if it is both h^s- and h^u-saturated. A section Ψ is *bi essentially saturated* if there exist an h^s-saturated section Ψ^s and a h^u-saturated section Ψ^u such that $\Psi = \Psi^s = \Psi^u$ almost everywhere with respect to volume on M.

Examples:

[5]This can be reformulated, in view of (1), as requiring that $(x,y,\eta)\mapsto h^*_{x,y}(\eta)$ is continuous when we restrict x and y to belong to *local* $\mathcal{W}^*$ leaves.

1) Let $\mathcal{X} = M \times \{0,1\}$ and set $h^*_{x,y}(\eta) = \eta$. In this trivial example, if $A \subset M$ is a ($\mathcal{W}^s$/$\mathcal{W}^u$/bi) - saturated set, then $x \mapsto (x, 1_A(x))$ is an (h^s/h^u/bi) - saturated section. If A is bi essentially saturated, then so is the associated section.
2) (cf. [W], Proposition 4.7) Every Hölder-continuous function $\psi\colon M \to \mathbb{R}$ determines stable and unstable holonomy maps on the bundle $M\times\mathbb{R}$, invariant under the skew product $(x,\eta) \mapsto (f(x), \eta + \psi(x))$.

   If $\Psi\colon M \to \mathbb{R}$ is a continuous solution to the cohomological equation

$$\psi = \Psi \circ f - \Psi, \tag{6.1}$$

   then Ψ is a bisaturated section. Moreover, if f is C^2, volume-preserving and ergodic, $\Psi\colon M \to \mathbb{R}$ is measurable, and the equation (6.1) holds almost everywhere with respect to volume, then Ψ is a bi essentially saturated section.
3) (cf. [ASV]) Let $A\colon M \to SL(n,\mathbb{R})$ be a Hölder-continuous matrix-valued cocycle. If this cocycle is dominated (in the sense of [ASV]), then it determines in a natural way stable and unstable holonomies on the refinable fiber bundle $\mathcal{X} = M\times\mathcal{M}(\mathbb{R}P^{n-1})$, where $\mathcal{M}(\mathbb{R}P^{n-1})$ is the space of probability measures on the projective space $\mathbb{R}P^{n-1}$.

   Suppose that the Lyapunov exponents of $A_n(f) = (A\circ f^{n-1})(A\circ f^{n-1})\cdots A$ vanish almost everywhere. Then A determines a bi essentially saturated section of the bundle $\mathcal{X}$. These results are proved in [ASV] and used to show that the generic such cocycle over an accessible, center-bunched partially hyperbolic diffeomorphism has a nonvanishing exponent.

Our main result expands Theorem B to include bi esentially saturated sections. Following [ASV], we introduce an analogue for measurable sections of the notion of density point for measurable sets.

Let $\pi\colon \mathcal{X} \to M$ be a refinable bundle. We say that $p \in M$ is a *point of measurable continuity* for a section $\Psi\colon M \to \mathcal{X}$, if there exists $\eta \in \mathcal{X}$ such that p is a Lebesgue density point of $\Psi^{-1}(V)$, for every open neighborhood V of η in $\mathcal{X}$. If such an η exists, it is unique, and is called the *density value* of Ψ at p.

Let $MC(\Psi)$ be the set of points of measurable continuity of Ψ. We define a measurable section $\tilde{\Psi}\colon MC(\Psi) \to \mathcal{X}$ by setting $\tilde{\Psi}(p)$ to be the density value of Ψ at p. Then $MC(\Psi)$ has full volume in M, and $\tilde{\Psi} = \Psi$ almost everywhere, with respect to volume (see Lemma 7.10, [ASV]).

Theorem D (cf. Theorem 7.6, [ASV])**.** *Let f be C^2 and partially hyperbolic, and let $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies.*

Then, for any bi essentially saturated section $\Psi\colon M \to \mathcal{X}$:

1) *$MC(\Psi) \cap CB^+$ is $\mathcal{W}^s$-saturated, and the restriction of $\tilde{\Psi}$ to $MC(\Psi) \cap CB^+$ is h^s-saturated;*
2) *$MC(\Psi) \cap CB^-$ is $\mathcal{W}^u$-saturated, and the restriction of $\tilde{\Psi}$ to $MC(\Psi) \cap CB^-$ is h^u-saturated;*

Proof. The proof follows the same lines as Theorem 7.6 in [ASV]. The proof there adapts the proof of the main result in [BW], and we correspondingly adapt the proof of Theorem B here.

We first prove the theorem under the assumption that the bundle $\mathcal{X}$ has stable and unstable holonomies. We prove the first part of the theorem; the second part follows from the first, replacing f by f^{-1}. Let $\pi\colon \mathcal{X} \to M$ be a refinable bundle with stable and unstable holonomies. The holonomy maps h^s and h^u define foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ of $\mathcal{X}$; the leaf of $\mathcal{F}^*$ through a point $\eta \in \mathcal{X}$ is:

$$\mathcal{F}^*(\eta) = \{h^*_{\pi(\eta),y}(\eta)\colon y \in \mathcal{W}^*(\pi(\eta))\}.$$

We similarly define for $r > 0$ the local leaf:

$$\mathcal{F}^*(\eta, r) = \{h^*_{\pi(\eta),y}(\eta) \colon y \in \mathcal{W}^*(\pi(\eta), r)\}.$$

Observe that a section Φ is $*$-saturated if and only if its image $\Phi(M) \subset \mathcal{X}$ is a union of whole leaves of $\mathcal{F}^*$.

