PREVALENCE OF NON-LIPSCHITZ ANOSOV FOLIATIONS

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Abstract. We give sharp regularity results for the invariant subbundles of hyperbolic dynamical systems in terms of contraction and expansion rates and prove optimality in a strong sense: we construct open dense set of codimension one systems where this regularity is not exceeded. We furthermore exhibit open dense sets of symplectic, geodesic, and codimension one systems where the analogous regularity results of [PSW] are optimal. Then we produce *open sets* of symplectic Anosov diffeomorphisms and flows with low transverse Hölder regularity of the invariant foliations *almost everywhere.* Prevalence of low regularity of conjugacies is a corollary. We also establish a new connection between the transverse regularity of foliations and their tangent subbundles.

1. Introduction

A diffeomorphism f of a compact Riemannian manifold M is called *Anosov* if the tangent bundle splits (necessarily uniquely) as the sum of two Df-invariant subbundles $TM = E^u \oplus E^s$, and there exist constants C and $a < 1$ such that

$$\|Df^n(v)\| \leq Ca^n\|v\| \text{ and } \|Df^{-n}(u)\| \leq Ca^n\|u\| \quad \text{for } v \in E^s,\ u \in E^u,\ n \in \mathbb{N}.$$

This implies that for two nearby points either positive or negative iterates move apart exponentially fast. Any hyperbolic matrix $A \in SL(m, \mathbb{Z})$ induces an Anosov diffeomorphism of the m-torus $\mathbb{T}^m$: the bundles E^u and E^s are defined by the expanding and contracting root spaces of A. All examples on $\mathbb{T}^m$ are topologically conjugate to a linear automorphism. Anosov systems and hyperbolic sets have motivated ideas that have seen far-reaching developments in weakly and partially hyperbolic systems and in questions such as ergodicity of a gas of hard spheres.

Anosov studied these diffeomorphisms in [A] and showed that those that preserve volume are ergodic, meaning that the only f-invariant sets in M have full or zero measure. The central and hard fact used to prove this is absolute continuity of the invariant foliations (which we touch upon in Theorem 4). To establish this, Anosov showed that E^u and E^s, called the *unstable* and *stable* bundles, are Hölder continuous. For E^u this means that there exist $0 < \alpha \leq 1$ and $C, d > 0$ such that

$$d_G(E^u(p), E^u(q)) \leq C d_M(p, q)^\alpha \text{ whenever } d_M(p, q) \leq d,$$

where d_G is an appropriate metric on subbundles of TM. We say that E^u is C^α or α-Hölder; in case $\alpha = 1$ we say E^u is C^{Lip} or Lipschitz continuous.

In Anosov's argument, absolute continuity comes for free if the subbundles are Lipschitz. Since this is not always the case [A, p. 201] he proved absolute continuity

Date: February 16, 1998.

1991 *Mathematics Subject Classification.* Primary: 58F15; Secondary: 53C12.

Key words and phrases. Anosov system, hyperbolic system, invariant foliations, stable foliation, Anosov splitting, horospheric foliations, holonomy, Hölder structures, conjugacy.

in a way that does not even require an estimate of the exponent α [A2]. Since then estimates have been made for α, based on contraction and expansion rates of Df.

An optimal estimate is interesting for several reasons. E^u and E^s are invariants of a smooth system (f may be assumed C^∞ in this discussion) that show a marked lack of smoothness. The exponent α is a measure of how nonsmooth they are—if they are C^1 then α may always be chosen to be 1. (Conversely, when $\alpha = 1$ the subbundles are often C^1.) It can be related to geometric invariants such as Hausdorff dimension (see, e.g., [B, Corollary 3.19], where estimates of the form $\alpha d \leq$ dimension $\leq \alpha^{-1}d'$ for various dimensions of an Axiom A basic set appear that clearly improve for larger α). Recent work on stable ergodicity in partially hyperbolic systems [GPS, W, PS] initially used invariant distributions for which α is close to 1. Finally, the primary example of an Anosov flow φ^t (defined at the end of this section) is the geodesic flow on the unit tangent bundle of a compact Riemannian manifold with negative sectional curvature [A, §22], [KH, Theorem 17.6.2]. Its invariant subbundles are tangent to the horospheric foliations whose Hölder exponent bears directly on the regularity of the Busemann functions studied in [G] and the structure at infinity. Also, high smoothness characterizes locally symmetric metrics [K, BFL, BCG].

Estimates of α based on global information were made by Hirsch-Pugh-Shub [HPS]. (Earlier Fenichel [F] gave sharp estimates based on local data, but only for $r-1$ when E^u is C^r, $r \geq 1$.) Hasselblatt [H1] used local data to obtain better estimates of α (which, according to our Theorem 4 follow partly from the slightly later [SS]) and showed that these estimates were sharp on a large class of Anosov systems: They are not exceeded for a generic symplectic Anosov diffeomorphism or flow, or the geodesic flow of a generic negatively curved metric.

There is an issue closely related to the regularity of E^u and E^s. Tangent to E^u and E^s are foliations W^u, W^s with smooth leaves [A, Theorem 8], [F]. The regularity of these foliations can be measured by that of their tangent distributions, but there is another natural measure of regularity; namely, between two sufficiently close smooth disks transverse to the foliation, there is a natural locally-defined homeomorphism (determined by following the leaves) called the holonomy map. One can ask whether it is Hölder continuous, and if so, is there an exponent α that works for every pair of transversals chosen from a suitable compact family of transversals. Siegmund-Schulze–Schmeling and Pugh–Shub–Wilkinson [SS, PSW] showed that in fact these holonomy maps are Hölder, with the same predicted optimal exponent α as found for the corresponding tangent distributions in [H1]. The exponent was not shown to be optimal.

This paper aims to give optimal regularity results for the subbundles and foliations by estimating the Hölder exponent. Furthermore, we show not only that those estimates are sharp in the strongest possible sense, but that extreme nonsmoothness of these invariant structures is quite common, both in the manifold in question and in the space of Anosov systems. We incidentally make an analogous observation for topological conjugacies. Concretely, we realize three major aims:

- To sharpen the regularity results in [H1, H3] and extend the sharpness statement to a wider class of Anosov systems (Theorem 6).
- To show that the holonomy estimates in [PSW] are generically sharp (Theorem 6 via Proposition 14).

- To find *large sets* of Anosov systems on which the regularity of E^u, E^s, W^u *and* W^s is low *on a large set of points* (Theorems 1 and 2, Proposition 12).

