Stable ergodicity and Anosov flows

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Abstract

In this note we prove that if M is a 3-manifold and $\varphi_t : M \to M$ is a C^2, volume-preserving Anosov flow, then the time-1 map φ_1 is stably ergodic if and only if φ_t is not a suspension of an Anosov diffeomorphism.

1 Introduction

A volume-preserving diffeomorphism is *stably ergodic* if it and all sufficiently C^2-close volume-preserving diffeomorphisms are ergodic. Until recently, the only known examples were hyperbolic — namely Anosov diffeomorphisms [1]. Grayson, Pugh and Shub found the first example of a nonhyperbolic stably ergodic diffeomorphism. They proved in [4] that if S is a surface of constant negative curvature and φ_t is the geodesic flow on the unit tangent bundle of S, then the time-1 map φ_1 is stably ergodic (with respect to Liouville measure).

Wilkinson later generalized this result to the case where S has variable negative curvature [14]. Pugh and Shub proved it for higher dimensional manifolds of constant, or nearly constant, negative curvature [11]. In all of these examples, φ_1 is the time-1 map of an Anosov flow. It is not true, however, that every volume-preserving Anosov flow has a stably ergodic time-1 map: the time-1 map for a suspension of an Anosov diffeomorphism is not even ergodic. More generally the time-t_0 map of a special flow under a constant height function with value h_0 cannot be ergodic if t_0 and h_0 are rationally related. It follows that the time-1 map for a special flow constructed from an Anosov diffeomorphism under a constant height function is never stably ergodic (although it will be ergodic if the suspension height is irrational and the diffeomorphism is volume preserving).

This paper shows that, in dimension 3 at least, such special flows give the *only* examples of volume preserving Anosov flows whose time-1 maps are not stably ergodic. We prove:

Theorem 1.1 *Let $\varphi_t : M \to M$ be a C^2, volume-preserving topologically mixing Anosov flow on a compact 3-manifold. Then the time-1 map φ_1 is stably ergodic. More precisely, any C^2 volume-preserving diffeomorphism that is close enough to φ_1 in the C^1 topology is ergodic.*

Since Plante [9] showed that the only non-mixing codimension 1 Anosov flows are special flows under constant height functions, we obtain:

Corollary 1.2 *Let $\varphi_t : M \to M$ be a C^2, volume-preserving Anosov flow on a compact 3-manifold. The time-1 map φ_1 is stably ergodic if and only if the flow ϕ_t is not a special flow under a constant height function.*

Thus

virtually all volume-preserving 3-dimensional Anosov flows have stably ergodic time-1 map.

Theorem 1.1 is a consequence of a theorem of Pugh and Shub [11], which we state in section 2, results of Plante [9], and the following theorem.

Theorem 1.3 *Let φ_t be a C^1 Anosov flow. Suppose that the stable and unstable foliations for the time-1 map φ_1 are not jointly integrable. Then any C^1 diffeomorphism g that is close enough to φ_1 in the C^1 topology has the property that any two points can be joined by a u, s-path.*

A u, s-path is a path that is a concatenation of a finite number of arcs each of which lies in a single leaf of either the stable foliation or the unstable foliation for g. Joint integrability was defined by Plante in [9]. The stable and unstable foliations for an Anosov flow φ_t are jointly integrable if the holonomy along unstable leaves between weak stable manifolds carries stable leaves to stable leaves.

Theorem 1.3 was proved by Katok and Kononenko ([7], Proposition 5.2) in the case of a contact Anosov flow. Their result motivated the present paper and their argument is an important step in our proof; see Proposition 3.4 and the subsequent remarks. We would like to thank Viorel Niţică for explaining this argument to us.

Section 3 contains the proof of a slightly generalized version of Theorem 1.3. In Section 4 we give the proof of Theorem 1.1 and also discuss what can be said for Anosov flows in higher dimensions. The difficulty in higher dimensions is that it is not known whether a topologically mixing flow must satisfy the hypothesis of Theorem 1.3. We do, however, have the following result, whose proof is outlined at the end of Section 4.

