Local density of diffeomorphisms with large centralizers

Densité locale des difféomorphismes ayant un gros centralisateur

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Abstract

Given any compact manifold M, we construct a non-empty open subset $\mathcal{O}$ of the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that the centralizer of every diffeomorphism in $\mathcal{D}$ is uncountable, hence non-trivial.

Résumé : Pour toute variété M compacte, de dimension quelconque, nous construisons une partie $\mathcal{O} \subset \mathrm{Diff}^1(M)$ non vide, ouverte dans l'espace $\mathrm{Diff}^1(M)$ des C^1-difféomorphismes de M, et un sous-ensemble $\mathcal{D} \subset \mathcal{O}$ dense en $\mathcal{O}$, constitué de difféomorphismes dont le centralisateur est non dénombrable, donc non trivial.

Key words: Trivial centralizer, trivial symmetries, Mather invariant.

Mots clés : Centralisateur trivial, symétries triviales, invariant de Mather.

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Introduction

The *centralizer* of a C^r diffeomorphism $f \in \mathrm{Diff}^r(M)$ is the group of diffeomorphisms commuting with f:

$$C(f) := \{g \in \mathrm{Diff}^r(M) : fg = gf\}.$$

The centralizer $C(f)$ always contains the group $< f >$ of all the powers of f. For this reason, we say that f has a *trivial centralizer* if $C(f) =< f >$. If f is the time one map of a C^r vector field X, then $C(f)$ contains the flow of X and hence contains a subgroup diffeomorphic to $\mathbb{R}$ (or $S^1 = \mathbb{R}/\mathbb{Z}$ if f is periodic).

The elements of $C(f)$ are transformations of M which preserve the dynamics of f: in that sense they are the symmetries of f. How large is, in general, this symmetry group?

- On one hand, the structure on M given by a diffeomorphism is very flexible, so that one might expect that any symmetry could be broken by a small perturbation of the diffeomorphism.
- On the other hand, the symmetries are sought in the very large group $\mathrm{Diff}^r(M)$, which makes the problem harder. For example, one can easily show that the group $C^0(f)$ of homeomorphisms commuting with a Morse-Smale diffeomorphism f is always uncountable.

Nevertheless it is natural to guess that general diffeomorphisms have no non-trivial smooth symmetries. Making this intuition explicit, Smale asked the following:

Question I ([S$_1$, S$_2$]). *Let $\mathcal{T}^r(M) \subset \mathit{Diff}^{\,r}(M), r \geq 1$ denote the set of C^r diffeomorphisms of a compact manifold M with trivial centralizer.*

1. *Is $\mathcal{T}^r(M)$ dense in $\mathit{Diff}^{\,r}(M)$?*
2. *Is $\mathcal{T}^r(M)$ residual in $\mathit{Diff}^{\,r}(M)$? That is, does it contain the intersection of countably many dense open subsets?*
3. *Is $\mathcal{T}^r(M)$ a dense open subset of $\mathit{Diff}^{\,r}(M)$?*

We think it's natural to reformulate the third part of Smale's question as:

Question II. *Does $\mathcal{T}^r(M)$ contain a dense and open subset of $\mathit{Diff}^{\,r}(M)$?*

This question has many parameters, the most obvious being the regularity r of the diffeomorphisms and the dimension $\dim(M)$ of the manifold. The question has been answered in varying degrees of generality for specific parameters. For instance, Kopell [K] proved that $\mathrm{Diff}^r(S^1)$, $r \geq 2$, contains a dense and open subset of diffeomorphisms with trivial centralizers. Many authors subsequently gave partial answers in higher dimension (see [BCW$_1$] for an attempt to list these partial results).

The present paper and [BCW$_2$] together give a complete answer to Smale's problem for $r = 1$. More precisely:

- [BCW$_2$] proves that C^1-generic diffeomorphisms have a trivial centralizer[1], giving a positive answer to the first two parts of Smale question. [BCW$_3$] shows that C^1-generic conservative (volume preserving or symplectic) diffeomorphisms have a trivial centralizer.

- In this paper, we answer in the negative (for $r = 1$) the third part of Smale's question (and to Question II) on any compact manifold.

Main theorem. *Given any compact manifold M, there is a non-empty open subset $\mathcal{O} \subset \mathit{Diff}^1(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is C^∞ and its C^∞-centralizer $C^\infty(f)$ is uncountable (hence not trivial).*

We will see below (see Theorem 5) that this statement also holds for symplectic diffeomorphisms on a symplectic manifold.

The uniform presentation of this result (*Given any compact manifold,...*) hides very different situations, arguments and results according to the dimension: namely, whether $\dim(M) < 3$ or $\dim(M) \geq 3$. We discuss this breakdown of the results below.

Our paper also deals with the question of how large is the class of diffeomorphisms that can be embedded in a flow. This is a natural question, since the studies of the dynamical systems defined either by diffeomorphisms or by vector fields are in fact closely related. In the paper [P] titled *Vector fields generate few diffeomorphisms*, Palis proved that C^1-generic diffeomorphisms cannot be embedded in a flow. Our results somehow counterbalance Palis' statement: diffeomorphisms that are the time one map of a flow are C^1-locally dense in dimension 1 and 2.

Theorem 1. *There is a dense subset $\mathcal{D} \subset \mathit{Diff}^{\,1}(S^1)$ such that every $f \in \mathcal{D}$ commutes with the flow of a C^∞ Morse-Smale vector field X. More precisely, f is Morse-Smale and f^q is the time one map of the flow of X, where $q = 2$, if f is orientation reversing, and q is the period of the periodic orbits of f otherwise. Furthermore, $C(f)$ is isomorphic to $\mathbb{R} \times (\mathbb{Z}/q\mathbb{Z})$.*

(Section 1.6 presents open questions on centralizers of diffeomorphisms in $\mathrm{Diff}^1(S^1)$.)

Among compact surfaces, the case of the sphere is special because of the existence north pole-south pole diffeomorphisms. The symmetries of these dynamics allow us to get a centralizer isomorphic to $S^1 \times \mathbb{R}$ and this is one of the reasons why we present this case separately.

Another specific feature of the north-south diffeomorphisms of the sphere is that for these maps it is possible to generalize the so-called *Mather invariant*, introduced by Mather in the one-dimensional case. Such an invariant plays a

[1] [BCW$_1$] is an announcement which gives the structure of the detailed proof written in [BCW$_2$].

fundamental role in our constructions: the Mather invariant of a diffeomorphism f is trivial if and only if f can be perturbed to become the time one map of a vector field.

Theorem 2. *Let $\mathcal{O} \subset \mathit{Diff}^{\,1}(S^2)$ denote the (open) subset of Morse-Smale diffeomorphisms f such that the nonwandering set $\Omega(f)$ consists of two fixed points, one source N_f and one sink S_f, such that the derivatives $D_{N_f}f$ and $D_{S_f}f$ have each a complex (non-real) eigenvalue.*

Then there is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is the time one map of a Morse-Smale C^∞-vector field. Furthermore $C(f)$ is isomorphic to $\mathbb{R} \times S^1$.

Theorem 2 is a bridge between the one-dimensional case and the general two-dimensional case. One the one hand, north pole-south pole dynamics on the sphere and Morse-Smale dynamics on the circle share the Mather invariant; on the other hand, other features of these dynamics on the sphere occur in simple dynamics on a general compact surface. The general case is solved by a combination of the arguments used for the sphere in a neighborhood of the sinks and the sources, together with an analysis of the local situation in a neighborhood of the saddles.

Theorem 3. *Let S be a connected closed surface. Let $\mathcal{O} \subset \mathit{Diff}^{\,1}(S)$ be the set of Morse-Smale diffeomorphisms f such that:*

- *any periodic point is a (hyperbolic) fixed point,*
- *f has at least one hyperbolic saddle point,*
- *for any hyperbolic saddle x, every eigenvalue of $Df(x)$ is positive,*
- *for any sink or source x, the derivative $Df(x)$ has a complex (non-real) eigenvalue,*
- *there are no heteroclinic orbits: if $x \neq y$ are saddle points then $W^s(x) \cap W^u(y) = \emptyset$.*

Then $\mathcal{O}$ is a non-empty open subset of $\mathit{Diff}^{\,1}(S)$ and there is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is the time one map of a Morse-Smale C^∞-vector field. Furthermore, $C(f)$ is the flow of this vector field, hence isomorphic to $\mathbb{R}$.

An important hypothesis in Theorems 2 and 3 (which holds trivially in Theorem 1) is that the derivative at each sink and source is conjugate to the composition of a homothety with a (non-trivial) rotation. This condition is open in dimension 2, but is nowhere dense in higher dimension. This explains why we are not able to obtain the local density of the embeddability in a flow in higher dimension, and naturally leads to the following question:

Question III. *Let M be a compact manifold of dimension $d \geq 3$. Denote by $\mathcal{O}$ the C^1-interior of the C^1-closure of the set of diffeomorphisms which are the time one map of a flow. Is $\mathcal{O}$ empty?*

In low dimension we find large centralizers among the simplest dynamical systems (the Morse-Smale systems). By contrast, in higher dimension we will use C^1-open subsets of *wild diffeomorphisms* to obtain periodic islands where the return map is the identity map. The resulting large centralizers for these wild diffeomorphisms are completely different. In low dimension, we embed the diffeomorphisms in a flow, and the centralizer is precisely the flow; hence all the diffeomorphisms in the centralizer have the same dynamics. In higher dimension, the diffeomorphisms we exhibit in the centralizer will be equal to the identity map everywhere but in the islands, in restriction to which they can be anything. This explains our result:

Theorem 4. *Given any compact manifold M of dimension $d \geq 3$, there is a non-empty open subset $\mathcal{O} \subset \mathit{Diff}^{\,1}(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ has non-trivial centralizer.*

More precisely, for $f \in \mathcal{D}$ the centralizer $C(f)$ contains a subgroup isomorphic to the group $\mathit{Diff}^{\,1}(\mathbb{R}^d, \mathbb{R}^d \setminus \mathbb{D}^d)$ of diffeomorphisms of $\mathbb{R}^d$ which are equal to the identity map outside the unit disc $\mathbb{D}^d$.