We now elaborate on the last and main point. We say that a function $g\colon X \to Y$ between metric spaces is C^α *at* $x \in X$ if there exist $C, d > 0$ such that

$$d_Y(g(x), g(x')) \le C d_X(x, x')^\alpha \text{ whenever } d_X(x, x') \le d. \tag{1}$$

Then g is C^α *on a set* S if it is C^α at every $x \in S$. We say g is $C^{\alpha-}$ if it is C^β for every $\beta < \alpha$. Note that if g is continuous and S is dense, then g is C^α on S if and only if (1) holds for all $x, x' \in S$ (with C, d possibly depending on x). Thus, if

$$H^\alpha_{C,d} := \{p \in M \mid d_G(E^u(p), E^u(q)) \le C d_M(p,q)^\alpha \text{ if } d_M(p,q) \le d\}$$

then E^u is uniformly C^α if $M = H^\alpha_{C,d}$ for some C, d and E^u is almost nowhere C^α with respect to a measure μ if $\mu(H^\alpha_{C,d}) = 0$ for every C, d.

By [H1] there generically is a periodic point in M where α can not exceed the predicted value. This leaves open the possibility that a larger α might work on the rest of the manifold. By contrast, Anosov [A, p. 201] has an example of an Anosov diffeomorphism where E^u is almost nowhere $C^{2/3+\epsilon}$, for any $\epsilon > 0$. In this paper we construct examples, where, as in Anosov's examples, regularity is low on a large set, but whose perturbations also have low regularity on a large set. In particular, we answer the question "Are the invariant foliations generically Lipschitz continuous almost everywhere?" in the negative. Indeed, for any $\alpha \in (0,1)$ we construct an open set of (symplectic) Anosov systems whose foliations fail almost everywhere to be α-Hölder. The construction is elementary; it relies on a simple locally-defined obstruction (4). The main result is Proposition 12. For $k \in \mathbb{N} \cup \{\infty\}$ it implies

Theorem 1. *For $\alpha \in (0,1)$ there is a C^k-open set of symplectic Anosov flows and diffeomorphisms whose unstable* and *stable holonomies are almost nowhere C^α.*

Say that $S \subset M$ is *negligible* if $M \smallsetminus S$ is residual and $\mu(S) = 0$ for any ergodic invariant probability measure μ that is fully supported, i.e., positive on open sets.

Theorem 2. *For $\alpha \in (0,1)$ there is a C^k-open set of symplectic Anosov flows and diffeomorphisms for which E^u* and *E^s are C^α only on a negligible set.*

Many homotopy classes of Anosov systems contain such open sets.

Among our motivations was to point to potential difficulties in using an a.e. Lipschitz property to get absolute continuity of holonomies or feasible dimension leafwise calculations (as in [B]). While the preceding results demonstrate a failure of such regularity in an extremely strong sense there is good news for dimension calculations: These are most interesting for fractal hyperbolic sets, and in that case the situation is quite the opposite according to forthcoming work by Hasselblatt–Schmeling.

An *Anosov flow* $\varphi^t\colon M \to M$ is a fixed point free flow for which there exists a $D\varphi^t$-invariant splitting $TM = E^{su} \oplus E^{ss} \oplus \langle\dot\varphi\rangle$, where $D\varphi^t$ expands and contracts E^{su} and E^{ss}, respectively. Examples are suspensions of Anosov diffeomorphisms [KH, Section 0.3] and geodesic flows of negatively curved manifolds. We define *weak* stable and unstable bundles by $E^\ell := E^{s\ell} \oplus \langle\dot\varphi\rangle$. They are tangent to weak foliations W^ℓ, $\ell = s, u$ [A, Theorem 8]. In the proof of the above results we show low regularity of the weak unstable subbundle/foliation, which implies low regularity of the strong unstable one because the flow direction $\langle\dot\varphi\rangle$ is smooth. Thus, for flows, we need only discuss the weak subbundles and foliations.

A nice consequence of Theorem 1 was brought to our attention by John Franks. Anosov showed that Anosov diffeomorphisms are *structurally stable*: For any two sufficiently C^1-close Anosov diffeomorphisms f and g of M there is a (unique) homeomorphism $h\colon M \to M$ close to the identity such that $h \circ f = g \circ h$. It is uniformly bi-C^β for some $\beta > 0$ [KH, Theorem 19.1.2], i.e., there are $C, d > 0$ with

$$(1/C)d_M(x,x')^{1/\beta} \le d_M(h(x),h(x')) \le Cd_M(x,x')^\beta \text{ for } d_M(x,x') \le d.$$

Theorem 3. *For $\alpha \in (0,1)$ there is a linear symplectic Anosov diffeomorphism A and a C^k-neighborhood U of A of symplectic diffeomorphisms such that for an open dense set of $f \in U$ the conjugacy to A is almost nowhere bi-C^α.*

A foliation tangent to a Hölder continuous subbundle may not have Hölder continuous holonomy maps, even when the leaves are uniformly smooth [W]. This is closely related to the fact that a Hölder vector field is not always uniquely integrable; trajectories near non-unique ones can move apart arbitrarily rapidly. Surprisingly, perhaps, the converse implication holds; Hölder regularity of holonomy maps implies (essentially) the same regularity for the tangent subbundles:

Theorem 4. *If the holonomies of a foliation $\mathcal{F}$ of a Riemannian manifold M with uniformly C^{n+1} (C^∞) leaves are $C^{\alpha-}$ then $T\mathcal{F}$ is $C^{\alpha n/(n+1)-}$ ($C^{\alpha-}$).*

Suppose λ is a Borel probability measure on M whose $\mathcal{F}$-conditionals are absolutely continuous and such that for some family $\{D_x\}_{x\in M}$ of smooth transversals and for λ–almost every x, y with $y \in \mathcal{F}(x)$, the $\mathcal{F}$-holonomy map

$$h\colon D_x \to D_y, \quad h(p) = \mathcal{F}(p) \cap D_y$$

is absolutely continuous and $C^{\alpha-}$ a.e. (with respect to Riemannian volume on D_x). Then $T\mathcal{F}$ is $C^{\alpha n/(n+1)-}$ ($C^{\alpha-}$) λ-a.e.

This follows from the stronger Proposition 14, which is proved by an induction on derivatives along leaves. It is possible that there is a true difference in regularity between holonomy and tangent distribution when the leaves are not C^∞ but this question remains open.