Fix a bi essentially saturated section $\Psi \colon M \to \mathcal{X}$. Recall that bi essential saturation of Ψ means that there exist an h^s-saturated section Ψ^s and a h^u-saturated section Ψ^u such that $\Psi^s = \Psi^u = \Psi$ almost everywhere.

Fix $x \in MC(\Psi) \cap CB^+$, and let $\eta = \tilde{\Psi}(x)$ be the density value of Ψ at x. Note that η is also a density value for Ψ^s and Ψ^u. We will show that for every $y \in \mathcal{W}^s(x, 1)$, $h^s_{x,y}(\eta)$ is a (the) density value of Ψ at y. Since CB^+ is $\mathcal{W}^s$-saturated, this will simultaneously establish that $MC(\Psi) \cap CB^+$ is $\mathcal{W}^s$-saturated and that the restriction of $\tilde{\Psi}$ to $MC(\Psi) \cap CB^+$ is h^s-saturated.

To this end, fix $y \in \mathcal{W}^s(x, 1)$, and let V be a neighborhood of $h^s_{x,y}(\eta)$ in $\mathcal{X}$. Note that $h^s_{x,y}(\eta)$ lies on the local leaf $\mathcal{F}^s(\eta, 1)$. To show that $h^s_{x,y}(\eta)$ is a density value for Ψ at y, we must show that y is a density point of $\Psi^{-1}(V)$.

Continuity of the stable holonomy maps in $\mathcal{X}$ and stable saturation of Ψ^s together imply that $(\Psi^s)^{-1}(V)$ is $\mathcal{W}^s$-saturated at y; recall this means that there exist $0 < \delta_0 < \delta_1$ such that for any $z \in B(y, \delta_0) \cap (\Psi^s)^{-1}(V)$, we have $\mathcal{W}^s(z, \delta_1) \subset \Psi_s^{-1}(V)$. Similarly, $(\Psi^u)^{-1}(V)$ is $\mathcal{W}^u$-saturated at y, and so $\Psi^{-1}(V)$ is bi essentially saturated at y.

Fix $\varepsilon > 0$ and $\delta > 0$ such that $\pi^{-1}(B(y, \varepsilon)) \cap N_\delta \subset V$, where

$$N_\delta = \bigcup_{z \in B(\eta, \delta)} \mathcal{F}^s(z, 1)$$

is the union of the local $\mathcal{F}^s$ leaves through $B(\eta, \delta)$ in $\mathcal{X}$. Since N_δ is saturated by local $\mathcal{F}^s(\cdot, 1)$ leaves, and the section Ψ^s is h^s-saturated, it follows that the set $(\Psi^s)^{-1}(N_\delta)$ is saturated by local $\mathcal{W}^s(\cdot, 1)$-leaves. The set $\Psi^{-1}(B(\eta, \delta))$ is bi essentially saturated at x and coincides mod 0 with the set $(\Psi^s)^{-1}(B(\eta, \delta))$, which is h^s-saturated at x. Since $x \in MD(\Psi)$, it is a Lebesgue density point of $\Psi^{-1}(B(\eta, \delta))$. But x is also an element of CB^+, and so Proposition 5.13 implies that x is a *cu*-julienne density point of $(\Psi^s)^{-1}(B(\eta, \delta))$, and hence of $(\Psi^s)^{-1}(N_\delta)$ as well.

Now, since $(\Psi^s)^{-1}(N_\delta)$ is saturated by local $\mathcal{W}^s(\cdot, 1)$-leaves, and $x \in CB^+(f)$, Proposition 5.14 implies that y is also a *cu*-julienne density point of $(\Psi^s)^{-1}(N_\delta)$. Thus y is a *cu*-julienne density point of $B(y, \varepsilon) \cap (\Psi^s)^{-1}(N_\delta)$. But

$$B(y, \varepsilon) \cap (\Psi^s)^{-1}(N_\delta) = (\Psi^s)^{-1}(\pi^{-1}(B(y, \varepsilon)) \cap N_\delta) \subset (\Psi^s)^{-1}(V);$$

since the latter set is $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated at y, and since $y \in CB^+(f)$, Proposition 5.13 implies that y is a Lebesgue density point of $(\Psi^s)^{-1}(V)$. Finally, since $(\Psi^s)^{-1}(V) = \Psi^{-1}(V) \mod 0$, we obtain that y is a Lebesgue density point of $\Psi^{-1}(V)$. This completes the proof of Theorem D. □

7. Examples

Here we will be interested first in the construction of a C^2-open class of maps which are not uniformly center bunched, but display *nonuniform center bunching* in the sense that the set CB of center bunched points has full Lebesgue measure. We then show, using Corollary C, that this class contains C^2-stably ergodic maps, and describe an application of Theorem D to the cohomological equation.

All of the following constructions can be carried out in the volume-preserving setting. We do it in the symplectic setting, as the arguments are slightly more subtle.