The large centralizer we build for a diffeomorphism in Theorem 4 consists of diffeomorphisms which have a very small support, and which are therefore C^0-close to the identity. It would be interesting to know if this is always the case. Let us formalize this question:

Question IV. *Let M be a compact manifold with $\dim(M) \geq 3$ and $\varepsilon > 0$. Let $\mathcal{O}_\varepsilon \subset \mathit{Diff}^1(M)$ be the set of diffeomorphisms f such that, for every $g \in C(f)$ there exists $n \in \mathbb{Z}$ such that $g \circ f^n$ is ε-close to the identity map for the C^0-distance. Does $\mathcal{O}_\varepsilon$ contain a dense open subset of $\mathit{Diff}^1(M)$ for every ε?*

If for non-conservative diffeomorphisms the existence of periodic islands depends on wild dynamics, the same islands appear in a more natural way for symplectic diffeomorphisms in a neighborhood of totally elliptic points. In order to state precisely this last result we need some notations. Let M be a compact manifold with even dimension $\dim(M) = 2d$. If M carries a symplectic form ω, then we denote by $\mathrm{Symp}^1_\omega(M)$ the space of C^1 diffeomorphisms of M that preserve ω (these diffeomorphisms are called *symplectomorphisms*).

Recall that a periodic point x of period n of a symplectomorphism f is called *totally elliptic* if all the eigenvalues of $Df^n(x)$ have modulus equal to 1. If $e^{i\alpha}$ is an eigenvalue of x then $e^{-i\alpha}$ is also an eigenvalue. Assume that $0 < \alpha_1 < \cdots < \alpha_d < \pi$ are the absolute values of the argument of the eigenvalues of x. Then x is *C^1-robustly totally elliptic*: every symplectomorphism g that is C^1-close to f has a totally elliptic point x_g of period n close to x.[2]

Theorem 5. *Let (M, ω) be a symplectic manifold and let $\mathcal{O} \subset Symp^1_\omega(M)$ denote the non-empty open subset of the symplectomorphisms having a C^1-robust totally elliptic periodic point.*

Then there is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that such that every $f \in \mathcal{D}$ has a non-trivial centralizer. More precisely, for $f \in \mathcal{D}$ the symplectic centralizer $C_\omega(f)$ contains a subgroup isomorphic to the group $Symp^1_\omega(\mathbb{R}^{2d}, \mathbb{R}^{2d} \setminus \mathbb{D}^{2d})$ of symplectomorphisms of $\mathbb{R}^{2d}$ equal to the identity map on the complement of the unit disc $\mathbb{D}^{2d}$.

We remark that, according to [ABC], C^1-generic symplectomorphisms of connected manifolds are transitive, and the manifold is a unique homoclinic class. In particular, there is a C^1-dense and open subset of symplectomorphisms having a hyperbolic periodic point with some transverse homoclinic intersection. Such a transverse homoclinic intersection prevents the diffeomorphism from being embedded in a flow: thus non-embeddability in a flow is satisfied on a C^1-open and dense subset of $\mathrm{Symp}^1_\omega(M)$. The same argument holds for volume preserving diffeomorphisms, showing that, if μ is a smooth volume form on a manifold M of dimension $\dim(M) \geq 2$, then the non embeddability in a flow is satisfied on a C^1-open and dense subset of $\mathrm{Diff}^1_\mu(M)$. However we have not been able to extend our main theorem for volume preserving diffeomorphisms:

Question V. *Let M be a closed manifold endowed with a smooth volume form μ. Does there exist a non-empty open subset $\mathcal{O} \subset Diff^1_\mu(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that for every $f \in \mathcal{D}$ the centralizer $C_\mu(f)$ is not trivial ?*

1 Local embeddability in a flow on the circle

1.1 Preparation of diffeomorphisms of the circle

The following proposition summarizes some very classical properties of diffeomorphisms of the circle.

Proposition 1.1. *Let $\mathcal{D}_0 \subset Diff^{\,1}(S^1)$ be the set of diffeomorphisms f satisfying the following properties:*

- *f is a C^∞ Morse-Smale diffeomorphism (i.e. the non wandering set consists of finitely many hyperbolic periodic points, alternately attracting or repelling);*

[2] This uses the fact that if λ is an eigenvalue of a symplectic matrix, then $\overline{\lambda}$, $\frac{1}{\lambda}$, $\frac{1}{\overline{\lambda}}$ are also eigenvalues.

- *for every periodic point $x \in Per(f)$, there is a neighborhood U_x of x such that the restriction $f|_{U_x}\colon U_x \to f(U_x)$ is an affine map (for the natural affine structure on $S^1 = \mathbb{R}/\mathbb{Z}$);*
- *if $x, y \in Per(f)$ are periodic points with distinct orbits, then $Df^q(x) \neq Df^q(y)$, where q is the period of x.*

Then $\mathcal{D}_0$ is dense in $\mathrm{Diff}^1(S^1)$.

For $\alpha > 1$ and $\beta \in (0,1)$, we introduce the set $D_{\alpha,\beta}$ of orientation-preserving C^∞ diffeomorphisms of the interval $[0,1]$ with the following properties:

- $\{0,1\}$ is the set of fixed points of f, and $f(x) > x$ for $x \in (0,1)$;
- $f(x) = \alpha x$ for small x and $f(x) = 1 + \beta(x-1)$ for x close to 1.

Applying Proposition 1.1, we will prove Theorem 1 by working inside the space $D_{\alpha,\beta}$.

1.2 The Mather invariant

We recall here a construction introduced by J. Mather [M] which associates to any diffeomorphism $f \in D_{\alpha,\beta}$ a class of diffeomorphisms of S^1.

Let us fix $\alpha > 1$ and $\beta \in (0,1)$ and introduce a C^∞ orientation preserving diffeomorphism $\varphi\colon (0,1) \to \mathbf{R}$ such that $\varphi(x) = \ln x/\ln(\alpha)$ for x small, and $\varphi(x) = \ln(1-x)/\ln(\beta)$ for x close to 1. Observe that there exists a constant $K_0 > 0$ such that $\varphi^{-1}(x) = \exp(\ln(\alpha)\,x)$ for $x < -K_0$ and $\varphi^{-1}(x) = 1 - \exp(\ln(\beta)\,x)$, for $x > K_0$.

For any $f \in D_{\alpha,\beta}$ the conjugated diffeomorphism $\theta_f = \varphi \circ f \circ \varphi^{-1}$ of $\mathbb{R}$ satisfies $\theta_f(x) > x$ for all x; furthermore, $\theta_f(x)$ agrees with $x+1$ when $|x|$ is larger than a constant $K_f > K_0$.

The space $\mathbb{R}/\theta_f$ of the orbits of θ_f is a smooth circle S_f which has two natural identifications with the (affine) circle $S^1 = \mathbb{R}/\mathbb{Z}$: two points $x, y \in (-\infty, -K_f]$ (resp. $x, y \in [K_f, +\infty)$) are in the same orbit for θ_f if and only if they differ by an integer. This leads to two diffeomorphisms $\pi_+\colon S_f \to S^1$ and $\pi_-\colon S_f \to S^1$, respectively. Let $\Delta_{f,\varphi} = \pi_+ \circ \pi_-^{-1}\colon S^1 \to S^1$.

Proposition 1.2. *The diffeomorphism f is the time one map of a C^1-vector field if and only if $\Delta_{f,\varphi}$ is a rotation.*

Proof. Note that $f \in D_{\alpha,\beta}$ agrees with the time one map of the vector field $X^- = \ln(\alpha)\,x\,\frac{\partial}{\partial x}$ in a neighborhood of 0 and with $X^+ = \ln(\beta)\,(x-1)\,\frac{\partial}{\partial x}$ in a neighborhood of 1. Furthermore, if f is the time one map of a C^1-vector field X on $[0,1]$, then $X = X^-$ in a neighborhood of 0 and $X = X^+$ in a neighborhood of 1. The hypothesis on φ implies $\varphi_*(X_-) = \frac{\partial}{\partial x}$ on some interval $(-\infty, L_-)$ and $\varphi_*(X_+) = \frac{\partial}{\partial x}$ on an interval (L_+, ∞).

Assume that $\Delta_{f,\varphi}$ is a rotation. Then we define a vector field Y on $\mathbb{R}$ as follows: consider $n > 0$ such that $\theta_f^n(x) > K_f$. Now let $Y(x) = \left(D_x\theta_f^n\right)^{-1}(\frac{\partial}{\partial x})$. This vector does not depend of n (because θ_f is the translation $t \mapsto t+1$ for $t \geq K_f$).

Claim. *if $x < -K_f$ then $Y(x) = \frac{\partial}{\partial x}$.*

Proof. Consider the natural projection $\pi_f : \mathbb{R} \to S_f$ that maps each point to its orbit for θ_f. Since Y is invariant under θ_f, the vector field $(\pi_f)_*(Y)$ is well-defined. Since on $(K_f, +\infty)$ the vector $Y(x)$ is equal to $\frac{\partial}{\partial x}$, the map $\pi_+ \circ \pi_f$ agrees with the natural projection $\mathbb{R} \to S^1$, and we have $(\pi_+ \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$. As $\Delta_{f,\varphi}$ is a rotation, and as the rotations preserve the vector field $\frac{\partial}{\partial x}$, we obtain that $(\pi_- \circ \pi_f)_*(Y(x)) = (\Delta_{f,\varphi}^{-1} \circ \pi_+ \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$. As θ_f agrees with the translation $t \mapsto t+1$ on $(-\infty, -K_f]$, the projection $\pi_- \circ \pi_f$ agrees on $(-\infty, -K_f]$ with the natural projection $\mathbb{R} \mapsto \mathbb{R}/\mathbb{Z}$. Hence $(\pi_- \circ \pi_f)_*(Y(x)) = \frac{\partial}{\partial x}$ implies $Y(x) = \frac{\partial}{\partial x}$. □

Notice that, by construction, the vector field Y is invariant under θ_f; furthermore, θ_f is the time one map of Y: this is true on a neighborhood of $\pm\infty$, and extends on $\mathbb{R}$ because Y is θ_f-invariant.