Proof of Theorem 1 from Theorem 2. In an Anosov system with almost nowhere $C^{\alpha-\epsilon}$ unstable subbundle the invariant set where the holonomies are C^α cannot have full measure by Theorem 4, hence is a null set by ergodicity of volume. □

Theorem 4 can be similarly combined with work of Gerber and Niţică [GN] to give novel results about geodesic flows:

Theorem 5. *Suppose $k \in \mathbb{N}$ and S is a complete C^∞ surface of nonpositive curvature, γ a closed geodesic such that the curvature vanishes to order exactly $2k-1$ on γ. Then the horocycle foliation has Hölder exponent at most $k/k+1$. Even if the curvature is negative off γ and vanishes to infinite order on γ the horocycle foliation may not be Lipschitz.*

We now turn to the regularity result and its sharpness. If f is Anosov let $B^u(f) := \sup\{\inf_{p\in M}(\log\mu_s - \log\nu_s)/\log\mu_f \mid \mu_f < \mu_s < 1 < \nu_s,\ \mu_f^n\|v\|/C \le \|Df^n(v)\| \le C\mu_s^n\|v\|,\ \|Df^{-n}(u)\| \le C\nu_s^{-n}\|u\| \text{ for } v \in E^s(p),\ u \in E^u(p),\ n \in \mathbb{N}\}$. For example, for f symplectic $\nu_s\mu_s = 1$, so $B^u(f) = 2\sup\inf_p \log\mu_s(p)/\log\mu_f(p)$ is close to 2 if and only if the contraction rates μ_s and μ_f are close together.

Theorem 6. *Denote by C^r the space of $C^{\lfloor r\rfloor}$ maps whose $\lfloor r\rfloor$th derivatives have modulus of continuity $O(x^{r-\lfloor r\rfloor})$.*

1. *If* $B^u(f) \notin \mathbb{N}$ *then* $E^u \in C^{B^u(f)}$*; if* $B^u(f) \in \mathbb{N}$ *then* $E^u \in C^{B^u(f)-1,O(x|\log x|)}$.
2. *1 holds for flows and hyperbolic sets.*
3. *For an open dense set of symplectic diffeomorphisms and flows the regularity of* E^u *and* W^u *is at most that asserted in 1.*
4. *For an open dense set of diffeomorphisms and flows with* $\operatorname{codim}(E^s) = 1$ *the regularity of* E^u *and* W^u *is at most that asserted in 1.*
5. *Of the metrics on a compact manifold with sectional curvature* $\leq -k^2$ *and injectivity radius* $\geq \log 2/k$ *an open dense set has horospheric foliations (hence structure at infinity) of at most the regularity claimed in 1.*

Proof. 1 and 2 are in [H3]. 3 and 5 for E^u are in [H1] hence follow from Theorem 4. 4 is proved in the proof of Proposition 12. □

Theorem 6 encompasses some known results. For example, an area-preserving diffeomorphism or geodesic flow in dimension 2 has $C^{1,O(x|\log x|)}$ foliations [HK], a volume preserving codimension one diffeomorphism (i.e., $\operatorname{codim}(E^s) = 1$) has both foliations $C^{1+\epsilon}$ [A, p. 11], [H1, Corollary 1.9].

Our techniques for proving Theorems 2 and 1 also apply to geodesic flows and codimension one Anosov diffeomorphisms and flows (Proposition 12). They need widespread failure of bunching rather than failure at a periodic point (Theorem 6), i.e., that no orbit satisfies a given α-bunching condition (Definition 8). This can be checked for small perturbations of a linear (or algebraic) system. Theorems 1–3 are proved in Section 4 and Section 5 proves Theorem 4.

2. Adapted coordinates

In the examples of the next section, the leaves of W^u and W^s are themselves foliated invariantly. We say that $\{\mathcal{F}_i \mid i \in I \subset \{1,\dots,n\}\}$ is a *filtration* of foliations if the leaves of $\mathcal{F}_i$ are smooth and every leaf of $\mathcal{F}_i$ is foliated by leaves of $\mathcal{F}_j$ for $j \leq i$. In the cases we study, E^s (and similarly E^u) has Df-invariant subbundles E^s_i, $i \in I \subset \{1,\dots,n\}$ with $E^s_j \subset E^s_i$ and $\|Df_{\restriction_{E^s_j}}\| < \|Df_{\restriction_{E^s_i}}\|$ for $j \leq i$. This gives an f-invariant filtration $\{W^s_i \mid i \in I\}$ with W^s_i tangent to E^s_i [KH, Theorem 6.2.8]. The bundle $E^s_{\min I}$ ($W^{fs} := W^s_{\min I}$) is sometimes called the *fast-stable* distribution (foliation). Just as W^s has local foliation charts, a filtration $\{W^s_i\}$ has *adapted coordinates*, i.e., a family of smooth local coordinates depending continuously on the point and in which the filtration of the stable leaf is given by coordinate planes [KH, Lemma A.3.16]. When f is symplectic they can be taken symplectic:

Lemma 7. *If* (M,ω) *is a symplectic manifold of dimension* $2n$ *and* $f\colon M \to M$ *a symplectic Anosov diffeomorphism whose stable and unstable foliations admit global filtrations* $\{W^\ell_i \mid i \in I \subset \{1,\dots,n\}\}$ $(\ell = u,s)$ *then there is a continuous family* (h_x, U_x) *of smooth symplectic coordinates with respect to which*

$$W^s_i(x) \text{ is coordinatized by points } (0,\dots,0;0,\dots,0,q_{n-i+1},\dots,q_n)$$
$$W^u_i(x) \text{ is coordinatized by points } (0,\dots,0,p_{n-i+1},\dots,p_n;0,\dots,0).$$

Proof. Extend $\{W^\ell_i(x) \mid i \in I\}$ to isotropic filtrations $\{W^\ell_i(x) \mid 1 \leq i \leq n\}$, $\dim W^\ell_i(x) = i$ $(\ell = u,s)$. Take a submanifold $M_{n-1} \supset W^s_{n-1}(x) \cup W^u_{n-1}(x)$ of a neighborhood U_x of x and $p_1\colon U_x \to \mathbb{R}$ with $p_1(W^s(x) \cup M_{n-1}) = \{0\}$ and P_1 defined by $dp_1 = \omega(P_1,\cdot)$ transverse to a hypersurface $N_n \supset W^u(x) \cup M_{n-1}$. Let P^t_1 be the Hamiltonian flow of P_1 and q_1 such that $P_1^{-q_1(z)}(z) \in N_n$. This can be done continuously in x. Now $M_{n-1} = \{z \in U_x \mid p_1(z) = q_1(z) = 0\}$, $\{q_1, p_1\} = 0$, and $q_1(N_n) = \{0\}$. If $n > 1$ continue similarly in M_{n-1}, etc., to get adapted coordinates $\{p_i, q_i\}_{i=1}^n$. These are symplectic [Ar, 8.43E]. ☐

Adapted coordinates can be varied continuously with the Anosov diffeomorphism. For flows they consist of smooth coordinate systems on transversals, with continuous dependence on the point. In codimension one these are straightforward, for symplectic flows (in odd dimension, with a preserved form that restricts to symplectic forms on transversals) they are constructed as above [H1].