7.1. **A nonuniformly, but not uniformly, center bunched example.** Let P, Q and S be compact symplectic manifolds, and let $F\colon P \to P$, $G\colon Q \to Q$ and $H\colon S \to S$ be symplectic C^2 diffeomorphisms with the following properties:

1) F and G are Anosov diffeomorphisms.
2) We have
$$\sup_Q \|DG|_{E^s_G}\| < \inf_S \mathbf{m}(DH)^2 \le \sup_S \|DH\|^2 < \inf_Q \|DG|_{E^u_G}\|,$$
so that $G \times H\colon Q \times S \to Q \times S$ is partially hyperbolic and center bunched, with center bundle tangent to the S factor.
3) We have
$$\sup_P \|DF|_{E^s_F}\| < \inf_S \mathbf{m}(DH) \le \sup_S \|DH\| < \inf_P \|DF|_{E^u_F}\|,$$
so that $F \times G \times H$ is partially hyperbolic on $M = P \times Q \times S$, with center bundle tangent to the S factor.
4) Indicating by m_P the normalized volume measure induced by the symplectic form on P, we have
$$\int \log \|DF|_{E^s_F}\| \, dm_P < 2 \inf_S \log \mathbf{m}(DH) \le 2 \sup_S \log \|DH\| < \int \log \mathbf{m}(DF|_{E^u_F}) \, dm_P \,.$$
5) There exists a point $p \in P$ of period k under F such that:
$$\|D_p F^k|_{E^s_F}\| < \mathbf{m}(DG|_{E^s_G})^k < \|DG|_{E^u_G}\|^k < \mathbf{m}(D_p F^k|_{E^u_F}) \,,$$
which implies that $\bigcup_{j=0}^{k-1} \{F^j(p)\} \times Q \times S$ is normally hyperbolic and contained in CB.

Let ω be the symplectic form in $M = P \times Q \times S$ given by the sum of the forms on P, Q and S. Then $f_0 = F \times G \times H : M \to M$ is symplectic.

Lemma 7.1. *If f is a C^2 volume-preserving (C^2-small) perturbation of f_0, then f is nonuniformly center bunched.*

Proof. To show that almost every orbit is forward center bunched, it is enough to prove that for any f-invariant set W of positive Lebesgue measure, we have
$$\frac{1}{m(W)} \int_W \log \|Df|_{E^s_f}\| \, dm < 2 \inf_M \log \mathbf{m}(Df|_{E^c_f}).$$
We notice that E^s_f is close to $E^s_F \oplus E^s_G$ and E^c_f is close to TS everywhere. Thus the right hand side is close to $2 \inf_S \log \mathbf{m}(DH)$, while the left hand side is bounded, up to small error, by the maximum of $\sup_Q \|DG|_{E^s_G}\|$ and $\int \log \|DF|_{E^s_F}\| \, d\pi_* \mu$, where μ is the normalized restriction of the Lebesgue measure m to W and $\pi : M \to P$ is the coordinate projection. By (2) and (4), we are reduced to showing that $\pi_* \mu$ is weak-* close to m_P.

An f-invariant probability measure which is absolutely continuous with respect to the unstable foliation $\mathcal{W}^u_f$ will be called an u-state for f. One defines s-states analogously. Let $\mathcal{U}(f)$ be the set of u-states for f and $\mathcal{S}(f)$ be the set of s-states for f. An u-state that is also an s-state will be called an su-state. Since f is C^2, the $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ foliations are absolutely continuous, thus μ is an su-state. We are going to show that this already implies that $\pi_* \mu$ is close to m_P.

The uniform expansion in the unstable direction as we iterate forward has a regularization effect which implies that there is an apriori bound on the densities of the disintegration of an u-state for f along $\mathcal{W}^u_f$: the quotient between the densities at different points in the same unstable leaf is bounded by K^d where K is a constant (uniform in a C^2 neighborhood of f_0) and d is the distance between the points inside the leaf. (Recall that the density is defined, in each leaf, only up to scaling but the quotient is well defined and given by the Anosov-Sinai cocycle; see formula (11.4) in [BDV].)

This bound has the important consequence that $\mathcal{U}(f)$ is closed (and hence compact) in the weak-$*$ topology. Moreover, in a C^2 neighborhood $\mathcal{V}$ of f_0, the set $\bigcup_{f\in\mathcal{V}}\{f\}\times\mathcal{U}(f)$ is also closed. We call this fact the upper-semicontinuity in f of the set of u-states.

Analogous considerations show that the set of s-states is upper-semicontinuous in f. Thus $\bigcup_{f\in\mathcal{V}}\{f\}\times(\mathcal{S}(f)\cap\mathcal{U}(f))$ is closed as well, so the set of su-states also depends upper-semicontinuously on f.

The product structure of the foliations implies that an su-state for f_0 projects onto an su-state for $F\times G$, which is C^2 Anosov, and the absence of a central direction for $F\times G$ implies that the projection is absolutely continuous. Since $F\times G$ is Anosov, it is ergodic so the projection is Lebesgue on $P\times Q$. Projecting again, we conclude that $\pi_*\nu = m_P$ whenever ν is an su-state for f_0 (in fact, an su-state for f_0 is just the product of Lebesgue on $P\times Q$ by an arbitrary invariant probability measure on S). By upper-semicontinuity, if f is close to f_0, the projection of any su-state for f is weak-$*$ close to m_P. The result follows. □

Notice also that we may construct the map f_0 so that no f nearby is center bunched. For example, one can arrange that the conditions above hold and in addition there are hyperbolic periodic points $p' = F^\ell(p')$, $q = H^m(q)$ such that

$$\rho(D_{p'}F^\ell|_{E^s_F})^{1/\ell} > \rho(D_qH^m)^{-2/m},$$

where ρ denotes spectral radius. Note that the main theorem in [BW] does not apply to such an example, nor to its perturbations.