Now, the vector field $X = \varphi_*^{-1}(Y)$, defined on $(0,1)$, agrees with X_- and X_+ in a neighborhood of 0 and 1, respectively, hence induces a smooth vector field on $[0,1]$. Finally, f is the time one map of X.

Conversely, if f is the time one map of a C^1-vector field X on $[0,1]$ then θ_f is the time one map of the vector field $Y = \varphi_*(X)$, which agrees with $\partial/\partial x$ in the neighborhood of $\pm\infty$ (because X agrees with X_- and X_+ in a neighborhood of 0 and 1, respectively). Hence the projections $(\pi_- \circ \pi_f)_*(Y)$ and $(\pi_+ \circ \pi_f)_*(Y)$ are both equal to the vector field $\partial/\partial x$ on S^1. This implies that $(\Delta_{f,\varphi})_*(\partial/\partial x) = \partial/\partial x$, which implies that $\Delta_{f,\varphi}$ is a rotation. □

Remark 1.3. *The function $\Delta_{f,\varphi}$ defined here seems to depend on the choice of φ. There is a more intrinsic way to define the diffeomorphism $\Delta_{f,\varphi}$ "up to composition with rotations:"*

The vector fields X_- and X_+ defined in a neighborhood of 0 *and* 1*, respectively, are the unique vector fields such that f is the time one map of the corresponding flows, in the neighborhood of* 0 *and* 1*, respectively. Each of these vector fields induces a parametrization of the orbit space $(0,1)/f = S_f$, that is, up to the choice of an origin, a diffeomorphism $\pi_f^{\pm} : S_f \to S^1$. The change of parametrization $\pi_f^+ \circ (\pi_f^-)^{-1}$ is well-defined, up to the choice of an origin of the circle, i.e. up to composition, at the right and at the left, by rotations. This class of maps is called the* Mather invariant *of f.*

1.3 Forcing the Mather invariant to vanish

In this subsection we prove the following result.

Proposition 1.4. *Let f be a diffeomorphism in $D_{\alpha,\beta}$. Then any C^1-neighborhood $\mathcal{U}$ of f in $\mathit{Diff}^1([0,1])$ contains a diffeomorphism g such that $g = f$ in a neighborhood of $\{0,1\}$ and g is the time one map of a C^∞-vector field on $[0,1]$.*

We retain the notation from the previous subsection. Fix $f \in D_{\alpha,\beta}$ and $K_f > 0$ such that $\theta_f = \varphi \circ f \circ \varphi^{-1}$ agrees with $x \mapsto x+1$ on $(-\infty, -K_f] \cup [K_f, +\infty)$.

Given a diffeomorphism $h\colon \mathbb{R} \to \mathbb{R}$, the *support of* h, denoted by *supp*(h) is the closure of the set of points x such that $h(x) \neq x$.

Lemma 1.5. *Consider a number $a > K_f$ and a diffeomorphism $\tilde{\psi}\colon \mathbb{R} \to \mathbb{R}$ whose support is contained in $(a, a+1)$. Let h denote the diffeomorphism $\varphi^{-1} \circ \tilde{\psi} \circ \varphi$, and let ψ denote the diffeomorphism of $S^1 \simeq [a, a+1]/a \sim a+1$ induced by $\tilde{\psi}$.*

Then the diffeomorphism $g = f \circ h$ belongs to $D_{\alpha,\beta}$, and $\Delta_{g,\varphi} = \psi \circ \Delta_{f,\varphi}$.

Proof. The diffeomorphism g agrees with f in neighborhoods of 0 and 1, which proves that $g \in D_{\alpha,\beta}$. Furthermore, by construction, one may choose $K_g = a+1$.

For $x < -a$, there is a (unique) integer such that $\theta_f^n(x) = \theta_g^n(x) \in [a, a+1)$, and by construction of $\Delta_{f,\varphi}$, the projection of $\theta_f^n(x)$ on S^1 is $\Delta_{f,\varphi(x)}$. Now the projection on S^1 of $\theta_g^{n+1}(x) = \theta_f \circ \tilde{\psi} \circ \theta_f^n(x)$ is $\psi \circ \Delta_{f,\varphi}(x)$, by construction. As $\theta_g = \theta_f = y \mapsto y+1$ for $y \geq a+1$, one gets that the projection on S^1 of $\theta_g^{n+k}(x)$ is $\psi \circ \Delta_{f,\varphi}(x)$, for all $k > 0$; hence $\Delta_{g,\varphi} = \psi \circ \Delta_{f,\varphi}$. □

Iterating the process described in Lemma 1.5, we obtain:

Corollary 1.6. *Consider a finite sequence of numbers $a_i > K_f$, $i \in \{1, \dots, \ell\}$, such that $a_{i+1} > a_i + 1$ for all $i \in \{1, \dots, \ell-1\}$. For every $i \in \{1, \dots, \ell\}$, fix a diffeomorphism $\tilde{\psi}_i\colon \mathbb{R} \to \mathbb{R}$ whose support is contained in (a_i, a_i+1). Let h_i denote the diffeomorphism $\varphi^{-1} \circ \tilde{\psi}_i \circ \varphi$, and let ψ_i denote the diffeomorphism of S^1 induced by $\tilde{\psi}_i$. (Note that the diffeomorphisms h_i have disjoint support, so that they are pairwise commuting.)*

Then the diffeomorphism $g = f \circ h_1 \circ h_2 \circ \cdots \circ h_\ell$ belongs to $D_{\alpha,\beta}$, and we have:

$$\Delta_{g,\varphi} = \psi_\ell \circ \cdots \circ \psi_1 \circ \Delta_{f,\varphi}.$$

Definition 1.7. *Let $a \in \mathbb{R}$, and let $\overline{a}$ be its projection on $S^1 = \mathbb{R}/\mathbb{Z}$. Given a diffeomorphism $\psi\colon S^1 \to S^1$ with support in $S^1 \setminus \{\overline{a}\}$ we call* the lift of ψ in $(a, a+1)$ *the diffeomorphism $\tilde{\psi}_a\colon \mathbb{R} \to \mathbb{R}$ with support in $(a, a+1)$ such that for any $x \in (a, a+1)$ the image $\psi_a(x)$ is the point of $(a, a+1)$ which projects to $\psi(\overline{x})$ where $\overline{x}$ is the projection of x.*

We denote by $\Theta_a(\psi)$ the diffeomorphism of $[0,1]$ whose expression in $(0,1)$ is $\Theta_a(\psi) = \varphi^{-1} \circ \psi_a \circ \varphi$.

Lemma 1.8. *For any C^1-neighborhood $\mathcal{U}$ of f there is a neighborhood $\mathcal{V}$ of $Id_{S^1} \in Diff^1(S^1)$ with the following property:*

Given any finite sequence $a_i > K_f$, $i \in \{1, \dots, \ell\}$, such that $a_{i+1} > a_i + 1$ for all $i \in \{1, \dots, \ell - 1\}$, we denote by $\overline{a}_i$ the projection of a_i on S^1. For any i, let $\psi_i \in \mathcal{V}$ be a diffeomorphism of S^1 with support in $S^1 \setminus \{a_i\}$. Then the diffeomorphism $g = f \circ \Theta_{a_1}(\psi_1) \circ \cdots \Theta_{a_\ell}(\psi_\ell)$ belongs to $\mathcal{U}$.

Proof. We fix a neighborhood $\mathcal{U}_0$ if the identity map of $[0, 1]$ such that, if $g_1, \dots, g_n \in \mathcal{U}_0$ and if the support of the g_i are pairwise disjoint, then $f \circ g_1 \circ h_2 \circ \cdots g_n$ belongs to $\mathcal{U}$. Now the lemma is a direct consequence of Lemma 1.9 below. □

Lemma 1.9. *For any C^1-neighborhood $\mathcal{U}_0$ of f there is a neighborhood $\mathcal{V}$ of $Id_{S^1} \in Diff^1(S^1)$ with the following property:*

Consider any $a > K_f$, its projection $\overline{a}$ on S^1 and any diffeomorphism $\psi \in \mathcal{V}$ with support in $S^1 \setminus \{a\}$. Then the diffeomorphism $\Theta_a(\psi)$ belongs to $\mathcal{U}_0$.

Proof. Notice that there exists $\varepsilon > 0$ such that $\mathcal{U}_0$ contains any diffeomorphism h of $[0, 1]$ with $\sup_{x \in [0,1]} |D_x h - 1| < \varepsilon$.

Now consider $a > K_f$ and an integer $n > 0$. Then for any diffeomorphism ψ of S^1 with support in $S^1 \setminus \{\overline{a}\}$, the lifts ψ_a and ψ_{a+n} are conjugated by the translation $x \mapsto x + n$. As a consequence, $\Theta_{a+n}(\psi)$ is obtained from $\Theta_a(\psi)$ by the conjugacy by the homothety of ratio β^n. As a consequence we obtain that $\sup_{x \in [0,1]} |D_x \Theta_{a+n}(\psi) - 1| = \sup_{x \in [0,1]} |D_x \Theta_a(\psi) - 1|$.