3. An obstruction to high regularity

Our main argument is based on [H1] where a generically false condition was found at periodic points where E^u is smoother than in Theorem 6. We show that if an orbit encounters a set of excessive Hölder regularity (i.e., where the optimal α exceeds expectations) then E^u satisfies a nongeneric condition along its fast stable leaf. This fails for fractal hyperbolic sets.

Adapted coordinates at $x \in M$ split as $\mathbb{R}^u \times \mathbb{R}^s$, with $\mathbb{R}^u \times \{0\}$ corresponding to $W^u(x)$ and $\{0\} \times \mathbb{R}^s$ to $W^s(x)$. Accordingly, the differential of f at $y \in W^s(x)$ is

$$D_y f = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{2}$$

because $W^s(x)$ is preserved by f. Represent $E^u(y)$ as the graph of a linear map $D\colon \mathbb{R}^u \to \mathbb{R}^s$ or the image of $\begin{pmatrix} I \\ D \end{pmatrix} : \mathbb{R}^u \to \mathbb{R}^u \times \mathbb{R}^s$. Then $D_y f(E^u)$ is the image of

$$D_y f \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A \\ B + CD \end{pmatrix} = \begin{pmatrix} I \\ (B + CD)A^{-1} \end{pmatrix} A.$$

The image of this map is unchanged when we reparametrize the preimage by A^{-1}, so by invariance the graph of $f^* D := (B+CD)A^{-1}$ is the unstable direction at $f(y)$. If z_i are the coordinates of $f^i(y)$ in the coordinate system at $f^i(x)$ we can thus write $D(z_1) = (B(z_0) + C(z_0)D(z_0))A(z_0)^{-1}$ or $D(z_{i+1}) = (B(z_i) + C(z_i)D(z_i))A(z_i)^{-1}$. To refine our analysis we now assume the following condition for some $\alpha \le 1$.

Definition 8. f is called *α-u-spread* if there are $E^{fs} \subseteq E^{ms} \subsetneq E^s$ and constants μ_f, μ_s, ν, with $\nu\mu_f^\alpha < \mu_s$, such that $\|Df^n_{\restriction_{E^{fs}}}\| < \text{cst}.\mu_f^n$, $\|Df^{-n}(v)\| < \text{cst}.\mu_s^{-n}\|v\|$ for $v \in E^s \setminus E^{ms}$, and $\|Df^{-n}_{\restriction_{E^u(x)}}\| > \text{cst}.\nu^{-n}$ for all x.

This means that the Mather spectrum has rings in the regions $\{|z| < \mu_f\}$ and $\{\mu_s < |z| < 1\}$ and a ring overlapping $\{1 < |z| < \nu\}$, and is an open condition.

Assume $\operatorname{codim}(E^s) = 1$ or f is symplectic (in which case we take $\nu = \mu_s^{-1}$). For $y \in W^{fs}_{\text{loc}}(x)$ (the local fast stable leaf of x defined by the adapted coordinate neighborhood) the square matrix C in (2) is lower block triangular. Denote the upper left $k \times k$ block corresponding to the complement of E^{ms} by c. In the symplectic case A, B, and D are of the same size as C and we denote by a, b, and d the corresponding blocks. If $\operatorname{codim}(E^s) = 1$ then $A =: a$ is scalar and B and D are column vectors whose top k entries define column vectors b and d. In either case

$$d(z_{i+1}) = \big(b(z_i) + c(z_i)d(z_i)\big)a(z_i)^{-1}. \tag{3}$$

(In the symplectic case we used that $A = C^{t^{-1}}$ is upper block triangular.) This decoupling is one of two central ideas of this proof: Df stretches horizontally and compresses vertically, making E^u closer to horizontal (smoothing) while the base point y of E^u moves closer to the orbit of x. High regularity of E^u along $W^s(x)$ results from smoothing fast while y approaches the orbit of x slowly. But $y \in W^{fs}_{\text{loc}}(x)$ approaches the orbit of x at the fastest rate, and (3) isolates the slowest smoothing action. So d represents those parts of E^u likely to have the lowest regularity. The other central ingredient is the existence of an obstruction: If

$$\xi^n_{z_0} := \prod_{i=0}^{n-1} c(z_{n-i-1}),\ \eta^n_{z_0} := \prod_{i=0}^{n-1} a(z_i)^{-1},\ \text{and}\ \Delta^n_{z_0} := -\sum_{i=0}^{n-1} (\xi^{i+1}_{z_0})^{-1} b(z_i) (\eta^i_{z_0})^{-1}$$

then $\|(\eta^i_{z_0})^{-1}\| \le \text{cst}.\nu^i$ (this is obvious for codimension one and follows from $a^{-1} = c$ in the symplectic case), $\|(\xi^i_{z_0})^{-1}\| \le \text{cst}.\mu_s^{-i}$, and $\|b(z_i)\| \le \text{cst}.\|z_i\| \le \text{cst}.\mu_f^i$. Now $\nu\mu_f^\alpha < \mu_s$ and $\alpha \le 1$, so $\mu_s^{-1}\mu_f\nu < 1$ and the continuous functions $\Delta^n_{z_0}$ converge uniformly. The limit $\Delta^f_{z_0}$ is then continuous in z_0 and f. Thus, the obstruction

$$O(x) := \sup\{\|d(z) - \Delta^f_z\| \mid z \in W^{fs}_{\text{loc}}(x)\}, \tag{4}$$

where we use any coordinate norm, is continuous in x and f.