Theorem 1.4 *Let φ_t be a C^2, volume preserving Anosov flow on a compact manifold. Suppose that the stable and unstable foliations for the time-1 map φ_1 are not jointly integrable. Then φ_1 is stably ergodic.*

It is natural to consider stability of stronger stochastic properties than ergodicity. Anosov showed in [1] that the time-1 map of any C^2 volume-preserving topologically mixing Anosov flow is a K-system, which means that these maps have no factors of zero entropy and implies that they are mixing of all orders. These flows are also Bernoulli, as was shown by Bunimovich [3] and Ratner [13]. The techniques in [1], [3] and [13] do not extend to perturbations of φ_1, as much of the structure and regularity is lost when φ_1 is perturbed.

Some previously developed machinery can nonetheless be brought to bear on this problem. Brin and Pesin showed in [2] that if a partially hyperbolic system has ergodic stable and unstable foliations, then it is in fact a K-system. The proof of the main result that we use, Theorem 2.1 below, shows that the systems we consider have ergodic stable and unstable foliations. Thus we have

Corollary 1.5 *Let $\varphi_t : M \to M$ be a volume-preserving topologically mixing Anosov flow on a compact 3-manifold. The time-1 map φ_1 is stably a K-system.*

We do not know whether Corollary 1.5 is true when "K-system" is replaced by "Bernoulli." The arguments for showing that a partition is very weak Bernoulli in [8] break down when the restriction of the dynamical system to the center foliation exhibits any nontrivial dynamical behavior. When φ_1 is perturbed, sources and sinks, among other things, can develop on the new center foliation.

2 Structure of the Proof

We take the approach of a recent paper by Pugh and Shub [11]. The general structure of their argument goes back to Hopf [5] and it shares some features with the arguments in a paper of Brin and Pesin [2]. It applies to many diffeomorphisms that have stable and unstable foliations.

Here is a rough sketch of the argument: the leaves of the stable and unstable foliations of the diffeomorphism stratify, respectively, the asymptotic past and future behavior of its iterates. The Birkhoff Ergodic Theorem implies that the past and future behavior are, in a measure-theoretic sense, the same.

Given a point p, the stable and unstable manifolds through p thus have the same asymptotic behavior. In fact all points that are connected to p by a chain of stable and unstable manifold segments have the same asymptotic behavior. If all points have the same asymptotic behavior, then the diffeomorphism is ergodic.

When the stable and unstable manifolds are transverse, as with an Anosov diffeomorphism, the density of a single unstable leaf implies that all points can be connected to p by such a chain. When they are not transverse, which is our situation, we must prove there is always such a chain. For the time-1 map of the suspension of an Anosov diffeomorphism, such chains only exist for pairs of points that are at the same height above the base. This paper is devoted to proving that chains always exist in the non-suspension case, a property we call u, s-transitivity.

The assertions in the argument above are really only correct almost everywhere, a fact which introduces serious technical complications. This is one of the reasons it was 30 years from the time Hopf proved that the geodesic flow of a hyperbolic surface is ergodic before Anosov generalized the result to higher dimensional manifolds with variable negative curvature. This also accounts for several of the technical hypotheses in Theorem 2.1 below.

Let φ_1 be the time-1 map of a C^2 volume-preserving topologically mixing Anosov flow and let g be a C^2 volume-preserving diffeomorphism of M that is close to φ_1 in the C^1 topology. We shall show that g satisfies the hypotheses of the following theorem.

Theorem 2.1 [11] *Let $g : M \to M$ be a C^2 volume-preserving diffeomorphism that is partially hyperbolic and dynamically coherent. Suppose that g*

is u, s-transitive and its invariant bundles are sufficiently Hölder. Then g is ergodic.

First we explain the terms in this theorem.

We denote the length with respect to a Riemannian structure of $v \in T_pM$ by $\|v\|_p$. If $A_p : T_pM \to T_{f(p)}M$ is a linear map, we denote the *norm* and *co-norm* of A_p by

$$\|A\|_p = \sup_{\|v\|_p=1} \|Av\|_{f(p)}, \qquad \text{and} \qquad m(A)_p = \inf_{\|v\|_p=1} \|Av\|_{f(p)},$$

respectively. If $A : TM \to TM$ covers f and is linear on fibers, we denote the norm and co-norm of A by

$$\|A\| = \sup_{p\in M} \|A\|_p, \qquad \text{and} \qquad m(A) = \inf_{p\in M} m(A)_p,$$

respectively.