Hence it suffices to prove the lemma for $a \in [K_f, K_f + 1]$. This is a direct consequece of the facts that the derivatives of φ and of φ^{-1} are bounded on $\varphi^{-1}([K_f, K_f+2])$ and $[K_f, K_f+2]$ respectively, and that for any ψ with support in $S^1 \setminus \{a\}$, one has:

$$\sup_{x \in [a, a+1]} |D_x \psi_a - 1| = \sup_{x \in S^1} |D_x \psi - 1|.$$

□

Let us now recall a classical result from [B] which is the key point of our proof.

Theorem 6 (Fragmentation lemma). *Let M be a closed Riemannian manifold, let $r > 0$ and let $\mathcal{U}$ be a C^1-neighborhood of the identity map. Then for any smooth diffeomorphism f of M isotopic to the identity, there exist $k \geq 1$ and $g_1, \dots, g_k \in \mathcal{U}$ such that $g_i = id$ on the complement of a ball $B(x_i, r)$, and*

$$f = g_1 \circ \cdots \circ g_k.$$

Here we use Theorem 6 on the circle S^1, where it is an easy consequence of the result, by M. Herman, that any smooth diffeomorphism is the product of a rotation by a diffeomorphism smoothly conjugate to a rotation. In Section 2, we will also use Theorem 6 on the torus T^2 and on compact surfaces.

Proof of Proposition 1.4. Given a C^1-neighborhood $\mathcal{U}$ of f, we choose a C^1-neighborhood $\mathcal{V}$ of the identity map of S^1 given by Lemma 1.8. Using Theorem 6, we can write $\Delta_{f,\varphi}$ as a finite product $\Delta_{f,\varphi} = \psi_1^{-1} \circ \cdots \circ \psi_\ell^{-1}$ such that $\psi_i \in \mathcal{V}$, and the support of ψ_i is contained in an interval of length $\frac{1}{2}$ in S^1 (and in particular is not all of S^1). Now we choose a finite sequence $a_i > K_k$ such that $a_{i+1} > a_i + 1$, and such that the projection $\overline{a}_i$ does not belong to the support of ψ_i. Let $h_i = \Theta_{a_i}(\psi_i)$.

Applying Lemma 1.8, we obtain that the diffeomorphism

$$g = f \circ h_1 \circ h_2 \circ \cdots \circ h_\ell$$

belongs to $\mathcal{U}$; applying Corollary 1.6, we get that

$$\Delta_{g,\varphi} = \psi_\ell \circ \cdots \circ \psi_1 \circ \Delta_{f,\varphi} = Id_{S^1}.$$

□

1.4 Existence of an invariant vector field

In this section we prove the first part of Theorem 1.

Proposition 1.10. *Let $\mathcal{D}_0^*$ be the set of diffeomorphisms $f \in \mathcal{D}_0$ which preserve a C^∞ Morse-Smale vector field X and such that f^q is the time one map of X, where q is the period of the connected components of $S^1 \setminus Per(f)$. Then $\mathcal{D}_0^*$ is dense in $Diff^1(S^1)$.*

Proof. By Proposition 1.1, it is enough to prove that $\mathcal{D}_0^*$ is dense in $\mathcal{D}_0$. Consider $f \in \mathcal{D}_0$. The set $Per(f)$ is finite. Let $\mathcal{I}$ be the set of segments joining two successive periodic points of f; in other words, every element $I \in \mathcal{I}$ is the closure of a connected component of $S^1 \setminus \mathrm{Per}(f)$. Notice that f induces a permutation on $\mathcal{I}$. Furthermore, all the elements of $\mathcal{I}$ have the same period denoted by $q > 0$, under this action (this period is equal to 2 if f reverses the orientation, and is equal to the period of the periodic orbits in the orientation preserving case).

Now consider a segment $I \in \mathcal{I}$. The endpoints of I are the fixed points of the restriction $f^q|_I$; moreover, one endpoint (denoted by a) is a repeller and the other (denoted by b) is an attractor. Let $h_I \colon I \to [0,1]$ be the affine map such that $h_I(a) = 0$ and $h_I(b) = 1$, and let $\varphi_I \colon [0,1] \to [0,1]$ denote the diffeomorphism $h_I \circ f^q|_I \circ h_I^{-1}$.

According to Proposition 1.4, there is a sequence $(\psi_{I,n})_{n\in\mathbb{N}}$, of diffeomorphisms converging to φ_I in the C^1-topology when $n \to +\infty$, and a sequence $(Y_{I,n})_{n\in\mathbb{N}}$ of C^∞ vector fields on $[0,1]$ such that $\psi_{I,n}$ agrees with φ_I in a small neighborhood of $\{0,1\}$ and is time one map of $Y_{I,n}$. We define $g_{I,n} = h_I^{-1} \circ \psi_{I,n} \circ h_I$. Notice that each diffeomorphism $g_{I,n}$ agrees with f^q in neighborhoods of the endpoints of I and converges to $f^q|_I$ when $n \to \infty$.

We now define a diffeomorphism $f_{I,n}$ of S^1 as follows:

$$f_{I,n} = \begin{cases} f & \text{on } S^1 \setminus f^{q-1}(I) \\ g_{I,n} \circ f^{-q+1} & \text{on } f^{q-1}(I). \end{cases}$$

This is a C^∞ diffeomorphism since it agrees with f in a neighborhood of the periodic orbits. Moreover, $(f_{I,n})$ converges to f as n approaches $+\infty$.

We denote by $X_{I,n}$ the vector field, defined on the orbit $\bigcup_0^{q-1} f^i(I)$ of the segment I, as follows:

- $X_{I,n} = (h_I^{-1})_*(Y_{I,n})$ on I;
- for all $i \in \{1, \dots, q-1\}$ and all $x \in I$:

$$X_{I,n}(f^i(x)) = f^i_*(X_{I,n}(x)).$$

Finally, we fix a family $I_1, \dots, I_\ell \subset \mathcal{I}$ such that for $i \neq j$ the segments I_i and I_j have distinct orbits, and conversely every orbit of segment in $\mathcal{I}$ contains one of the I_i.

We denote by f_n the diffeomorphism of S^1 coinciding with $f_{I_i,n}$ on the orbit of I_i for all $i \in \{1, \dots, \ell\}$. This diffeomorphism is well-defined because all the $f_{I_i,n}$ agree with f in a small neighborhood of the periodic points (the endpoints of the segments in $\mathcal{I}$). We denote by X_n the vector field on S^1 that agrees with $X_{I_i,n}$ on the orbit of I_i, for all $i \in \{1, \dots, \ell\}$.

It is easy to see that X_n is a smooth vector field on S^1, invariant under f_n, and such that f_n^q is the time one map of X_n: the unique difficulty consists in checking the continuity and smoothness of the vector field X_n at the periodic points. As f_n is affine in the neighborhood of the periodic orbits, it follows that, at both sides of a periodic point x, the vector field X_n is the affine vector field vanishing at x and whose eigenvalue at x is $\ln D_x f$. We hence have built a sequence (f_n) in $\mathcal{D}_0^*$ converging to f in the C^1-topology, as required. $\square$

1.5 Centralizer of a diffeomorphism $f \in \mathcal{D}_0^*$

To complete the proof of Theorem 1, it remains to exhibit the centralizer $C(f)$, for $f \in \mathcal{D}_0^*$. Let X be the smooth vector field such that $f^q = X^1$. Denote by $\{x_i\}$ the set of periodic points of f indexed in an increasing way for a cyclic order (according to the natural orientation of the circle $S^1 = \mathbb{R}/\mathbb{Z}$). The flow $\{X^t\}_{t\in\mathbb{R}}$ of X is contained in the centralizer $C(f)$. Let $h_0 = f \circ X^{-\frac{1}{q}}$; it belongs to $C(f)$, it acts on $Per(f)$ as f, and $h_0^q = id$. The group G_f generated by the flow of X and by h_0 is isomorphic to $\mathbb{R} \times \mathbb{Z}/q\mathbb{Z}$ and is contained in $C(f)$. We will prove:

Proposition 1.11. *With the notation above, $C(f)$ is isomorphic to $\mathbb{R}\times\mathbb{Z}/q\mathbb{Z}$ or $\mathbb{R}\times\mathbb{Z}/q\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. More precisely, $C(f)=G_f\simeq\mathbb{R}\times\mathbb{Z}/q\mathbb{Z}$, except in the special case where f preserves the orientation and has exactly two periodic orbits; in this special case, either $C(f)=G_f\simeq\mathbb{R}\times\mathbb{Z}/q\mathbb{Z}$ or $C(f)=G_f\times\mathbb{Z}/2\mathbb{Z}\simeq\mathbb{R}\times\mathbb{Z}/q\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.*

Consider now $g\in C(f)$. Let $Y^t=g\circ X^t\circ g^{-1}$. Then $\{Y^t\}_{t\in\mathbb{R}}$ is a one parameter group of C^1-diffeomorphisms commuting with f, and $f^q=Y^1$. One easily deduces that Y^t fixes every (oriented) segment $[x_i,x_{i+1}]$. As f^q is an affine contraction or dilation in the neighborhood of x_i and x_{i+1}, one also easily deduces that $Y^t=X^t$ for every t, so that g commutes with the flow of X. In particular, this shows that the group G_f is contained in the center of $C(f)$ (i.e. every element of G_f commutes with every element of $C(f)$).