Lemma 9. *For α-u-spread symplectic or codimension one Anosov diffeomorphisms O is continuous in x and f. $O(x) = 0$ if $f^n(x) \in H^\alpha_{C,d}$ for infinitely many $n \in \mathbb{N}$.*

Proof. Iterating (3) gives $d(z_n) = \xi^n_{z_0} \cdot (d(z_0) - \Delta^n_{z_0}) \cdot \eta^n_{z_0}$ and if $f^n(x) \in H^\alpha_{C,d}$ then $\|d(z_0) - \Delta^n_{z_0}\| = \|(\xi^n_{z_0})^{-1} d(z_n)(\eta^n_{z_0})^{-1}\| \le \text{cst}.\mu_s^{-n}\mu_f^{\alpha n}\nu^n = \text{cst}.(\mu_s^{-1}\mu_f^\alpha\nu)^n$. ☐

Proposition 10. *For an α-u-spread symplectic or codimension one Anosov diffeomorphism $H^\alpha := \bigcup_{C,d} H^\alpha_{C,d}$ is negligible if $O \ne 0$.*

Proof. Otherwise either some $H^\alpha_{C,d}$ has nonempty interior and $O = 0$ on any dense orbit. Or $\mu(H^\alpha) > 0$ for a fully supported ergodic invariant probability measure, so $\mu(H^\alpha_{C,d}) > 0$ for some C, d and almost every positive semiorbit encounters $H^\alpha_{C,d}$ infinitely many times. Then $O = 0$ a.e., i.e., on a dense set. ☐

For flows the same arguments work and the calculations are identical. Consequently

Proposition 11. *Lemma 9 and Proposition 10 hold for flows.*

4. Construction of examples

Proposition 12. *In a small neighborhood of an α-u-spread symplectic or codimension one Anosov diffeomorphism or flow the systems whose unstable subbundle is C^α only on a negligible set are C^1-open C^k-dense.*

If a metric of negative curvature on a compact manifold has α-u-spread geodesic flow, then in a small neighborhood of this metric there is a C^3-open C^{k+2}-dense set of metrics whose horospheric subbundles are C^α on a negligible set only.

Proof. In [H1] symplectic diffeomorphisms and flows and geodesic flows are perturbed so that $O \ne 0$. (For geodesic flows one can drop the condition on the injectivity radius in [H1] which yields a "negatively nonreturning" point on the fast stable leaf of a periodic point; we instead pick a point whose fast stable manifold contains a point heteroclinic to different periodic points. It never returns to a small neighborhood on the base manifold, so the perturbation in [H1] has the desired effect.) Suppose $\operatorname{codim}(E^s) = 1$. For an α-u-spread periodic point x take $y \in W^{fs}_{\mathrm{loc}}(x)$ with a neighborhood U such that $U \cap \{f^{-n}(y) \mid n \in \mathbb{N}\} = \varnothing$, $f^n(y) \notin U$ for $n \in \mathbb{N}$, and $x \notin U$ [H1, Proposition 4.1]. Let J be a perturbation of the identity on M supported in $U \setminus \{y\}$ such that $D_y J = I + \epsilon e$ in adapted coordinates for x, where the only nonzero entry of e is $e_{21} = 1$. Then $J \circ f$ has x periodic and $y \in W^{fs}_{\mathrm{loc}}(x)$ not returning to U with unstable subspace $J\begin{pmatrix}1\\D\end{pmatrix} = \begin{pmatrix}1\\D\end{pmatrix} + (0, \epsilon, 0, \dots, 0)^t$, so $d(y) \neq \Delta^f_y = \Delta^{J\circ f}_y$. This is stable, hence proves 4 of Theorem 6. Clearly $O(x) \neq 0$. ☐

Proof of Theorem 2. The matrix $A_\alpha := \begin{pmatrix}B & 0\\ 0 & B^{\lfloor 2/\alpha \rfloor + 1}\end{pmatrix}$, where $B = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$, induces a symplectic Anosov automorphism of $\mathbb{T}^4$. Both A_α and A_α^{-1} are α-u-spread. Proposition 12 gives Theorem 2 for E^u or E^s and the intersection of open sets is open. Suspending these examples proves the result for flows. ☐

Proof of Theorem 3. Take $A := A_{\alpha^2}$ as above, f symplectic Anosov C^1-near A. If the conjugacy h is bi-C^α at x then so is $h \circ f = A \circ h$. Thus the set U where h is bi-C^α is f-invariant. Suppose it has positive, hence full measure. For $x \in U$ and $y \in W^u_{\mathrm{loc}}(x) \cap U$ the unstable holonomy $\pi^u \colon W^s_{\mathrm{loc}}(x) \to W^s_{\mathrm{loc}}(y)$ is C^{α^2} at x because

$$\begin{aligned} d(\pi^u(x), \pi^u(x')) &\leq C d(h(\pi^u(x)), h(\pi^u(x')))^\alpha \\ &= C d(\rho^u(h(x)), \rho^u(h(x')))^\alpha \leq C' d(h(x), h(x'))^\alpha \leq C'' d(x, x')^{\alpha^2}, \end{aligned}$$

where $\rho^u \colon W^s_{\mathrm{loc}}(A, h(x)) \to W^s_{\mathrm{loc}}(A, h(y))$ is the unstable holonomy for A. Now for

$$x \in U' := \{x \in U \mid W^s_{\mathrm{loc}}(x) \text{ is essentially } U\text{-saturated}\}$$

with $W^u_{\mathrm{loc}}(x)$ essentially U'-saturated, a.e. $y \in W^u_{\mathrm{loc}}(x)$, $z \in W^s_{\mathrm{loc}}(x)$ we have z, $\pi^u(z) \in U$, so the unstable holonomy for f is C^{α^2} a.e. Now use Proposition 12. ☐

No α-u-spread geodesic flow is known to us because it is hard to control contraction and expansion rates through curvature information, except in the opposite direction to get bunching from curvature pinching [H2]. The closest are nonconstantly curved locally symmetric metrics, whose geodesic flow is $1+\epsilon$-u-spread for every $\epsilon > 0$, but it is not clear whether they can be perturbed to give a 1-u-spread geodesic flow. Our Arguments do include implicitly a construction corresponding to failure of $C^{1+\alpha}$ regularity of the foliations (see [H1]). This implies

Proposition 13. *For any $\epsilon > 0$ and any nonconstantly curved locally symmetric metric (on a compact manifold) there is a C^∞ neighborhood in which metrics whose horospheric subbundles are $C^{1+\epsilon}$ on a negligible set only are C^3-open C^{k+2}-dense.*

Perturbations of volume-preserving, in particular linear, codimension one flows and diffeomorphisms have C^1 subbundles [A, p. 11], hence are not examples.