For a diffeomorphism $f : M \to M$ to be *partially hyperbolic*, there must exist a continuous, Tf-invariant, direct sum decomposition of the tangent bundle

$$TM = E^u \oplus E^c \oplus E^s$$

in which E^u and E^s are non-trivial bundles. Furthermore it must be possible to choose a Riemannian structure on M so that

$$m(T^u f) > \|T^c f\| \geq m(T^c f) > \|T^s f\|,$$

$$m(T^u f) > 1 > \|T^s f\|,$$

where $T^a f$ the restriction of Tf to E^a ($a = s$, c or u).

A partially hyperbolic diffeomorphism is *dynamically coherent* if the distributions E^c, E^u, E^s, $E^c \oplus E^u$ and $E^c \oplus E^s$ are all uniquely integrable, with the integral manifolds of $E^c \oplus E^u$ and $E^c \oplus E^s$ foliated, respectively, by the integral manifolds of E^c and E^u and by the integral manifolds of E^c and E^s.

We denote the foliation tangent to the distribution E^a by $\mathcal{W}^a$ for $a = u, c, s, cu, cs$.

A partially hyperbolic, dynamically coherent diffeomorphism g has *sufficiently Hölder* invariant bundles if

(a) The Hölder exponents of the three bundles E^u, E^c and E^s are greater than a certain number $\theta < 1$ that is close to 1[1].

(b) The foliation $\mathcal{W}^u$ is locally uniformly C^1 when restricted to each center unstable leaf $\mathcal{W}^{cu}$, and the foliation $\mathcal{W}^s$ is locally uniformly C^1 when restricted to each center unstable leaf $\mathcal{W}^{cs}$.

(c) The unstable holonomy maps between center stable leaves are uniformly θ-Hölder continuous as are the stable holonomy maps between center unstable leaves.

Condition (c) is used but not stated in [11].

Proposition 2.2 *Let g be a C^2 diffeomorphism that is partially hyperbolic and dynamically coherent. Suppose there is a Riemannian structure with respect to which the expansion and contraction of g on the invariant subbundles satisfy the following inequalities at every point p, where $\theta < 1$ is the number described above:*

$$\|T_p^c f\|_p < m(T_p^u f)_p m(T_p^s f)_p^\theta \quad \textit{and} \quad \|T_p^s f\|_p \|T_p^u f\|_p^\theta < m(T_p^c f)_p.$$

Then the invariant subbundles for g are sufficiently Hölder.

Proof. Properties (b) and (c) are proved in [12]. Property (a) is proved in [14] in the case when g is the time-1 map of the geodesic flow for a surface of variable negative curvature; the proof extends easily to the general case. □

We now show that if φ_t is a volume-preserving Anosov flow on a compact three dimensional manifold, then φ_1 satisfies partial hyperbolicity, dynamical coherence, and sufficient bunching.

The map φ_1 leaves invariant a C^2, 1-dimensional foliation $\mathcal{W}^c$ of M by orbits of the flow φ_t. Because φ_t is an Anosov flow, it leaves invariant stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, each of which is jointly integrable with $\mathcal{W}^c$, giving rise to center-stable and center-unstable foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

Tangent to $\mathcal{W}^u$, $\mathcal{W}^c$, and $\mathcal{W}^s$ are continuous distributions E^u, E^c and E^s. They give the $T\varphi_1$-invariant partially hyperbolic splitting $TM = E^u \oplus E^c \oplus E^s$. The joint integrability of $\mathcal{W}^c$ with the stable and unstable distributions means that φ_1 is dynamically coherent.

[1]The value of θ given in [11] is $(\sqrt{10m^2+1}-1)/10m$, where $m = \dim M$. This is not the smallest possible θ, nor is it claimed to be. For $m = 2$ we can take $\theta = .96$.

In order to show that the invariant bundles for φ_1 are sufficiently Hölder, we verify that φ_1 satisfies the hypotheses of Proposition 2.2. We begin by noting that, since E^u, E^c and E^s are all one dimensional, $\|T_p^a\varphi_1\|_p = m(T_p^a\varphi_1)_p$ for $a = u$, c or s at all p. We next change the Riemannian structure so that

1. the distributions E^u, E^c and E^s are pairwise orthogonal;
2. the generating vector field of the flow has unit length;
3. the volume defined by the adapted metric is φ_t-invariant;
4. $\|T^s\varphi_1\| < 1 < m(T^u\varphi_1)$.