The diffeomorphism g preserves the (finite) set $Per(f)$, so that $Per(f)$ consists in periodic orbits of g. Let $\ell>0$ be the smallest positive integer such that g^ℓ preserves the orientation and has a fixed point. This implies that every $x_i\in Per(f)$ is a fixed point of g^ℓ. As f^q is an affine contraction or dilation in a neighborhood of x_i, and g^ℓ commutes with f^q, it follows that g^ℓ is an affine map in the neighborhood x_i: in other words, g^ℓ agrees in the neighborhood of x_i with the time-t_i map X^{t_i} of the flow of X. As a consequence, g^ℓ agrees with X^{t_i} on the basin (stable or unstable manifold) of x_i. Since the basin of x_i meets the basin of x_{i+1}, an inductive argument shows that the time t_i does not depend on i. This proves that there exists a t such that $g^\ell=X^t$. Let $h_g=g\circ X^{-\frac{t}{\ell}}$. Then h belongs to $C(f)$, induces the same permutation of the periodic points of f as g, and (as g commutes with $X^{-\frac{t}{\ell}}$), we have $h_g^\ell=id$.

The proposition now follows from two remarks:

- Since the derivatives (at the period) of the periodic orbits $f\in\mathcal{D}_0$ are pairwise distinct, any diffeomorphism $g\in C(f)$ preserves each periodic orbit of f.
- An orientation preserving diffeomorphism of S^1 which is periodic and has a fixed point is the identity map.

We can now complete the proof:

<u>Case a:</u> **f reverses the orientation.** Then f has two fixed points. Assume $g\in C(f)$ preserves the orientation; then so does h_g. By the first remark, h_g fixes the fixed points of f. By the second remark, $h_g=id$, and so $g=X^{\frac{t}{\ell}}\in G_f$.

If g reverses the orientation, then $f^{-1}\circ g\in C(f)$ preserves the orientation: so there exists $r\in\mathbb{R}$ such that $g=X^r\circ f\in G_f$.

This shows that $C(f)=G_f\simeq\mathbb{R}\times\mathbb{Z}/2\mathbb{Z}$.

<u>Case b:</u> **f preserves the orientation and has at least 3 periodic orbits.** Then the periodic orbits of f are cyclically ordered. Since $g\in C(f)$ preserves every periodic orbit, it preserves this order, and hence g preserves the orientation.

Let $x \in Per(f)$. Then $h_g(x)$ belongs to the f-orbit of x, hence to the h_0 orbit of x. So there exists r such that $h_g \circ h_0^r$ has a fixed point. As h_0 belongs to the center of $C(f)$, it commutes with h_g. As h_0 and h_g are periodic we deduce that $h_g \circ h_0^r$ is periodic, hence is the identity map, because it is orientation-preserving and has a fixed point. So $h_g = h_0^r$. This proves that $g \in G_f$, and so $C(f) = G_f \simeq \mathbb{R} \times \mathbb{Z}/q\mathbb{Z}$.

<u>Case c:</u> f preserves the orientation and has exactly 2 periodic orbits. Notice that this case represents an open subset $\mathcal{D}_1 \subset \mathcal{D}_0^*$. We will prove that every $f \in \mathcal{D}_1$ satisfies $C(f) = \mathbb{R} \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $C(f) = G_f \simeq \mathbb{R} \times \mathbb{Z}/q\mathbb{Z}$ and that both behaviors are dense in $\mathcal{D}_1$. The argument in the previous case shows that every $g \in C(f)$ preserving the orientation belongs to G_f. The two possible behaviors depend on the existence of an orientation-reversing $g \in C(f)$. We will exhibit an invariant (of differentiable conjugacy) that vanishes if and only if $C(f)$ contains an orientation-reversing element.

The space of orbits of h_0 is a circle $S_0 = S^1/h_0$, and f induces on that circle a diffeomorphism $f_0 = f/h_0$ with exactly 2 fixed points: one attractor and one repellor; moreover, there are smooth coordinates on $S_0 \simeq \mathbb{R}/\mathbb{Z}$ such that the fixed points of f_0 are a source at 0 and a sink at $\frac{1}{2}$, and such that f_0 is affine in the neighborhood of its fixed points. We fix $\varepsilon > 0$ such that f_0 is affine on $[-\varepsilon, \varepsilon]$ and on $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$.

The vector field X induces on S_0 a vector field Y, invariant by f_0, and f_0^q is the time 1 map of Y. It follows that $f_0 = Y^{\frac{1}{q}}$.

For $r, s \in (0, \varepsilon]$ let $t^+(r,s)$ and $t^-(r,s)$ be the time length of the Y-orbit from r to $\frac{1}{2} - s$ and from $-r$ to $\frac{1}{2} + s$. One easily checks that $t^+(r,s) - t^-(r,s)$ does not depend on $r, s \in (0, \varepsilon]$; let $\theta_f = t^+(r,s) - t^-(r,s)$.

Lemma 1.12. *$C(f)$ contains an orientation-reversing element if and only if $\theta_f = 0$.*

Proof. If $g \in C(f)$ is orientation-reversing, then h_g projects on S_0 to a diffeomorphism commuting with Y and agreeing with $r \mapsto -r$ on $[-\varepsilon, \varepsilon)$ and with $\frac{1}{2} - s \mapsto \frac{1}{2} + s$ on $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$. This implies $t^-(r,s) = t^+(r,s)$. Conversely, if $\theta_f = 0$, then it is possible to build an orientation-reversing symmetry for the flow Y, agreeing with $r \mapsto -r$ on $[-\varepsilon, \varepsilon)$ and with $\frac{1}{2} - s \mapsto \frac{1}{2} + s$ on $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$. This symmetry lifts on S^1 to a element $g \in C(f)$. □

We conclude the proof of the proposition by proving

Lemma 1.13. *The subsets $\{f \in \mathcal{D}_1, \theta_f = 0\}$ and $\{f \in \mathcal{D}_1, \theta_f \neq 0\}$ are both dense in $\mathcal{D}_1$.*

The proof is very analoguous to the argument that forces the Mather invariant to vanish. One proves that C^1-small perturbations can give arbitrary values of θ_f.

If $\mathcal{D}_1^*$ denotes the second set that appears in lemma 1.13, then Theorem 1 is now proved with the set

$$\mathcal{D} = (\mathcal{D}_0^* \setminus \mathcal{D}_1) \cup \mathcal{D}_1^*.$$

1.6 Open questions on the circle

Theorem 1 presents a dense subset of $\mathrm{Diff}^1(S^1)$ of smooth diffeomorphisms whose centralizer is not trivial, and more precisely is isomorphic to a group $\mathbb{R} \times \mathbb{Z}/q\mathbb{Z}$, for some integer q. However, it is known that C^1-diffeomorphisms of S^1 may have very different centralizers. For instance, [FF] presents faithful actions on S^1 of any finitely generated nilpotent (non-abelian) group G. This nilpotent group G has a non-trivial center $Z(G)$, and for every element $f \in Z(G)$ of this center, the centralizer $C(f)$ contains the whole group G, hence is not abelian.

Conjecture 1. *The set of diffeomorphisms $f \in \mathrm{Diff}^1(S^1)$ with a non-abelian centralizer is dense in $\mathrm{Diff}^1(S^1)$.*

It could be interesting to build examples of diffeomorphisms with irrational rotation number and having a non-abelian centralizer.

Theorem 1 uses the fact that C^1-small perturbations allow us to change arbitrarily the Mather invariant of any smooth diffeomorphism of the interval $[0,1]$ whose fixed points are precisely 0 and 1. This proves that, in the set $D_{\alpha,\beta}$ we defined, every class of smooth conjugacy is C^1-dense. This suggests another question. For $\alpha > 1$ and $0 < \beta < 1$, consider the set $\hat{D}_{\alpha,\beta} \subset \mathrm{Diff}^1([0,1])$ of diffeomorphisms f whose fixed points are precisely 0 and 1 and such that $Df(0) = \alpha$ and $Df(1) = \beta$ (this set contains $D_{\alpha,\beta}$, and it differs from $D_{\alpha,\beta}$ because we do not require f to be affine in the neighborhood of 0 and 1). Notice that $\hat{D}_{\alpha,\beta}$ is invariant by conjugacy by orientation preserving diffeomorphisms of $[0,1]$.

Conjecture 2. *Every C^1-conjugacy class is dense in $\hat{D}_{\alpha,\beta}$. In other words, given any two elements $f, g \in \hat{D}_{\alpha,\beta}$, there is a diffeomorphism arbitrarily C^1-close to g that is conjugated to f by a diffeomorphism of $[0,1]$.*

A positive answer to this conjecture would allow us to show that every pathological behavior (in particular of the centralizer) built on a Morse-Smale example would appear densely in $\mathrm{Diff}^1(S^1)$. It would be interesting to understand the same question for diffeomorphisms with irrational rotation number:

Question VI. *Is every C^1-conjugacy class dense in the set of diffeomorphisms having a given rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$?*

2 Local embeddability in a flow on S^2

As in the one-dimensional case, the idea here is to measure how far certain diffeomorphisms of S^2 are from the time one map of a vector field. One obtains in this way a generalization of the Mather invariant, which in this setting is a diffeomorphism of $\mathbb{T}^2$. Such an invariant has already been constructed[3] in [AY] by V. Afraimovich and T. Young, and we now have to show that by a C^1-small perturbation of the dynamics, this invariant vanishes.

2.1 Preparation of diffeomorphisms in $\mathcal{O}$

Let $\mathcal{O}$ be as in Theorem 2. Let S^2 be the unit sphere in $\mathbb{R}^3$ endowed with the coordinates (x, y, z). We denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and the south poles of S^2. Notice that the coordinates x, y define local coordinates of S^2 in local charts U_N and U_S in neighborhoods of N and S.

The following straightforward lemma asserts that one may assume that the fixed points of any diffeomorphism f in the open set $\mathcal{O}$ are N and S and that the derivative at these points are conformal maps.