5. A connection between holonomy and bundle regularity

In [PSW, SS] it is shown that the unstable holonomy maps are uniformly C^α if the local contraction and expansion rates of a diffeomorphism satisfy the pointwise bunching condition $\nu_s(p)\mu_f(p)^\alpha > \mu_s(p)$, where

$$\|Df^{-n}(u)\| \leq C\nu_s^{-n}(p)\|u\| \quad \text{and} \quad \mu_f^n(p)\|v\|/C \leq \|Df^n(v)\| \leq C\mu_s^n(p)\|v\|$$

for $v \in E^s(p), u \in E^u(p)$ and $n \in \mathbb{N}$. This complements the corresponding earlier result for the unstable subbundle in [H1]. To prove sharpness (Theorem 6) and the non-C^α result for unstable holonomies (Theorem 1) we now prove Theorem 4.

Since the question is local and invariant under smooth coordinate changes, we may formulate the problem so that the manifold is $\mathbb{R}^{u+s} = \mathbb{R}^u \times \mathbb{R}^s$ and the leaf of $\mathcal{F}$ through $(0, y) \in \{0\} \times \mathbb{R}^s$ is the graph of a smooth function $g_y : \mathbb{R}^u \to \mathbb{R}^s$. Uniform smoothness of the leaves of $\mathcal{F}$ is equivalent to $g(x, y) := g_y(x) : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^s$ being C^k in the first u coordinates with all kth derivatives $\partial^k g/\partial x^k$ uniformly continuous in the last s coordinates. $\mathcal{F}$ has C^α holonomy if and only if $g(x, \cdot)$ is C^α for every $x \in \mathbb{R}^u$. One statement holds uniformly, or λ-a.e., if and only if the other does, since the (x, y) coordinate system is smooth. Similarly, $T\mathcal{F}$ is (a.e.) C^α if and only if $\partial g/\partial v$ is (a.e.) C^α for every $v \in \mathbb{R}^u$. Thus Theorem 4 boils down to:

Proposition 14. *If $g : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^s$ is $C^{\alpha-}$ (λ-a.e., λ as in Theorem 4), and $\partial^k g/\partial v^k$ is uniformly continuous for all $v \in \mathbb{R}^u$ and $1 \le k \le n+1$ then $\partial^k g/\partial v^k$ is $C^{\alpha(n+1-k)/(n+1)-}$ (λ-a.e.) for $0 \le k \le n$.*

Restricting to the coordinate functions of g and taking partial derivatives allows us to further assume that $u = s = 1$. The uniform version of Proposition 14 in the case $u = s = 1$ is Proposition 15; after proving it, we turn to the almost everywhere version, Proposition 16, below. Proposition 15 can be viewed as an extension of some of the results in [J, LMM] to the case where the function is less than differentiable in one of the variables. To simplify notation write $F_x^{(k)} := \partial^k F/\partial x^k$.

Proposition 15. *Suppose $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^{α_0-}, and $F_x^{(k)}$ is uniformly continuous in both variables for $1 \le k \le n+1$. Then $F_x^{(k)}$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ for $0 \le k \le n$. If $n = \infty$ then $F_x^{(k)}$ is C^{α_0-} for all k.*

Proof. The case $n = 1$ is illustrative: subtracting $F(x, 0) + xF_x^{(1)}(x, 0)$ from F we may assume $F(x, 0) = F_x^{(1)}(x, 0) = 0$ for all x. Expand $F(\cdot, y)$ about $(0, 0)$ to obtain

$$F(\epsilon, y) - F(0, y) = \epsilon F_x^{(1)}(0, y) + \epsilon^2 M(\epsilon, y)$$

with $M(\epsilon, y)$ uniformly continuous in both variables. Fix $\alpha < \alpha_0$ and $\epsilon = \epsilon(y) = |y|^{\alpha/2}$. Divide by $|y|^\alpha$ to get

$$\frac{F(\epsilon, y)}{|y|^\alpha} - \frac{F(0, y)}{|y|^\alpha} = \frac{F_x^{(1)}(0, y)}{|y|^{\alpha/2}} + M(\epsilon, y).$$

The left-hand side is uniformly bounded for $|y|$ near 0 because F is C^α. $M(\epsilon, y)$ is bounded in any bounded region of $\mathbb{R}^2$, and so $F_x^{(1)}(0, y)/|y|^{\alpha/2}$ is bounded for all y in a bounded region of $\mathbb{R}^2$. Since $\alpha < \alpha_0$ is arbitrary, this implies that $F_x^{(1)}$ is $C^{\alpha_0/2-}$ at $(0, 0)$. The origin was arbitrary so $\partial F/\partial x$ is $C^{\alpha_0/2-}$ everywhere.

Note that without changing the left-hand side choosing $\epsilon = |y|^{\alpha/3}$, say, gives a more promising first term on the right-hand side above, but, alas, an unbounded error term $M(\epsilon, y)/\epsilon$. This suggests controlling second derivatives to get a higher order of ϵ in $M(\epsilon, y)$. Control of these will, in turn, be improved by control of third derivatives, etc. Thus, we use induction on n:

Suppose the assertion holds for n; that is, if $\beta > 0$ and G is $C^{\beta-}$ with $G_x^{(k)}$ continuous for $1 \le k \le n+1$ then $G_x^{(k)}$ is $C^{\beta(n+1-k)/(n+1)-}$. We show this implies the assertion for $n+1$, i.e., if $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^{α_0-} with $F_x^{(k)}$ continuous for

$1 \le k \le n+2$ then $F_x^{(k)}(x_0, \cdot)$ is $C^{\alpha(n+2-k)/(n+2)-}$ at every (x_0, y_0). We may assume $x_0 = y_0 = 0$ and $F_x^{(i)}(\cdot, 0) = 0$ for $i \le n+1$.

For $0 \le d \le \dfrac{n+1}{n+2}$ let $g(d) := \dfrac{1}{2}\left(1 + \dfrac{nd}{n+1}\right)$ and $h(d) := 1 - g(d)$. Then

$$1 - kh(d) \le d\,\frac{n+2-k}{n+1} \text{ for } k \ge 2 \tag{5}$$

because both sides are linear in k with equality for $k = 2$ and reversed inequality for $k = 0$. Now $F(\,\cdot\,, y)$ is C^{n+2}, so $F_x^{(1)}$ is $C^{\alpha_0 n/(n+1)-}$ by the induction hypothesis.