It is always possible to choose a continuous Riemannian metric with these properties. The first two can be obtained by fiat, and (2) implies that $\|T_p^c\varphi_1\|_p = 1$ for all p. Now make the familiar construction of the adapted norm for vectors in E^s and choose the norm of vectors in E^u so that (3) holds.

Then $\|T^s\varphi_1\| < 1$ by construction and $1 < m(T^u\varphi_1)$, since (1), (2) and (3) imply that

$$\|T_p^s\varphi_1\|_p\|T_p^u\varphi_1\|_p = 1$$

for all p. These inequalities give us (4).

It is evident from the above equation and (4) that the inequalities in Proposition 2.2 hold for any $\theta < 1$ if we use the adapted metric. If we now fix an acceptable $\theta < 1$, these inequalities will still hold for our chosen θ and any smooth metric that is close enough to the adapted metric.

Partial hyperbolicity is stable under C^1-small perturbations. It follows from the corollary to Proposition 2.3 of [11] that dynamical coherence is stable under C^1-small perturbations if the center foliation of the original map is C^1. Furthermore the hypotheses of Proposition 2.2 are stable under C^1-small perturbations. Hence any g that is sufficiently C^1-close to φ_1 will satisfy the hypotheses in Theorem 2.1 about partial hyperbolicity, dynamical coherence, and sufficiently Hölder foliations.

The only condition left to consider is u, s-transitivity. We say that g is *u, s-transitive* if any two points can be joined by a u, s-path. As we explained in the introduction, such a path is the concatenation of a finite number arcs, each of which lies in a single leaf of either the stable or the unstable foliation for g.

Definition: The diffeomorphism g is *stably u, s-transitive* if every g sufficiently C^1-close to f is u, s-transitive.

It is clear from the above discussion that the only step remaining in the proof of Theorem 1.1 is to show the following

Lemma 2.3 (Transitivity Lemma) *A topologically mixing Anosov flow on a compact* 3*-manifold is stably u, s-transitive.*

This was proved by Katok and Kononenko for contact Anosov flows in any dimension ([7], Proposition 5.2). Their result also implies the Transitivity Lemma in the case when φ_t is a C^∞ Anosov flow on a 3-manifold and $E^u \oplus E^s$ is C^1. For then $E^u \oplus E^s$ defines a C^1 invariant transverse 1-form α (with kernel $E^u \oplus E^s$), and Theorem 2.3 in Hurder-Katok [6] implies that $\alpha \wedge d\alpha$ is C^∞ and nondegenerate; that is, the flow φ_t is contact and Proposition 5.2 in [7] applies.

In general, however, the distribution $E^u \oplus E^s$ will not be C^1. The discussion in [9] on pages 751–753 shows that $E^u \oplus E^s$ is not C^1 for the generic perturbation of a geodesic flow. Similarly, for any perturbation of a suspension that is not itself a suspension, $E^u \oplus E^s$ is not C^1.

3 Engulfing

In this section we shall give criteria for a diffeomorphism to be stably u, s-transitive. It is convenient to begin with the more general situation of a pair of topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on a compact connected manifold M with dimension n.

Definition: An *$\mathcal{F}_1, \mathcal{F}_2$-path* is a path $\psi : [0, 1] \to M$ consisting of a finite number of consecutive arcs — called *legs* — each of which is a curve that lies in a single leaf of one the two foliations[2]. The pair $\mathcal{F}_1, \mathcal{F}_2$ is *transitive* if any two points in M are joined by an $\mathcal{F}_1, \mathcal{F}_2$-path and is *stably transitive* if any pair of foliations sufficiently close to $\mathcal{F}_1, \mathcal{F}_2$ is transitive.