Proposition 2.1. *Consider a diffeomorphism $f \in \mathcal{O}$. Then there is a smooth diffeomorphism $h\colon S^2 \to S^2$ such that $h(N_f) = N$, $h(S_f) = S$ are the fixed points of $g = hfh^{-1}$; furthermore, the derivatives $D_N g$ and $D_S g$ are conformal linear maps, i.e., each of them is a composition of a rotation with a homothety of ratio $\alpha > 1$ and $\beta < 1$, respectively.*

Finally, any C^1-neighborhood of g contains a diffeomorphism $\tilde{g}$ such that there are neighborhoods $V_N \subset U_N$ and $V_S \subset U_S$ of N and S, respectively, such that the expression of $\tilde{g}$ in the coordinates (x, y) is $\tilde{g}(x, y) = D_N g(x, y)$ for $(x, y) \in V_N$ and $\tilde{g}(x, y) = D_S g(x, y)$ for $(x, y) \in V_S$.

2.2 Space of orbits of a conformal linear map

Let $A \in GL(\mathbb{R}, 2)$ be a conformal matrix of norm $\alpha \neq 1$. There exists $a \in [0, 2\pi)$ such that $A = R_a \circ h_\alpha$ where R_a is the rotation of angle a and b and h_α is the homothety of ratio α. Notice that, for all $n \in \mathbb{Z}$, the linear map A is the time one map of the vector field

$$X_{A,n} = \ln \alpha.(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + (a + 2\pi n).(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}).$$

The orbit space $T_A = \mathbb{R}^2 \setminus \{0\}/A$ (of the action of A on $\mathbb{R}^2 \setminus \{0\}$) is a torus (diffeomorphic to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$); we denote by π_A the canonical projection from

[3] In [AY], the authors write that the Mather invariant for a diffeomorphism of S^2 is always isotopic to the identity, but this is not correct (their Proposition 1 contains an error). For this reason, we choose here to build in detail the construction of this invariant on the sphere.

$\mathbb{R}^2 \setminus \{0\}$ onto T_A. Moreover, the vector fields

$$Z = 2\pi(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

and $X_{A,n}$ project on T_A to pairwise transverse commuting vector fields, which we also denote by Z and $X_{A,n}$; the orbits of both flows are periodic of period 1. Hence, for any pair $(Z, X_{A,n})$ there is a diffeomorphism $\mathcal{L}_{A,n} \colon T_A \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ sending Z to $\partial/\partial x$ and $X_{A,n}$ to $\partial/\partial y$; this diffeomorphism is unique up to composition with a translation of T^2. Furthermore, the diffeomorphisms $\mathcal{L}_{A,m} \circ \mathcal{L}_{A,n}^{-1}$ are affine maps of the torus T^2, for all $n, m \in \mathbb{Z}$, so that T_A is endowed with a canonical affine structure (indeed the affine map $\mathcal{L}_{A,m} \circ \mathcal{L}_{A,n}^{-1}$ on T^2 is the map induced by the matrix $\begin{pmatrix} 1 & n-m \\ 0 & 1 \end{pmatrix}$ composed with a translation).

Note that the orbits of Z correspond to the positive generator of the fundamental group of $\mathbb{R}^2 \setminus \{0\}$; we denote by σ the corresponding element of $\pi_1(T_A)$. Given any closed loop $\gamma \colon [0,1] \to T_A$, and any point $x \in \mathbb{R}^2 \setminus \{0\}$ with $\pi_A(x) = \gamma(0)$, there is a lift of γ to a path in $\mathbb{R}^2 \setminus \{0\}$ joining x to $A^k(x)$, where k is the algebraic intersection number of σ with γ. Finally, observe that the homotopy classes corresponding to the orbits of $X_{n,A}$, when $n \in \mathbb{Z}$ are precisely those whose intersection number with σ is 1: in other words, there is a basis of $\pi_1(T_A) = \mathbb{Z}^2$ such that $\sigma = (1,0)$ and the orbits of $X_{A,n}$ are homotopic to $(n,1)$.

2.3 A Mather invariant for diffeomorphisms of S^2

Denote by $D_{A,B} \subset \mathcal{O}$ the set of diffeomorphisms $f \in \mathcal{O}$ whose expression in the coordinates (x,y) coincides with some conformal linear maps A and B in neighborhoods U_f^N of N and U_f^S of S. The aim of this part is to build a Mather invariant for diffeomorphisms in $D_{A,B}$.

Consider $f \in D_{A,B}$. We retain the notations of the previous subsection and introduce the affine tori T_A and T_B. The orbit space $(S^2 \setminus \{N,S\})/f$ is a torus T_f and we denote by $\pi_f \colon S^2 \setminus \{N,S\} \to T_f$ the natural projection. Furthermore, as f agrees with A on U_f^N, the torus T_f may be identified with the torus T_A by a diffeomorphism $\pi_N \colon T_f \to T_A$, and in the same way, the fact that f coincides with B in a neighborhood of S induces a diffeomorphism $\pi_S \colon T_f \to T_B$.

Notice that the homomorphisms $\pi_{N*} \colon H_1(T_f,\mathbb{Z}) \to H_1(T_A,\mathbb{Z})$ and $\pi_{S*} \colon H_1(T_f) \to H_1(T_B)$ preserve the homology class of σ (corresponding to the positive homology generator of $S^2 \setminus \{N,S\}$ or of $\mathbb{R}^2 \setminus \{0\}$), and the homology intersection form with σ.

Consequently, for any $f \in D_{A,B}$, there is an integer $n(f)$ such that the map $\Delta_{f,0,0} = \mathcal{L}_{B,0} \circ \pi_S \circ \pi_N^{-1} \circ \mathcal{L}_{A,0}^{-1}$ is isotopic to the linear map of T^2 induced by the matrix $\begin{pmatrix} 1 & n(f) \\ 0 & 1 \end{pmatrix}$.

Lemma 2.2. *For any $f \in D_{A,B}$ there is a C^1-neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(S^2)$ such that for any $g \in \mathcal{U} \cap D_{A,B}$ one has $n(f) = n(g)$.*

Proof. Let us consider S^2 contained in $\mathbb{R}^3$. We can choose a neighborhood $\mathcal{U}$ such that, if $g \in \mathcal{U}$ then the map

$$f_t(x) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a smooth isotopy between f and g. Furthermore, by shrinking $\mathcal{U}$ if necessary, for any $g \in \mathcal{U}$, the isotopy f_t belongs to $\mathcal{O}$ (that is $\Omega(g) = \{N_g, S_g\}$).

If $g \in \mathcal{U} \cap D_{A,B}$ then there are discs D^N and D^S centered on N and S, respectively, such that $f_t = A$ on D^N and $f_t = B$ on D^S so that $f_t \in D_{A,B}$. In particular $f_t(D^S) \subset D^S$, and $f_t^{-1}(D^N) \subset D^N$. Furthermore, there exists $\ell > 0$ such that for any $x \in S^2 \setminus (D^N \cup D^S)$, $f_t^\ell(x) \in D^S$ and $f_t^{-\ell}(x) \in D^N$.

Let $x \in D^N$ such that $A(x) = f_t(x) \in D^N$ and $A^2(x) \notin D^N$. Hence $y_t = f_t^{\ell+2}(x) \in D^S$ and $f_t(y_t) = B(y_t) \in D^S$. Let γ be the segment of orbit of $X_{A,0}$ joining x to $A(x) = f_t(x)$, and let $\gamma_t = f_t^{\ell+2}(\gamma)$. For every t, γ_t is homotopic (relative to $\{y_t, B(y_t)\}$ in $S^2 \setminus \{N, S\}$) to a segment of orbit of $X_{B,n(f_t)}$. As a consequence, $n(f_t)$ varies continuously with t as t varies from 0 to 1. Hence $n(f_t)$ is constant; that is, $n(g) = n(f)$. □

Hence there is a partition of $D_{A,B}$ into open subsets $D_{A,B,n}$ such that $n(f) = n$ for $f \in D_{A,B,n}$. For $f \in D_{A,B,n}$, we define:

$$\Delta_f = \mathcal{L}_{B,n} \circ \pi_S \circ \pi_N^{-1} \circ \mathcal{L}_{A,0}^{-1}.$$

Then Δ_f is a diffeomorphism of T^2, isotopic to the identity.

Theorem 7 below justifies calling Δ_f *the Mather invariant of f.*

Theorem 7. *Let $f \in D_{A,B,n}$ be a smooth diffeomorphism such that Δ_f is a translation of the torus T^2. Then f leaves invariant two transverse commuting vector fields Z_f and X_f on S^2 such that $Z_f = Z$ in a neighborhood of $\{N, S\}$, $X_f = X_{A,0}$ in a neighborhood of N and $X_f = X_{B,n}$ in a neighborhood of S.*

As a consequence the centralizer of f is isomorphic to $S^1 \times \mathbb{R}$.

Proof. Fix two discs D^N and D^S centered at N and S, respectively, in which f coincides with A and B, respectively.

For any $x \neq S$ there exists $m(x) < 0$ such that $f^{m(x)}(x) \in D^N$. One defines $Z_f(x) = f_*^{-m(x)}(Z(f^{m(x)}))$ and $X_f(x) = f_*^{-m(x)}(X_{A,0}(f^{m(x)}))$. As Z and $X_{A,0}$ are invariant by A, one proves that the vectors $Z_f(x)$ and $X_f(x)$ are independent of the choice of $m(x)$. As a consequence, one deduces that they depend smoothly on $x \in S^2 \setminus \{S\}$ and that they commute on $S^2 \setminus \{S\}$. Furthermore the restrictions of Z_f and X_f to D^S are invariant by f, and hence by B, so that they induce two vector fields on T_B whose images by $\mathcal{L}_{B,n}$ are $\Delta_f(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$ and $\Delta_f(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$, respectively; that is, they agree with the projections of the restrictions Z and $X_{B,n}$ to D^S. Thus $Z_f = Z$ and $X_f = X_{B,n}$ on D^S, proving that the centralizer $C(f)$ contains the subgroup of linear conformal maps $\{Z_f^t \circ X_f^t,\ (t,s) \in S^1 \times \mathbb{R}\}$.