Claim 1. *If $F_x^{(1)}$ is $C^{d\alpha_0-}$, with $0 \le d \le (n+1)/(n+2)$, then $F_x^{(1)}$ is $C^{g(d)\alpha_0-}$.*

Proof. $F(\,\cdot\,, y)$ is C^{n+2}, so for all ϵ near 0

$$\frac{F(\epsilon, y)}{|y|^\alpha} - \frac{F(0,y)}{|y|^\alpha} = \frac{\epsilon F_x^{(1)}(0,y)}{|y|^\alpha} + \cdots + \frac{\epsilon^{n+1} F_x^{(n+1)}(0,y)}{(n+1)!|y|^\alpha} + \frac{\epsilon^{n+2} M(\epsilon, y)}{|y|^\alpha}, \tag{6}$$

where $M(\epsilon, y)$ is bounded uniformly in ϵ and y. Take $\alpha < \alpha_0$. Then the left-hand side is bounded for all ϵ and y because F is C^α at $(\epsilon, 0)$ and $F(\epsilon, 0) = 0$ for all ϵ. If $F_x^{(1)}$ is $C^{d\alpha_0-}$ then the induction hypothesis (applied to $F_x(\,\cdot\,, y)$, which is C^{n+1}) implies that $F_x^{(k)}$ is $C^{\beta-}$, for $k \ge 2$, where

$$\beta = (d\alpha_0)\left(\frac{n+2-k}{n+1}\right) \ge \alpha(1 - kh(d))$$

by (5). Let $\epsilon = \epsilon(y) := |y|^{h(d)\alpha}$. Then the kth term on the right-hand side of (6) is

$$\frac{F_x^{(k)}(0,y)}{k!|y|^{\alpha(1-kh(d))}} = \frac{F_x^{(k)}(0,y) - F_x^{(k)}(0,0)}{k!|y|^{\alpha(1-kh(d))}},$$

hence bounded for $2 \le k \le n+1$, since $F_x^{(k)}$ is $C^{\alpha(1-kh(d))}$. The last term $M(\epsilon, y)|y|^{-\alpha(1-(n+2)h(d))}$ is bounded because $1 - (n+2)h(d) \le 0$ by (5). Since all other terms in (6) are bounded as $|y| \to 0$, so is the term

$$\frac{F_x^{(1)}(0,y)}{|y|^{\alpha(1-h(d))}} = \frac{F_x^{(1)}(0,y) - F_x^{(1)}(0,0)}{|y|^{\alpha g(d)}}.$$

Hence $F_x^{(1)}$ is $C^{\alpha g(d)}$ at $(0,0)$, for all $\alpha < \alpha_0$. The point $(0,0)$ was arbitrary, so $F_x^{(1)}$ is $C^{\alpha_0 g(d)-}$. This proves Claim 1. $\square$

Since $F_x^{(1)}$ is $C^{\alpha_0 n/(n+1)-}$ Claim 1 iteratively shows that $F_x^{(1)}$ is $C^{\alpha_0 g^m(n/(n+1))-}$ for all $m \in \mathbb{N}$. The contraction g has fixed point $g(d_0) = d_0 = (n+1)/(n+2)$, so $F_x^{(1)}$ is $C^{\alpha_0 d_0-}$ and $F_x^{(k)}$ is $C^{\alpha_0 d_0(n+2-k)/(n+1)-} = C^{\alpha_0(n+2-k)/(n+2)-}$. $\square$

Remark. To see that Proposition 15 is sharp consider the function defined by $F(\cdot, 0) = 0$ and $F(x,y) = |y|^{\alpha_0} \sin(x \cdot |y|^{-\alpha_0/(n+1)})$ for $y \ne 0$.

At the heart of the proof of Proposition 15 is the choice of $\epsilon = \epsilon(y) = |y|^{\alpha h(d)}$ in Claim 1. On the one hand, $\epsilon(y)$ is chosen small relative to $|y|$ so that the terms $F_x^{(k)}(0,y)|y|^{-\alpha(1-kh(d))}$ and $M(\epsilon, y)|y|^{-\alpha(1-(n+2)h(d))}$ are uniformly bounded in y. On the other hand, we use that $F(\epsilon, y)/|y|^\alpha$ is uniformly bounded. When F is uniformly C^α, this is automatic; otherwise $\epsilon(y)$ must be chosen more carefully. If, for example, the Hölder constant of F at $(\epsilon(y),0)$ is unbounded as $y\to 0$, then so is $F(\epsilon,y)/|y|^\alpha$. So suppose F is C^{α_0-} λ-almost everywhere; that is, for $\alpha<\alpha_0$ and

$$B^\alpha_m := \{p\in\mathbb{R}^2 \mid d(p,q)\le 1 \implies |F(p)-F(q)|\le md(p,q)^\alpha\}$$

we have $\mathbb{R}^2=\bigcup_{m=1}^\infty B^\alpha_m$ a.e. Define the set of *$\mathcal{F}$-density points* of $A\subset\mathbb{R}^2$ by

$$\langle A\rangle := \Big\{(x,y)\in A \mid \lim_{r\to 0}\frac{\lambda_y\big(A\cap \operatorname{graph} F(\cdot,y)\restriction_{[x-r,x+r]}\big)}{\lambda_y\big(\operatorname{graph} F(\cdot,y)\restriction_{[x-r,x+r]}\big)}=1\Big\},$$

where λ_y is the conditional measure. By [LY, Lemma 4.1.2] and absolute continuity of the foliation $\lambda(A\smallsetminus\langle A\rangle)=0$, so for $\alpha<\alpha_0$ almost every (x,y) is in some $\langle B^\alpha_N\rangle$. We avoid the above problems by assuming that $(0,0)$ is in some $\langle B^\zeta_N\rangle$, and making a more delicate choice of $\epsilon(y)$ to bound the other terms. This gives

Proposition 16. *Suppose $F\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is C^{α_0-} λ-a.e., and $F^{(k)}_x$ is C^0 for $1\le k\le n+1$. Then $F^{(k)}_x$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ a.e. for $k\le n$. If $n=\infty$ then $F^{(k)}_x$ is C^{α_0-} a.e. for $k\in\mathbb{N}$.*

Proof. First consider $n=1$. Fix $\alpha<\alpha_0$ and $\delta<(\alpha_0-\alpha)/2$. Suppose $(0,0)$ is in some $\langle B^{\alpha+\delta}_N\rangle$ and divide $F(\epsilon,y)-F(0,y)=\epsilon F^{(1)}_x(0,y)+\epsilon^2M(\epsilon,y)$ by $|y|^{\alpha+\delta}$ to get

$$\frac{F(\epsilon,y)}{|y|^{\alpha+\delta}}-\frac{F(0,y)}{|y|^{\alpha+\delta}}-\frac{\epsilon^2}{|y|^{\alpha+\delta}}M(\epsilon,y)=\frac{\epsilon F^{(1)}_x(0,y)}{|y|^{\alpha+\delta}}.$$