Two foliations are close if compact pieces of the leaves of one them can be uniformly approximated by leaves of the other. To make this precise, we

[2]We make the convention that consecutive legs must be of opposite types, although it is permissible to have a leg of length 0. The initial leg may be of either type.

describe a basis for the topology on the space of foliations. Let us consider foliations with k dimensional leaves and say that a map $\tau : \mathbf{R}^n \to M$ is a local parametrization for a foliation if τ is a homeomorphism onto its image and maps each set $\{\text{const}\} \times \mathbf{R}^k$ into a single leaf. Given a finite collection $\mathcal{V} = \{V_1, \ldots, V_r\}$ of subsets of $C^0(\mathbf{R}^n, M)$ that are open in the compact-open topology, we define $\mathcal{U}_{\mathcal{V}}$ to be the set of all foliations $\mathcal{F}$ for which we can find maps $\tau_i \in V_i$ for $i = 1, \ldots, r$ such that each τ_i is a local parametrization for $\mathcal{F}$ and the images of the τ_i cover M. The collection of all sets of the form $\mathcal{U}_{\mathcal{V}}$ is the promised basis.

In the following I will be the unit interval $[0, 1]$ and D^k will be the closed unit disk in $\mathbf{R}^k$.

Definition: A compact set K can be $\mathcal{F}_1, \mathcal{F}_2$-*engulfed* from a point p_0 if we can find a neighborhood U of K such that the pair $(U, U \setminus K)$ has the homotopy type of $(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$ and a continuous map $\Psi : D^n \times I \to M$ such that:

1. For each $x \in D^n$, $\psi_x(\cdot) = \Psi(x, \cdot)$ is an $\mathcal{F}_1, \mathcal{F}_2$-path that starts at p_0.

2. There is a constant N such that every path ψ_x has at most N legs.

3. $\psi_x(1) \in U \setminus K$ for any $x \in \partial D^n$.

4. The map $H_n(D^n, \partial D^n) \to H_n(U, U \setminus K)$ induced by $x \mapsto \psi_x(1)$ is nontrivial.

The reader should picture K as a small ball sitting in the middle of a larger ball U and visualize an n parameter set of $\mathcal{F}_1, \mathcal{F}_2$-paths spreading out from p_0 and terminating on a set of points whose boundary is a sphere that lies in U and surrounds K.

Here are some obvious properties of engulfing.

1. If K can be $\mathcal{F}_1, \mathcal{F}_2$-engulfed from p_0, then every point in K can be reached from p_0 along an $\mathcal{F}_1, \mathcal{F}_2$-path with at most N legs.

2. If every $\mathcal{F}_1$ leaf passes through K, then every point of M can be reached from p_0 along an $\mathcal{F}_1, \mathcal{F}_2$-path with at most $N + 1$ legs. Any point can be reached from any other along a $\mathcal{F}_1, \mathcal{F}_2$-path with at most $2(N + 1)$ legs, by moving to p_0 from the first point via the set K and then moving back out to the second point.

3. Engulfing is stable under small perturbations of p_0, K and the foliations. In particular if $\{q_0\}$ can be $\mathcal{F}_1, \mathcal{F}_2$-engulfed from p_0, there is $\delta > 0$ such that the closed geodesic ball of radius $\delta > 0$ around q_0 can be $\mathcal{F}_1', \mathcal{F}_2'$-engulfed from p_0 for any pair of foliations $\mathcal{F}_1', \mathcal{F}_2'$ that are sufficiently close to $\mathcal{F}_1, \mathcal{F}_2$.

Observe also that if every leaf of $\mathcal{F}_1$ is dense in M, any $\mathcal{F}_1'$ that is sufficiently close to $\mathcal{F}_1$ will have the property that every leaf passes through the δ ball around q_0. Indeed, if we fix a Riemannian metric on our compact manifold M, we have

Lemma 3.1 *Let $\mathcal{F}$ be a foliation such that every leaf of $\mathcal{F}$ is dense. Let $\delta > 0$ be given. Then there is a neighborhood $\mathcal{U}$ of $\mathcal{F}$ in the space of foliations such that $\mathcal{F}' \in \mathcal{U}$ implies that each leaf of $\mathcal{F}'$ passes within δ of every point of M.*