Conversely, the following easy lemma shows that any diffeomorphism $g \in C(f)$ agrees with a linear conformal map h in a neighborhood of S. Since g and h both commute with f, one deduces that g and h coincide on the whole sphere. Hence $C(f)$ is the group of conformal linear maps, which is isomorphic to $S^1 \times \mathbb{R}$. □

Lemma 2.3. *Let $B = R_b \circ h_\beta$, with $b \in [0, 2\pi)$ and $\beta \neq 1$ be a linear conformal contraction whose angle b is different from 0 and π. Then, any local diffeomorphism g defined in a neighborhood of 0 and that commutes with B coincides with a conformal linear map.*

2.4 Vanishing of the Mather invariant

This part is now very close to the 1-dimensional case: we consider $f \in D_{A,B,n}$, a disk D_f^S centered on S on which $f = B$ and want to perturb the homeomorphism Δ_f.

Let $h\colon S^2 \to S^2$ be a diffeomorphism whose support is contained in a disk $D \subset D_f^S$, disjoint from all $B^m(D)$ for $m > 0$. The disk D projects homeomorphically onto a disk $D' \subset T_B$, and finally onto a disk $\tilde{D} = \mathcal{L}_{B,n}(D') \subset T^2$. Let ψ be the diffeomorphism of T^2 with support in $\tilde{D}$ whose restriction to $\tilde{D}$ is the projection of h. We says that ψ is the projection of h on T^2 and conversely, that h is the lift of ψ with support in D.

Fix $k > 0$ such that D is disjoint from $B^k(D_f^S)$.

Lemma 2.4. *Consider a disk D and a diffeomorphism h as above. Fix $k > 0$ such that D is disjoint from $B^k(D_f^S)$. Then, $f \circ h$ is a diffeomorphism in $D_{A,B,n}$ with $B^k(D_f^S) \subset D_{f\circ h}^S$, and whose Mather invariant is*

$$\Delta_{f\circ h} = \psi \circ \Delta_f.$$

Corollary 2.5. *Let $D_0, \ldots, D_\ell \subset D_f^S$ be a finite sequence of disks such that*

- *for every i, the disk D_i is disjoint from $B^k(D_i)$ for $k > 0$;*
- *for all $i < j$ the disk D_i is disjoint from $B^k(D_j)$, $k \geq 0$*

For every i, let h_i be a diffeomorphism of S^2 with support in D_i, and let ψ_i be the projection of h_i on T^2 (by $\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f$).

Then the Mather invariant of $f \circ h_0 \circ \cdots \circ h_\ell$ is

$$\Delta_{f \circ h_0 \circ \cdots \circ h_\ell} = \psi_\ell \circ \cdots \circ \psi_0 \circ \Delta_f.$$

Reciprocally, for any disk $\tilde{D} \subset T^2$ with diameter strictly less than 1, each connected component of $(\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f)^{-1}(\tilde{D})$ projects diffeomorphically onto $\tilde{D}$, and f induces a permutation of these components. For $i > 0$, let D_i denote the (unique) component of $(\mathcal{L}_{B,n} \circ \pi_S \circ \pi_f)^{-1}(\tilde{D})$ such that $f^{-i}(D_i) \subset D_f^S$ but $f^{-(i+1)}(D_i)$ is not contained in D_f^S. For any diffeomorphism ψ with support in $\tilde{D}$ we will denote by $\theta_i(\psi)\colon S^2 \to S^2$ the lift of ψ with support in D_i.

The next lemma is the unique reason we required that the derivative of f at N, S be complex, hence conjugate to conformal linear maps:

Lemma 2.6. *Let $\tilde{D} \subset T^2$ be a disk with diameter strictly less than 1 and let $i, j \in \mathbb{N}$. Then:*

$$\sup_{x \in S^2} \|D_x\theta_i(\psi) - Id\| = \sup_{x \in S^2} \|D_x\theta_j(\psi) - Id\|.$$

Proof. $\theta_i(\psi)$ is conjugated to $\theta_j(\psi)$ by B^{j-i}, which is the composition of a homothety with a rotation; the C^1-norm is preserved by conjugacy by isometries, and by homotheties, and hence is preserved by the conjugacy by B^{j-i}. ☐

Corollary 2.7. *For any $\varepsilon > 0$ there is a C^1-neighborhood $\mathcal{V}_\varepsilon \subset Diff(T^2)$ of the identity map such that for any diffeomorphism $\psi \in \mathcal{V}_\varepsilon$ with support in a disk $\tilde{D} \subset T^2$ with diameter strictly less than 1, and for any $i \geq 0$, the lift $\theta_i(\psi)$ satisfies :*

$$\sup_{x \in S^2} \|D_x\theta_i(\psi) - Id\| < \varepsilon.$$

Definition 2.8. *Let $\psi_1, \ldots, \psi_\ell$ be ℓ diffeomorphisms of T^2 such that the support of every ψ_i is contained in a disk $\tilde{D}_i$ with diameter strictly less than 1; a* lift of the sequence *$\psi_1, \ldots, \psi_\ell$ is a sequence of lifts $h_1 = \theta_{i_1}(\psi_1), \ldots, h_\ell = \theta_{i_\ell}(\psi_\ell)$ such that, for every $i < j$ the support of h_i is disjoint from all the iterates $B^k(supp(h_j))$, for $k \geq 0$.*

It is straightforward to check that, for any sequence $\psi_1, \ldots, \psi_\ell$ of diffeomorphisms of T^2 such that the support of every ψ_i is contained in a disk $\tilde{D}_i$ with diameter strictly less than 1, the sequence $h_i = \theta_i(\psi_i)$ is a lift.

Proof of Theorem 2. Consider $f \in D_{A,B,n}$ and a C^1-neighborhood $\mathcal{U}$ of f. Fix $\varepsilon > 0$ such that, if $g_1, \ldots, g_m$, $m > 0$, are diffeomorphisms of S^2 with pairwise disjoint supports in $S^2 \setminus \{N, S\}$, and such that $\sup_{x \in S^2} \|Dg_i(x) - Id\| < \varepsilon$, then $f \circ g_1 \circ \cdots \circ g_m \in \mathcal{U}$. Let $\mathcal{V}_\varepsilon$ be the C^1-neighborhood of the identity map of T^2 given by Corollary 2.7.

Using Theorem 6, we write

$$\Delta_f = \psi_1^{-1} \circ \cdots \circ \psi_\ell^{-1},$$

for some $\ell > 0$, where $\psi_i \in \mathcal{V}_\varepsilon$, and the support of ψ_i is contained in a disk $\tilde{D}_i$ with diameter strictly less than 1. Let $(h_1, \ldots, h_\ell)$ be a lift of the sequence $(\psi_1, \ldots, \psi_\ell)$; the h_i satisfy

$$\sup_{x \in S^2} \|D_x h_i - Id\| < \varepsilon,$$

by our choice of $\mathcal{V}_\varepsilon$.

Our choice of $\varepsilon > 0$ implies that $g = f \circ h_1 \circ \cdots \circ h_\ell$ is a diffeomorphism belonging to $D_{A,B,n} \cap \mathcal{U}$. Furthermore, its Mather invariant is $\Delta_g = \psi_\ell \circ \cdots \circ \psi_1 \circ \Delta_f = Id$.

We have just shown that any $f \in D_{A,B}$ is the C^1-limit of a sequence $g_k \in D_{A,B}$ whose Mather invariant is the identity map; in particular, the centralizer of g_k is isomorphic to $\mathbb{R} \times S^1$.

Since by Proposition 2.1, $\mathcal{O}$ contains a dense set of diffeomorphisms smoothly conjugate to elements of $D_{A,B,n}$, any diffeomorphism in $\mathcal{O}$ is the limit of diffeomorphisms g_k that are the time 1 map of Morse-Smale vector fields, ending the proof of Theorem 2. $\square$

3 Local embeddability in a flow on surfaces

The aim of this section is to prove Theorem 3: we consider a closed connected surface S, and let $\mathcal{O} \subset \mathrm{Diff}^1(S)$ be the set of Morse-Smale diffeomorphisms as defined in the statement of Theorem 3. Since Morse-Smale systems are structurally stable, we have that $\mathcal{O}$ is a non-empty open subset of $\mathrm{Diff}^1(S)$.

Let $\mathcal{D}_1 \subset \mathcal{O}$ be the dense subset of $\mathcal{O}$ such that for every $f \in \mathcal{D}_1$ one has:

- every fixed point x of f admits a neighborhood U_x and smooth local coordinates defined on U_x such that the expression of f in restriction to $U_x \cap f^{-1}(U_x)$ is linear (hence coincides with the derivative $Df(x)$);

- given any two fixed points x, y of f, one has $\det Df(x) \neq \det Df(y)$.

The eigenvalues of every sink or source q of a diffeomorphism $f \in \mathcal{D}_1$ are non-real. Hence we can choose the local coordinates on U_q in such a way that the restriction of f to $U_q \cap f^{-1}(U_q)$ is a conformal linear map (i.e. the composition of a rotation with a homothety). In the same way we can choose local coordinates around any saddle p so that $f\,|_{U_p \cap f^{-1}(U_p)}$ is described by a diagonal matrix.

Proposition 3.1. *Given any $f \in \mathcal{D}_1$ and any C^1-neighborhood $\mathcal{U}$ of f, there is $g \in \mathcal{U}$ such that g is the time one map of a smooth vector field and agrees with f outside an arbitrarily small neighborhood of the sinks and sources of f. Furthermore, the centralizer $C(g)$ of g is isomorphic to $\mathbb{R}$.*

Proposition 3.1 clearly implies Theorem 3. The proof of this proposition is the aim of the rest of Section 3. In particular, up to Subsection 3.3, we prove that $C(g)$ contains $\mathbb{R}$, while in Subsection 3.4 we prove the equality.