7.2. **Stable ergodicity.** Condition (5) implies that for any C^1 perturbation f of f_0, there exists a normally hyperbolic manifold N_f, C^1-close to $\bigcup_{j=0}^{k-1}\{F^j(p)\}\times Q\times S$, whose connected components are permuted under f.

Let us say that N_f is accessible if for any x and y in the same connected component of N_f, there is an su-path in N_f connecting x and y. We say that N_f is stably accessible if N_g is accessible for every g in a neighborhood of f in $\mathrm{Diff}^1_\omega(M\times N\times P)$. These properties are non-void:

Lemma 7.2. *For any neighborhood $\mathcal{Z}$ of f_0 in $\mathrm{Diff}^\infty_\omega(M)$ there exists $f\in\mathcal{Z}$ such that N_f is stably accessible.*

Proof. In [SW] it is shown that for every neighborhood $\mathcal{V}$ of the identity in $\mathrm{Diff}^\infty_{\omega_{Q\times S}}(Q\times S)$ there exists $\Phi\in\mathcal{V}$ such that $\Phi\circ(G\times H)$ is stably accessible. For such a Φ, define $\phi\in\mathrm{Diff}^\infty_\omega(M\times N\times P)$ by $\phi = Id_M\times\Phi$. Then $\phi\circ f_0$ is close to f_0 and satisfies the desired properties. □

Lemma 7.3. *If f is C^1 near f_0 and N_f is accessible, then we can join any two points in CB by an su-path with corners in CB.*

Proof. Fix $x\in N_f$. Obviously $\mathcal{W}^c(x)\subset N_f\subset CB$, and since N_f is stably accessible, any two points in $\mathcal{W}^c(x)$ can be joined by an su-path with corners in N_f and hence in CB. Thus it is enough to show that any $y\in CB$ can be joined to some point in $\mathcal{W}^c(x)$ through an su-path with corners in CB. The action of f on $M/\mathcal{W}^c_f$ is topologically conjugated to the Anosov map $F\times G$, and under this identification, the projection of any unstable or stable leaf of f is an unstable or stable leaf of $F\times G$. Obviously, for $F\times G$ any two points can be connected by an su-path with 2 legs. We conclude that for every $y\in M$ there exists $z\in\mathcal{W}^c(x)$ such that $\mathcal{W}^u(y)\cap\mathcal{W}^s(z)\neq\emptyset$. When $y\in CB$, $\mathcal{W}^u(y)\cap\mathcal{W}^s(z)\subset CB$ (since $y,z\in CB$ and CB^+ is $\mathcal{W}^s$-saturated while

CB^- is $\mathcal{W}^u$-saturated), showing that y is connected to $\mathcal{W}^c(x)$ by a 2-legged su-path with corners in CB. □

Putting together Lemmas 7.2, 7.3 and Corollary C we conclude:

Theorem E. *If f is C^2-close to f_0 and N_f is accessible then f is ergodic (and in fact, a K-system).*

7.3. **Continuity of bi saturated sections and the cohomological equation.**

Lemma 7.4. *Let $f : M \to M$ be a C^2 volume-preserving partially hyperbolic diffeomorphism and let Z be a bi essentially saturated set of positive Lebesgue measure. If $x \in \operatorname{supp}(m|Z)$ then any su-path starting at x can be approximated by an su-path with corners in Z.*

Proof. Let us say that $z \in Z$ is k-pretty if almost every $w \in \mathcal{W}^u(z) \cup \mathcal{W}^s(z)$ is $k-1$-pretty, where all points of Z are declared to be 0-pretty. Since $\mathcal{W}^u$ and $\mathcal{W}^s$ are absolutely continuous, it follows by induction that almost every $z \in Z$ is k-pretty for every k.

Consider an su-path connecting x_0 to x_n through $x_1, \dots, x_{n-1}$. Now just approximate x_0 by an n-pretty point z_0, and then successively x_i by an $n-i$-pretty point $z_i \in \mathcal{W}^*(z_{i-1})$. □

Theorem F. *Let f be C^2-close to f_0 and let $\mathcal{X}$ and Ψ be as in Theorem D. If N_f is accessible then Ψ coincides almost everywhere with a continuous bi invariant section.*

Proof. Since any two points of CB can be joined by an su-path with corners in CB, and CB has positive Lebesgue measure, it follows that $MC(\Psi)$ contains CB.

Let us show that we can define a bi saturated section that coincides with $\tilde{\Psi}$ on CB. By the argument of Section 8.2 of [ASV] (where center bunching does not play a role), the accessibility of f implies that such a section is necessarily continuous, and since $m(CB) = 1$, Ψ must coincide almost everywhere with it.

We notice that, restricting the above considerations to $N_f \subset CB$, and using that N_f is accessible, we can already conclude that $\tilde{\Psi}|N_f$ is continuous.

Let $x \in N_f$. We are going to show that, for any su-path starting and ending at x, the composition of holonomies along the su-path fixes $\tilde{\Psi}(x)$. Since f is accessible, this allows us to define a bi saturated section: join x to any $y \in M$ by any su-path and apply the holonomy to $\tilde{\Psi}(x)$. If well defined, this new section automatically will coincide with $\tilde{\Psi}(x)$ on CB by Theorem D (since, by Lemma 7.3, x can be joined to any $y \in CB$ through an su-path with corners in CB).