Proof. Suppose that the leaves of $\mathcal{F}$ are k dimensional. For each point $p \in M$ we can choose a local parametrization $\tau_p : \mathbf{R}^n \to M$ for $\mathcal{F}$ such that $\tau_p(0) = p$ and for any $x \in D^{n-k}$ the set $\tau_p(\{x\} \times D^k)$, which lies in a single leaf of $\mathcal{F}$, passes within $\delta/2$ of every point of M. Since M is compact, we can choose a finite set $p_1, \ldots, p_r$ such that the interiors of the sets $\tau_{p_i}(D^{n-k} \times D^k)$ cover M. For each i, there is a neighborhood V_i of τ_{p_i} in the compact-open topology on $C^0(\mathbf{R}^n, M)$ such that $\tau(\{x\} \times D^k)$ passes within δ of every point of M for each $x \in D^{n-k}$ and each $\tau \in V_i$. Let $\mathcal{V} = \{V_1, \ldots, V_r\}$. Then every $\mathcal{F}'$ in the neighborhood $\mathcal{U}_{\mathcal{V}}$ of $\mathcal{F}$ has the desired property. $\square$

Combining the preceding lemma and the earlier observations gives us the following criterion for transitivity.

Proposition 3.2 *Two foliations $\mathcal{F}_1, \mathcal{F}_2$ are stably transitive if every leaf of $\mathcal{F}_1$ is dense and there are points p_0 and q_0 such that q_0 can be $\mathcal{F}_1, \mathcal{F}_2$-engulfed from p_0.*

This criterion can be simplified in the situation of interest in this paper, where $\mathcal{F}_1 = \mathcal{W}^u$, $\mathcal{F}_2 = \mathcal{W}^s$ and an $\mathcal{F}_1, \mathcal{F}_2$-path is a u, s-path

Lemma 3.3 *Let f be a dynamically coherent partially hyperbolic diffeormorphism of a manifold M whose central leaves have dimension c. Suppose we have two points p_0 and q_0, a neighborhood B of q_0 in $\mathcal{W}^c(q_0)$ that is homeomorphic to a ball, and a continuous map $\Psi : D^c \times I \to M$ with the following properties:*

1. *For each* $z \in D^c$, $\psi_z(\cdot) = \Psi(z, \cdot)$ *is a* u, s*-path that starts at* p_0.

2. *There is a constant* N *such that each path* ψ_z *has at most* N *legs.*

3. $\psi_z(1) \in B \setminus \{q_0\}$ *for each* $z \in \partial D^c$.

4. *The map* $H_c(D^c, \partial D^c) \to H_c(B, B \setminus \{q_0\})$ *induced by* $z \to \psi_z(1)$ *is nontrivial.*

Then q_0 *can be engulfed from* p_0.

The point of the lemma is that we only need to check the engulfing property in the central direction.

Proof. Let u and s be the dimensions of the unstable and stable leaves respectively and set $n = u + s + c = \dim M$. There is a homeomorphism ρ from $D^u \times D^s \times B$ onto a neighborhood V of q_0 in M such that

1. $\rho(0, 0, b) = b$ and $\rho(D^u \times \{(0, b)\}) \subset \mathcal{W}^u(b)$ for all $b \in B$;

2. $\rho(\{x\} \times D^s \times \{b\}) \subset \mathcal{W}^s(\rho(x, 0, b))$ for all $(x, b) \in D^u \times B$.

We can construct a continuous map $\overline{\Psi} : D^u \times D^s \times D^c \times I \to M$ such that, for all $(x, y, z) \in D^u \times D^s \times D^c$,

$$\overline{\Psi}(x, y, z, 0) = p_0 \quad \text{and} \quad \overline{\Psi}(x, y, z, 1) = \rho(x, y, \Psi(z, 1)),$$

and each curve $\overline{\Psi}(x, y, z, \cdot)$ is a u, s-path with at most $N + 2$ legs, consisting of the path $\Psi(z, \cdot)$ concatenated with a path in $\mathcal{W}^u(\Psi(z, 1))$ and a path in $\mathcal{W}^s(\rho(x, y, \Psi(z, 1)))$.

It is obvious that the map

$$H_n(D^u \times D^s \times D^c, \partial(D^u \times D^s \times D^c)) \to H_n(V, V \setminus \{q_0\})$$

induced by $(x, y, z) \mapsto \overline{\Psi}(x, y, z, 1)$ is nontrivial. $\square$

Further simplification is possible if the leaves of $\mathcal{W}^c$ are one dimensional. We say that $\mathcal{W}^u$ and $\mathcal{W}^s$ are jointly integrable at a point if they are jointly integrable in some neighborhood of the point.