The density of $[-|y|^{(\alpha+2\delta)/2},|y|^{(\alpha+2\delta)/2}]\times\{0\}$ in $[-|y|^{(\alpha+\delta)/2},|y|^{(\alpha+\delta)/2}]\times\{0\}$ is bounded away from 1 as $|y|\to 0$ (by absolute continuity of conditionals) and $(0,0)\in\langle B^{\alpha+\delta}_N\rangle$, so the density of $B^{\alpha+\delta}_N$ in the same interval approaches 1; thus we can choose $(\epsilon,0)\in B^{\alpha+\delta}_N$ such that $|y|^{(\alpha+2\delta)/2}<|\epsilon|<|y|^{(\alpha+\delta)/2}$. Then the left-hand side is bounded for $|y|\ne 0$, and hence so is the right-hand side, which bounds $F^{(1)}_x(0,y)/|y|^{\alpha/2}$. So $F^{(1)}_x$ is $C^{\alpha/2}$ at $(0,0)$. The point $(0,0)$ was essentially arbitrary, so $F^{(1)}_x$ is $C^{\alpha/2}$ a.e. for every $\alpha<\alpha_0$, hence $C^{\alpha_0/2-}$ a.e.

Assume the statement holds for every function that is C^{α_0-} a.e. and C^{n+1} in x. Suppose F is C^{n+2} in x, and $F^{(k)}_x$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ a.e. for $k\le n+1$. We show $F^{(k)}_x$ is $C^{\alpha_0(n+2-k)/(n+2)-}$ a.e. for $k\le n+2$. Claim 1 becomes

Claim 2. *If $\dfrac{n}{n+1}\le d<\dfrac{n+1}{n+2}$ and $F^{(1)}_x$ is $C^{d\alpha_0-}$ a.e. then $F^{(1)}_x$ is $C^{g(d)\alpha_0-}$ a.e.*

Proof. The bounds on d imply that $1/(n+2)<h(d)\le(2n+1)/(2(n+1)^2)$. Choose $\alpha,\beta<\alpha_0$ such that $0<\beta-\alpha<\alpha(n+1)\big(1-(1/(n+2)h(d))\big)$. Then

$$0<\alpha-\frac{\beta-\alpha}{n+1}<\gamma:=\alpha-(\beta-\alpha)\left(\frac{1}{(n+1)h(d)}-1\right)\le\alpha-\frac{\beta-\alpha}{2n+1}<\alpha.$$

In particular $\alpha-\gamma<(\beta-\alpha)/(n+1)<\alpha\big(1-(1/(n+2)h(d))\big)$, so

$$\alpha-(n+2)\gamma h(d)=\alpha(1-(n+2)h(d))+(\alpha-\gamma)(n+2)h(d)<0. \tag{7}$$

For $k\le n+1$ the definition of γ implies

$$\alpha-k\gamma h(d)=\alpha+(\beta-\alpha)\frac{k}{n+1}-k\beta h(d)\le\beta-k\beta h(d). \tag{8}$$

Let G_N be the set where $F^{(k)}_x$ is $C^{d\beta(n+2-k)/(n+1)-}$ with constant N for $k\le n+1$. By the inductive hypothesis applied to $F^{(1)}_x$, which is C^{n+1} in x, almost every (x,y)

is in some $\langle G_N\rangle$. Assume $(0,0)\in\langle G_N\rangle$. The density of $[-|y|^{\alpha h(d)},|y|^{\alpha h(d)}]\times\{0\}$ in $[-|y|^{\gamma h(d)},|y|^{\gamma h(d)}]\times\{0\}$ is bounded away from 1 as $|y|\to 0$, so

$$\lambda_0\left(\left([-|y|^{\gamma h(d)},-|y|^{\alpha h(d)}]\cup[|y|^{\alpha h(d)},|y|^{\gamma h(d)}]\right)\cap G_N\right)>0$$

and we can take $(\epsilon,0)\in G_N$ with $|y|^{\alpha h(d)}<|\epsilon|<|y|^{\gamma h(d)}$. By (8) we find that in

$$\frac{F(\epsilon,y)}{|y|^\alpha}-\frac{F(0,y)}{|y|^\alpha}=\frac{\epsilon F_x^{(1)}(0,y)}{|y|^\alpha}+\cdots+\frac{\epsilon^{n+1}F_x^{(n+1)}(0,y)}{(n+1)!|y|^\alpha}+\frac{\epsilon^{n+2}M(\epsilon,y)}{|y|^\alpha}$$

the right-hand terms

$$\frac{\epsilon^k F_x^{(k)}(0,y)}{k!|y|^\alpha}\le\frac{F_x^{(k)}(0,y)}{k!|y|^{\alpha-k\gamma h(d)}}\le\frac{F_x^{(k)}(0,y)}{k!|y|^{\beta(1-kh(d))}}\text{ and }\frac{\epsilon^{n+2}M(\epsilon,y)}{|y|^\alpha}\le\frac{M(\epsilon,y)}{|y|^{\alpha-(n+2)\gamma h(d)}}$$

are bounded in y using (7) and that $F_x^{(k)}$ is $C^{\beta(1-kh(d))}$ at $(0,0)$ with constant N for $2\le k\le n+1$ by (5). Since $(\epsilon,0)$ and $(0,0)$ are in $G_N\subset B_N^\beta\subset B_N^\alpha$, the left-hand terms $F(0,y)/|y|^\alpha$ and $F(\epsilon,y)/|y|^\alpha$ are bounded. Hence the remaining term

$$\frac{\epsilon F_x^{(1)}(0,y)}{|y|^\alpha}>\frac{F_x^{(1)}(0,y)}{|y|^{\alpha-\alpha h(d)}}=\frac{F_x^{(1)}(0,y)}{|y|^{g(d)\alpha}},$$

is bounded, and $F_x^{(1)}$ is $C^{g(d)\alpha}$ at $(0,0)$ for $\alpha<\alpha_0$. Since $(0,0)$ is essentially arbitrary, this proves Claim 2. □

Iteratively $F_x^{(1)}$ is $C^{\alpha_0 g^m(n/(n+1))-}$ a.e. for $m\in\mathbb{N}$, hence $C^{\alpha_0(n+1)/(n+2)-}$ a.e. □

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