Let us consider thus an su-path starting and ending in x_0, and its composed holonomy map h. Assume that $h(\tilde{\Psi}(x)) \neq \tilde{\Psi}(x)$. By the previous lemma, it is approximated by an su-path with corners in CB. Apriori, the extremes of the latter path do not belong to N_f, but by adding at most 4 short legs to the latter (two at the beginning and two at the end), we obtain an su-path starting and ending at points $y, z \in N_f$. Since the corners of this path all belong to CB, the corresponding composed holonomy map $\tilde{h}$ takes $\tilde{\Psi}(y)$ to $\tilde{\Psi}(z)$. Since $y, z \in N_f$ are close to x, we can use the continuity of holonomy maps, and of $\tilde{\Psi}|N_f$, to conclude that $\tilde{h}(\tilde{\Psi}(y))$ is close to $h(\tilde{\Psi}(x))$ and $\tilde{\Psi}(z)$ is close to $\tilde{\Psi}(x)$. Since we assumed that $h(\tilde{\Psi}(x)) \neq \tilde{\Psi}(x)$, this implies that $h(\tilde{\Psi}(y)) \neq \tilde{\Psi}(z)$, contradiction. □

One particular interesting application is the case of the cohomological equation (see Example 2 in Section 6): if $\psi : M \to \mathbb{R}$ is a Hölder continuous function then a measurable solution of the cohomological equation $\psi = \Psi \circ f - \Psi$ coincides almost everywhere with a continuous one.

One can also deduce non-degeneracy of the Lyapunov spectrum of generic bunched cocycles over f (see Example 3 of Section 6). However, the application of those ideas to the analysis of the central Lyapunov exponents of f themselves is more subtle since this cocycle is not bunched (but only nonuniformly bunched), and will be carried out elsewhere: we will show for instance that in the case that S is a surface then stably Bernoulli, nonuniformly hyperbolic examples like above can be obtained.

7.4. **Further examples.** The mechanism for ergodicity implemented above can be abstracted somewhat to a criterion for ergodicity, which we quickly describe.

Let $f : M \to M$ be a C^2 accessible partially hyperbolic volume preserving diffeomorphism. Let $N \subset M$ be a normally hyperbolic compact (not necessarily connected) submanifold.[6] It is easy to see that $T_xN = (E^s(x) \cap T_xN) \oplus (E^c(x) \cap T_xN) \oplus (E^u(x) \cap T_xN)$ at every $x \in N$. If those three subbundles are non-trivial, then this splitting is partially hyperbolic. We are interested on the case that N is c-saturated in the sense that $T_xN \supset E^c(x)$ for every $x \in N$. Assume that $f|N$ is center bunched and has some open accessibility class. We will show that *the restriction of Lebesgue measure to the set CB is either null or ergodic.*

Let us first note that, since N is normally hyperbolic, the condition that $f|N$ is center bunched implies that $N \subset CB$.

We claim that for almost every $x \in M$, there exists $y \in N$ such that $\mathcal{W}^u(x) \cap \mathcal{W}^s(y) \neq \emptyset$. Indeed the set U of all x with this property is clearly an f-invariant set with non-empty interior (its interior must contain all open accessibility classes of $f|N$), and since f is accessible Theorem 3.4 implies that U must be a full measure set.

If $x \in CB$, then $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ also belongs to CB, for any $y \in N$. Thus any two points in $CB \cap U$ can be connected by a su-path with all corners in CB as well. We can thus apply Corollary C to get the conclusion.

One can also apply the argument of the previous section to conclude, for instance, that if $\psi : M \to \mathbb{R}$ is a Hölder continuous function then any measurable solution of the cohomological equation $\psi = \Psi \circ f - \Psi$ defined over CB coincides almost everywhere with a continuous solution defined in the whole M. We notice that here it is only needed to assume that $m(CB) > 0$, and apriori the system could even be non-ergodic as far as the current theory goes.

Appendix A. Reobtaining Some Results from [BDP]

A variation of the method presented in Section 4 allows one to obtain various [BDP]-like (topological) conclusions from [BV]-like (ergodic) results. To illustrate, we will reobtain the following:

Theorem A.1 ([BDP]). *A diffeomorphism that has a non-dominated homoclinic class can be perturbed to display a nearby periodic orbit with all eigenvalues of the same modulus.*

Let us explain the result from [BV] that we need. Let (X, μ) be a non-atomic probability space, and let f be an ergodic automorphism of it. Fix a positive integer d and let L^∞ be the set of measurable maps (called *cocycles*) $A : X \to \mathrm{GL}(d, \mathbb{R})$ such that $\|A^{\pm 1}\|$ are μ-essentially bounded, where maps that differ on zero sets are identified. Notice L^∞ is a Baire space.