Proposition 3.4 *Let f be as in the previous lemma. Suppose that the leaves of the central foliation $\mathcal{W}^c$ are one dimensional and p_0 is a point at which $\mathcal{W}^u$ and $\mathcal{W}^s$ are not jointly integrable. Then there is a point in $\mathcal{W}^c(p_0)$ that can be engulfed from p_0.*

Proof. By Lemma 3.3, it suffices to find a continuous one parameter family of u, s-paths emanating from p_0 whose endpoints form an arc in $\mathcal{W}^c(p_0)$.

We can find a neighborhood U of p_0 and a homeomorphism $\sigma : \mathbf{R}^u \times \mathbf{R}^s \times \mathbf{R} \to U$ with $\sigma(0,0,0) = p_0$ such that each line $\{(x,y)\} \times \mathbf{R}$ maps into a single $\mathcal{W}^c$ leaf and slices of the form $\{x\} \times \mathbf{R}^s \times \mathbf{R}$ and $\mathbf{R}^u \times \{y\} \times \mathbf{R}$ map into single leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ respectively.

Define a quadrilateral to be an $\mathbf{R}^u, \mathbf{R}^s$-path $\psi : I \to \mathbf{R}^u \times \mathbf{R}^s \times \{0\}$ with 4 legs that begins and ends at $(0,0,0)$. Any short enough quadrilateral ψ has a unique lift to a u, s-path $\widehat{\psi}$ in U that begins at p_0 and has the property that ψ is the orthogonal projection to $\mathbf{R}^u \times \mathbf{R}^s$ of $\sigma^{-1} \circ \widehat{\psi}$. The curve $\widehat{\psi}$ ends at a point in $\mathcal{W}^c(p_0)$.

Since $\mathcal{W}^u$ and $\mathcal{W}^s$ are not jointly integrable at p_0, we can find a quadrilateral ψ_1 whose sides are as short as we wish such that the endpoint of $\widehat{\psi}_1$ is not p_0. Let $\psi_r(t) = r\psi(t)$. Then the family ψ_r, $0 \le r \le 1$, is a homotopy through quadrilaterals from the trivial quadrilateral, whose sides all have length 0, to ψ_1. We may assume that ψ_1 is short enough so that $\widehat{\psi}_r$ is defined for $0 \le r \le 1$. The lifts $\widehat{\psi}_r$ form the desired one parameter family. $\square$

Remarks: Proposition 3.4 can be strengthened slightly to say that p_0 can be engulfed from itself. We define ψ_{-r} be the curve ψ_r traversed backwards. Then ψ_{-r} is also a quadrilateral, and $\widehat{\psi}_{-r}(t)$ and $\widehat{\psi}_r(1-t)$ are in the same $\mathcal{W}^c$ leaf for all $t \in I$. These curves cannot cross, because that would force two different leaves of $\mathcal{W}^u$ or $\mathcal{W}^s$ to cross. Hence $r \mapsto \widehat{\psi}_r(1)$, $-1 \le r \le 1$, is a continuous arc in $\mathcal{W}^c(p_0)$ that joins points on opposite sides of p_0. It follows from Lemma 3.3 that p_0 can be engulfed from p_0.

Since the stable and unstable foliations of a partially hyperbolic map depend continuously on the map, Propositions 3.2 and 3.4 imply the following theorem.

Theorem 3.5 *Let f be a dynamically coherent partially hyperbolic diffeomorphism of a manifold M whose stable leaves are dense and whose central*