Given a cocycle $A \in L^\infty$, asymptotic information about the products

$$A^n_f(x) = A(f^{n-1}(x)) \cdots A(x)$$

[6]Our arguments would also work by taking N as a (non-compact) leaf of a normally hyperbolic lamination.

is given by Oseledets Theorem. So let $\mathbb{R}^d = E^1(x) \oplus \cdots \oplus E^k(x)$ be the Oseledets splitting, defined for μ-a.e. $x \in X$, and let $\lambda_1(A) \geq \cdots \geq \lambda_d(A)$ the Lyapunov exponents repeated according to multiplicity. (Notice that k and the Lyapunov exponents are constant μ-almost everywhere by ergodicity.) We also write $L_i(A) = \sum_{j=1}^{i} \lambda_j(A)$. We have

$$L_i(A) = \inf_{n \geq 1} \frac{1}{n} \int \log \|\wedge^i A_f^n(x)\| \, d\mu(x) \, .$$

As a consequence of this formula, the function $A \in L^\infty \mapsto L_i(A)$ is upper-semicontinous, and hence its points of continuity form a residual set. Another semi-continuity property that follows easily from the formula is:

Lemma A.2. *Given $A \in L^\infty$, $C > \|A^{\pm 1}\|_\infty$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $B \in L^\infty$ is such that if $\|B^{\pm 1}\|_\infty < C$ and $\mu[B \neq A] < \delta$ then $L_i(B) < L_i(A) + \varepsilon$.*

Let $\mathbb{R}^d = E(x) \oplus F(x)$ be a splitting defined for μ-a.e. x and invariant under a cocycle $A \in L^\infty$. Also assume that $\dim E$ is constant (called the *index* of the splitting). We say that the splitting is *dominated* (or, more precisely, that E *dominates* F) in the case that there exists $m \in \mathbb{N}$ such that

$$\frac{\left\|A^m(x)|_{F(x)}\right\|}{\mathbf{m}\left(A^m(x)|_{E(x)}\right)} \leq \frac{1}{2} \quad \text{for } \mu\text{-a.e. } x \in X. \tag{A.1}$$

It is not hard to check the following elementary properties[7]:

1) The angle between E and F is essentially bounded from below.
2) Fixed the index, the dominated splitting is unique over the points where it exists.
3) In the case that the space X is compact Hausdorff and A is a continuous map, then the splitting can be defined over each point of $\operatorname{supp} \mu$, and varies continuously.

We say that the Oseledets splitting of A is *trivial* if $k = 1$, and *dominated* if $k > 1$ and $E^1 \oplus \cdots \oplus E^i$ dominates $E^{i+1} \oplus \cdots \oplus E^k$ for all $i \in \{1, \ldots, k-1\}$.

Theorem A.3 ([BV])**.** *A cocycle $A \in L^\infty$ is a point of continuity of all L_i's if and only if the Oseledets splitting is trivial or dominated.*

Remark A.4. As shown in [BV], the statement of Theorem A.3 remains true if $\mathrm{GL}(d, \mathbb{R})$ is replaced by any Lie group of matrices that act transitively on the projective space, for example the symplectic group.

We will deduce from Theorem A.3 the following:

Proposition A.5. *If $A \in L^\infty$ has no dominated splitting then there exists $B \in L^\infty$ arbitrarily close to A whose Oseledets splitting is trivial.*

The proof of the proposition requires a few preliminaries.

Given a cocycle $A \in L^\infty$, we define μ_A as a probability measure on $\mathrm{GL}(d, \mathbb{R})^{\mathbb{Z}}$ by taking the push-forward of μ under the map $x \mapsto (A(f^n(x)))_n$. Let $\mathrm{Hull}(A) = \operatorname{supp} \mu_A$; this is a compact Hausdorff space. Let $\hat{A} : \mathrm{Hull}(A) \to \mathrm{GL}(d, \mathbb{R})$ be the projection on the zeroth coordinate, considered as a cocycle over the shift on $\mathrm{Hull}(A)$. This new cocycle has the advantage of being continuous. Using the elementary properties listed above, it is easy to see that a cocycle $A \in L^\infty$ has a dominated splitting if and only if so does $\hat{A}$. In particular, the existence of a dominated splitting for A depends only on $\mathrm{Hull}(A)$.

[7] Or see e.g. [BDV, Appendix B].

Let $\mathcal{N}$ indicate the set of cocycles $A \in L^\infty$ that have no dominated splitting. Then $\mathcal{N}$ is a G_δ subset[8] of L^∞, and thus a Baire space. Indeed, the set of $A \in L^\infty$ that have a dominated splitting with fixed index and fixed m as in (A.1) is easily seem to be a closed set.

Lemma A.6. *If $A \in \mathcal{N}$ is a point of continuity of $L_i|\mathcal{N}$ then A is a point of continuity of L_i.*

Proof. Assume that L_i is *not* continuous at some $A \in \mathcal{N}$; we will show that neither is $L_i|\mathcal{N}$.

Let $a^k = (a_n^k)_n$, $k \geq 0$ be a dense sequence in $\operatorname{Hull}(A)$. For every $k \geq 0$ and $j \geq 0$, choose a set $D_{k,j} \subset X$ with μ-measure at most 2^{-k-j} such that if $x \in D_{k,j}$ then $A(f^n(x))$ is 2^{-n}-close to a_n^k for $|k| \leq j$ (this is possible since μ is not atomic). Let $Z_j = \bigcup_{k \geq 0} D_{k,j}$. Then $\mu(Z_j) \to 0$ as $j \to \infty$. Moreover, if $B \in L^\infty$ is any cocycle that coincides with A on some Z_j then $\operatorname{Hull}(B) \supset \operatorname{Hull}(A)$ and therefore B belongs to $\mathcal{N}$.

Since L_i is upper-semicontinuous and not continuous at A, there exists a sequence $A_n \in L^\infty$ converging to A and $\varepsilon > 0$ such that $L_i(A_n) < L_i(A) - \varepsilon$ for each n. Let $B_{n,j}$ be the cocycle equal to A on Z_j and equal to A_n elsewhere. By Lemma A.2, for each n there exists j_n such that $L_i(B_{n,j_n}) < L_i(A_n) + \varepsilon/2$. Thus the sequence B_{n,j_n} is in $\mathcal{N}$, converges to A, and satisfies $L_i(B_{n,j_n}) < L_i(A) - \varepsilon/2$. This shows that $L_i|\mathcal{N}$ is not continuous at A, as desired. ◻

Now we can give the:

Proof of Proposition A.5. Let A be a element of $\mathcal{N}$, that is, a cocycle without dominated splitting. Since $\mathcal{N}$ is a Baire space and the functions L_i are upper-semicontinuous, we can find a point B of continuity of all functions $L_i|\mathcal{N}$ that is as close to A as desired. By Lemma A.6, B is a point of continuity of all L_i's, and thus, by Theorem A.3, its Oseledets splitting is either dominated or trivial. Since $B \in \mathcal{N}$, the latter alternative is forbidden and thus all Lyapunov exponents of B are equal. ◻

Now let us use these results to prove Theorem A.1. Our approach to needs a suitable measure to start with:

Lemma A.7. *For every homoclinic class H, there exists an ergodic invariant probability measure whose support is H.*

Proof. This is a simple consequence of the fact that any non-trivial homoclinic class H is contained in the closure of a countable union of *horseshoes* $H_1 \subset H_2 \subset \cdots$ (by a horseshoe we mean an invariant compact set restricted to which the dynamics is topologically conjugate to a transitive subshift of finite type).This allows one to construct a wealth of invariant measures with support H (for instance, with positive entropy), as suitable "infinite Markovian" measures, but below we will proceed by a somewhat less direct argument.

Given a compact invariant set $X \subset M$, let $\mathcal{M}(X)$ be the set of invariant probability measures μ with $\operatorname{supp} \mu \subset X$, endowed with the weak-star topology. Let $\mathcal{M}_e(X) \subset \mathcal{M}(X)$ be the set of ergodic measures, and for any compact subset $Y \subset X$, let $\mathcal{M}(X, Y)$ be the set of invariant measures whose support contains Y. It is easy to see that $\mathcal{M}_e(X)$ and $\mathcal{M}(X, Y)$ are always G_δ subsets of $\mathcal{M}(X)$.

Since H_i is a horseshoe, both $\mathcal{M}_e(H_i)$ and $\mathcal{M}(H_i, H_i)$ are dense (and hence residual) in $\mathcal{M}(H_i)$. Let $G_i = \mathcal{M}_e(H_i) \cap \mathcal{M}(H_i, H_i)$. Let $W \subset \mathcal{M}(H)$ be the closure of the union of the $\mathcal{M}(H_i)$. Let $W_i = W \cap \mathcal{M}_e(H) \cap \mathcal{M}(H, H_i)$, which is a G_δ-subset of W. Notice that W_i contains G_j for each $j \geq i$. Since G_j is a G_δ-dense subset of $\mathcal{M}(H_j)$, it

[8] More precisely, $\mathcal{N}$ is a closed set, but we will not need this.

follows that W_i is dense in $W = \overline{\bigcup_{j\geq i} \mathcal{M}(H_j)}$ for every i. Now, W is a compact Hausdorff and hence Baire space, and we conclude that $\bigcap W_i$ is a dense subset of W. Since $H = \overline{\bigcup H_i}$, the set $\bigcap W_i$ is precisely $W \cap \mathcal{M}_e(H) \cap \mathcal{M}(H,H)$. In particular, $\mathcal{M}_e(H) \cap \mathcal{M}(H,H)$ is non-empty, as desired. □

Proof of Theorem A.1. Let f be a diffeomorphism and let H be an homoclinic class that has no dominated splitting. Choose an ergodic probability measure μ whose support is H, using Lemma A.7.

We will consider L^∞-perturbations A of the derivative of f restricted to H. Such an object A is the assignment for μ-a.e. $x \in H$ of a linear map $A(x) : T_xM \to T_{f(x)}M$ that is close to $Df(x)$, and varies measurably. Now, using Proposition A.5 we can find such a perturbation A of the derivative whose Lyapunov exponents coincide μ-almost everywhere. Using Lusin's Theorem, we may alter A on a set of arbitrarily small μ-measure, while keeping it uniformly close to Df, to obtain a *continuous* perturbation B. It follows from Lemma A.2 that the Lyapunov exponents of B are all close to each other μ-almost everywhere. In other words, there is a small number $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|B^n_f(x)\|}{\mathbf{m}(B^n_f(x))} \leq \frac{\varepsilon}{2} \quad \text{for } \mu\text{-a.e. } x \in H,$$

where we indicate $B^n_f(x) = B(f^{n-1}(x)) \cdots B(x)$. Next we apply the Ergodic Closing Lemma (imitating the proof of Lemma 4.2) and find a C^1-perturbation $\tilde{f}$ of f that has a periodic point x of period p such that

$$\frac{\|B^{mp}_{\tilde{f}}(x)\|}{\mathbf{m}(B^{mp}_{\tilde{f}}(x))} < e^{\varepsilon mp} \quad \text{for some } m \geq 1.$$

The latter implies the moduli of the eigenvalues of $B^p_{\tilde{f}}(x)$ are all close to each other. By means of an (easier) dissipative analogue of Lemma 4.3, we can perturb B along the $\tilde{f}$-orbit of x to make the eigenvalues of $B^p_{\tilde{f}}(x)$ of the same moduli. By Franks' Lemma one can perturb the diffeomorphism again, keeping the periodic orbit and inserting the desired derivatives. This concludes the proof. □

References

[AV] F. Abdenur, M. Viana. Flavors of partial hyperbolicity. In preparation.