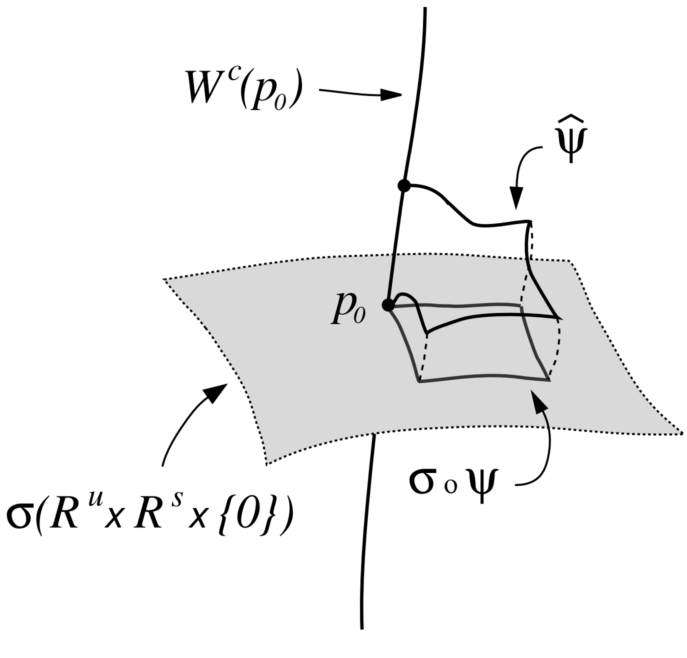

Figure 1: The paths $\sigma \circ \psi(t)$ and $\widehat{\psi}(t)$.

leaves have dimension 1*. Suppose that there is a point p_0 at which $\mathcal{W}^u$ and $\mathcal{W}^s$ are not jointly integrable. Then f is stably u, s-transitive.*

Theorem 1.3 is a corollary of the above theorem and a result of Plante.

Proof of Theorem 1.3 Proposition 1.7 of [9] states that if $\mathcal{W}^u$ and $\mathcal{W}^s$ for the time-1 map φ_1 (i.e. the strong stable and strong unstable foliations for the flow φ_t) are not jointly integrable, then all leaves of both $\mathcal{W}^u$ and $\mathcal{W}^s$ are dense. Hence φ_1 satisfies the hypotheses of Theorem 3.5. Theorem 1.3 now follows immediately. $\square$

4 Proof of the Transitivity Lemma

Let φ_t be any mixing Anosov flow on a 3-manifold. Let $\mathcal{W}^u$ and $\mathcal{W}^s$ be the unstable and stable foliations for φ_1; they both have one dimensional leaves.

In view of Theorem 1.3, we only need to show that there is a point at which $\mathcal{W}^u$ and $\mathcal{W}^s$ are not jointly integrable. Plante [9] (Theorem 3.1, p. 744) proved that for any Anosov flow, if $\mathcal{W}^u$ and $\mathcal{W}^s$ are jointly integrable, then φ_t is orbit equivalent to a suspension. When, in addition φ_t is a codimension-1 Anosov flow (meaning that at least one of $\mathcal{W}^u$ and $\mathcal{W}^s$ has one-dimensional leaves), then φ_t *is* a suspension ([9] Theorem 3.7, p. 746). Since we are assuming that φ_t is topologically mixing, and suspensions are not topologically mixing, this cannot happen. Thus $\mathcal{W}^u$ and $\mathcal{W}^s$ are not jointly integrable. $\square$

Remarks: It is possible that the Transitivity Lemma holds in all dimensions. Our proof shows that it holds for any codimension-1 Anosov flow. The only step where we the need the extra assumption that the flow is codimension-1 is when we claim that φ_t must be a suspension if $\mathcal{W}^u$ and $\mathcal{W}^s$ are jointly integrable. We know of no counterexample to this claim. An Anosov flow for which $\mathcal{W}^u$ and $\mathcal{W}^s$ are jointly integrable must be a special flow constructed over an Anosov diffeomorphism of a compact manifold, by Theorem 1.8 of [9]. The only known Anosov diffeomorphisms act on infranilmanifolds. Plante has shown that any special flow over an Anosov diffeomorphism on an infranilmanifold such that $\mathcal{W}^u$ and $\mathcal{W}^s$ are jointly integrable must be a suspension; see Section 3 of [9], in particular Corollary 3.6 and the remark after Theorem 3.2.

Since this paper was written, Pugh and Shub [10] have proved a stronger version of Theorem 2.1. In this stronger theorem, the sufficiently Hölder hypothesis is replaced by a condition called center bunching, which requires that the ratio

$$\|T^c g\|/m(T^c g)$$

be close to 1. Exactly how close is specified in [10]. The condition is stable under C^1-small perturbations and is automatically satisfied if $T^c g$ preserves the norm of vectors. In the case of the time one map of an Anosov flow, this latter property holds if the generating vector field of the flow has unit length. We observed in Section 2 that it is always possible to choose a Riemannian structure with this property. It follows that any C^2 volume preserving Anosov flow that does not have $\mathcal{W}^u$ and $\mathcal{W}^s$ jointly integrable must be stably ergodic. This proves Theorem 1.4.

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