[AK] D. V. Anosov, A. B. Katok. New examples in smooth ergodic theory. Ergodic diffeomorphisms. *Trans. Moscow Math. Soc.* 23 (1970), 1-35.

[AM] A. Arbieto, C. Matheus. A pasting lemma and some applications for conservative systems. *Ergod. Th. Dynam. Sys.* 27 (2007), no. 5, 1399–1417.

[Arna] M.-C. Arnaud. Le "closing lemma" en topologie C^1. *Mém. Soc. Math. Fr.* 74 (1998).

[ABC] M.-C. Arnaud., C. Bonatti, S. Crovisier. Dynamiques symplectiques génériques. *Ergod. Th. Dynam. Sys.* 25 (2005), 1401–1436.

[Arno] L. Arnold. *Random dynamical systems*. Springer Verlag, 2003.

[A] A. Avila. On the regularization of conservative maps. Preprint.

[ASV] A. Avila, J. Santamaria, M. Viana. Cocycles over partially hyperbolic maps. Preprint.

[B] J. Bochi. C^1-generic symplectic diffeomorphisms: partial hyperbolicity and zero center Lyapunov exponents. To appear in *J. Inst. Math. Jussieu*.

[BV] J. Bochi, M. Viana. The Lyapunov exponents of generic volume preserving and symplectic maps. *Annals of Math.* 161 (2005), 1423–1485.

[BC] C. Bonatti, S. Crovisier. Récurrence et généricité. *Invent. Math.* 158 (2004), 33–104.

[BDP] C. Bonatti, L. Díaz, E. Pujals. A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Math.* 158 (2003), 355–418.

[BDV] C. Bonatti, L. Díaz, M. Viana. *Dynamics beyond uniform hyperbolicity.* Springer, 2005.

[BMVW] C. Bonatti, C. Matheus, M. Viana, A. Wilkinson. Abundance of stable ergodicity. *Comment. Math. Helv.* 79 (2004), 753–757.

[Br] M. Brin. Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Functional Anal. Appl.* 9 (1975), 8–16.

[BuDP] K. Burns, D. Dolgopyat, Y. Pesin. Partial hyperbolicity, Lyapunov exponents, and stable ergodicity. *J. Stat. Phys.* 109 (2002), 927–942.

[BW] K. Burns, A. Wilkinson. On the ergodicity of partially hyperbolic systems. To appear in *Annals of Math.*

[DW] D. Dolgopyat, A. Wilkinson. Stable accessibility is C^1 dense. Geometric Methods in Dynamics II, *Astérisque* 287 (2003), 33–60.

[G] N. Gourmelon. Adapted metrics for dominated splittings. *Ergod. Th. Dynam. Sys.* 27 (2007), 1839–1849.

[HPS] M. Hirsch, C. Pugh, M. Shub. Invariant manifolds. *Lecture Notes in Mathematics*, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[HT] V. Horita, A. Tahzibi. Partial hyperbolicity for symplectic diffeomorphisms. *Annales Inst. H. Poincaré – Analyse non linéaire* 23 (2006), 641–661.

[M] R. Mañé. An ergodic closing lemma. *Annals of Math.* 116 (1982), 503–540.

[P] K. Petersen. *Ergodic Theory*. Cambridge Univ. Press, 1989.

[OU] J. C. Oxtoby, S. M. Ulam. Measure-preserving homeomorphisms and metrical transitivity. *Annals of Math.* 42, (1941), 874–920.

[PS] C. Pugh, M. Shub. Stable ergodicity and julienne quasiconformality. *J. Eur. Math. Soc.* 2 (2000), 125–179.

[PSW] C. Pugh, M. Shub, A. Wilkinson. Hölder foliations. *Duke Math. J.* 86 (1997), 517–546.

[RRTU] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, R. Ures. A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms. *Electron. Res. Announc. Math. Sci.* 14 (2007), 74–81.

[RRU] F. Rodriguez Hertz, M. A. Rodriguez Hertz, R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle. To appear in *Invent. Math.*

[SX] R. Saghin, Z. Xia. Partial hyperbolicity or dense elliptic periodic points for C^1-generic symplectic diffeomorphisms. *Trans. Amer. Math. Soc.* 358 (2006), 5119–5138.

[SW] M. Shub, A. Wilkinson. Stably ergodic approximation: two examples. *Ergod. Th. Dynam. Sys.* 20 (2000), 875–893.

[W] A. Wilkinson. The cohomological equation for partially hyperbolic diffeomorphisms. Preprint.

[Z] E. Zehnder. Note on smoothing symplectic and volume preserving diffeomorphisms. *Lect. Notes in Math.* 597 (1977), 828–854.

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