We fix now a diffeomorphism $f \in \mathcal{D}_1$ and and a neighborhood $\mathcal{U}$ of f. Let σ_f, α_f and ω_f denote the sets of saddles, sources and sinks of f, respectively.

3.1 Vector field in a neighborhood of any saddle

Recall that by assumption σ_f is non-empty. In this section, we shall build an invariant neighborhood of σ_f endowed with a flow which will be our local model around saddles (and their invariant manifolds).

Proposition 3.2. *There exists an invariant open neighborhood V_0 and a vector field Y_0 on V_0 such that*

- *the flow of Y_0 is complete (i.e. defined from $-\infty$ to $+\infty$);*
- *the diffeomorphism f coincides with the time one map of Y_0 on V_0.*

Proof. We will use the following property satisfied by Morse-Smale diffeomorphisms in $\mathcal{O}$ (recall that by definition of $\mathcal{O}$ in Theorem 3 there are no heteroclinic orbits):

(*) *For any two saddles $p_1, p_2 \in \sigma_f$, there exist neighborhoods B_{p_1}, B_{p_2} of p_1 and p_2, respectively, such that there is no point $x \in S \setminus (U_{p_1} \cup U_{p_2})$ whose backward orbit intersects B_{p_1} and whose forward orbit intersects B_{p_2}.*

Let $p \in \sigma_f$ be any saddle periodic point. One may assume that in the local coordinates (x, y) of U_p, the expression of the map f is $(x, y) \mapsto (\lambda_p^u x, \lambda_p^s y)$. Since the eigenvalues λ_p^u, λ_p^s are positive, f agrees in U_p with the time one map of the vector field

$$Y_p(x, y) = x \ln \lambda_p^u \frac{\partial}{\partial x} + y \ln \lambda_p^s \frac{\partial}{\partial y}.$$

For $T > 0$ large, we introduce the octagon Δ_p defined by the equations:

$$|x| < (\lambda_p^u)^{-T},\ |y| < (\lambda_p^s)^{T},\ \frac{\ln |x|^{-1}}{\ln \lambda_p^u} + \frac{\ln |y|^{-1}}{\ln (\lambda_p^s)^{-1}} > 3T.$$

By property (*), the forward orbit of $f(\Delta_p) \setminus \Delta_p$ does not intersect Δ_p. Hence, one can extend the vector field Y_p to $V_p = \cup_{k \in \mathbb{Z}} f^k(\Delta_p)$, so that it is equal to $f_*^k(Y_p)$ on $f^k(\Delta_p) \setminus f^{k-1}(\Delta_p)$ and to $f_*^{-k}(Y_p)$ on $f^{-k}(\Delta_p) \setminus f^{-k+1}(\Delta_p)$ for any

$k > 0$. The open set V_p is invariant by the flow of Y_p, and by construction the restriction of f to V_p coincides with the time one map of Y_p.

Using (*) again, we deduce that if each domain Δ_p has been chosen small enough, then the open sets V_p are pairwise disjoint. Hence, one can define on the union $V_0 = \bigcup_{p \in \sigma_f} V_p$ a vector field Y_0 as required that coincides with Y_p on any V_p. $\square$

3.2 Vector field on the punctured surface $S \setminus (\omega_f \cup \alpha_f)$

In this section we prove the following proposition:

Proposition 3.3. *There exists a vector field Y defined on the punctured surface $S \setminus (\omega_f \cup \alpha_f)$ such that:*

- *the flow of Y is complete;*
- *the diffeomorphism f is the time one map of Y on $S \setminus (\omega_f \cup \alpha_f)$.*

Let $q \in \omega_f$ be a sink of f. In the local coordinates we fixed on U_q, the expression for the diffeomorphism f is given by a conformal matrix $B = R_b \circ h_\beta$ with $b \in [0, 2\pi)$ and $0 < \beta < 1$. Hence the results of Subsection 2.2 apply: f is in the neighborhood of q the time one map of each of the vector fields $X_{B,n}$ for any $n \in \mathbb{Z}$. We also defined the vector field $Z = 2\pi(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$.

Denote by π_q the projection of $W^s(q) \setminus \{q\}$ on the orbit space $T_q = (W^s(q) \setminus \{q\})/f$, which is a torus T^2. The vector fields $X_{B,n}$ and Z project to vector fields whose orbits are all periodic. We can choose a basis for the homology $H_1(T_q, \mathbb{Z})$ such that the class of the orbits of Z is $\sigma = (1, 0)$ and the class of the orbits of $X_{B,n}$ is $(n, 1)$.

Now consider an invariant open neighborhood V_0 and a vector field Y_0 on V_0 as given by Proposition 3.2. We can also choose a smaller neighborhood V of σ_f that is invariant by Y_0 (by taking small balls centered at each saddle and saturating by the flow of Y_0). We want to focus on the traces of V and V_0 on the orbit spaces T_q. We emphasize the following facts.

- The set $\pi_q(V_0 \cap W^s(q) \setminus \{q\})$ is foliated by the orbits of the projection $(\pi_q)_* Y_0$, which are closed, have period 1 and define the same non-zero homology class.
- As f is not a north-south diffeomorphism on the sphere, there is at least one unstable separatrix of a saddle p that is contained in the basin of q.
- The set $\pi_q(V \cap W^s(q) \setminus \{q\})$ is a neighborhood of the projection in T_q of the unstable separatrices that are contained in the basin of q. Hence it is compactly contained in $\pi_q(V_0 \cap W^s(q) \setminus \{q\})$ and invariant by the flow of $(\pi_q)_* Y_0$.

This implies the following:

Lemma 3.4. *The orbit space T_q can be endowed with a vector field Q_q such that*

- *the restriction of Q_q to $\pi_q(V \cap W^s(q) \setminus \{q\})$ coincides with $(\pi_q)_*(Y_0)$;*
- *all the orbits of Q_q are closed and of period* 1.

Each vector field Q_q lifts to a vector field Y_q on the open set $W^s(q) \setminus \{q\} = \pi_q^{-1}(T_q)$, and by construction coincides with Y_0 on the intersection of $W^s(q) \setminus \{q\}$ with V. Hence, we have defined a vector field Y on $V \cup \bigcup_{q \in \omega_f} W^s(q)$ whose time one map agrees with f. Any point y of $S \setminus (\omega_f \cup \alpha_f \cup V)$ is a wandering point and its ω-limit set is a sink (otherwise y would belong to one of the invariant manifolds of a saddle, hence to V, which is a contradiction). This shows that Y is now defined on the whole punctured surface $S \setminus (\omega_f \cup \alpha_f)$, proving Proposition 3.3.

3.3 Gluing the vector fields around sinks and sources

The aim of this section is to perform a small perturbation of f in a small neighborhood of the sinks and sources of f (but keeping f unchanged in a smaller neighborhood of the sinks and sources) so that the vector field Y provided by Proposition 3.3 can be extended to a smooth vector field on S.

Fix a sink $q \in \omega_f$ and keep the notations of the previous section. The dynamics in a neighborhood of q agree with those of a conformal linear map B. By projecting the vector field Y on the torus T_q, we obtain a vector field $\hat{Y}_q = (\pi_q)_*(Y)$. Each orbit of Y is a path joining a point y to $f(y)$. consequently, the orbits of $\hat{Y}_q$ on T_q are (simple) curves and are in the same homology class as the orbits of a vector field $\hat{X}_q$ obtained by projecting the vector field X_{B,n_q}, for some n_q.

Lemma 3.5. *There exists a smooth diffeomorphism $\psi_{f,q}$ of T_q that is isotopic to the identity map and such that $(\psi_{f,q})_*(\hat{Y}_q) = \hat{X}_q$.*

Proof. The orbits of $\hat{Y}_q$ and $\hat{X}_q$ are all periodic of period 1 and are in the same homology class. Let σ be a cross-section of $\hat{X}_p$ obtained by projecting an orbit of the vector field $Z = 2\pi(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$. One chooses a complete smooth cross section σ_q of $\hat{Y}_q$ that is in the same homology class as σ and cuts every orbit of $\hat{Y}_q$ in exactly one point. We also choose an orientation-preserving diffeomorphism $\psi_{f,q}\colon \sigma_q \to \sigma$. This diffeomorphism extends in a unique way in the announced diffeomorphism of the torus T_q. □

Remark 3.6.
- *The diffeomorphism $\psi_{f,q}$ is not unique: its depends on the choice of Y and of the cross section σ_q in the neighborhood of q.*

- *If $\psi_{f,q}$ is the identity map, then the vector field Y agrees with the linear vector fields X_q in a neighborhood of q.*

Given any $f \in \mathcal{D}_1$, we will perform a perturbation $g_q \in \mathcal{D}_1$ of f whose associated diffeomorphism $\psi_{g,q}$ is the identity map.

Proposition 3.7. *Consider $f \in \mathcal{D}_1$, a smooth complete vector field Y on $S \setminus (\alpha_f \cup \omega_f)$ such that f is the time one map of the flow of Y and a sink $q \in \omega_f$. For every C^1-neighborhood $\mathcal{U}$ of f, and for every neighborhood O of q there is $g_q \in \mathcal{U} \cap \mathcal{D}_1$ with the following properties:*

- *there is neighborhood $O' \subset O$ of q such that $g_q = f$ on $O' \cup (S \setminus O)$;*
- *there is a smooth complete vector field $\tilde{Y}$ defined on $S \setminus (\alpha_f \cup \omega_f) \cup \{q\}$ coinciding with Y outside O and such that g_q is the time one map of the flow of $\tilde{Y}$.*

One obtains the first part of Proposition 3.1 by applying Proposition 3.7 to f for each sink and to f^{-1} for each source, successively.

The argument for modifying $\psi_{f,q}$ is almost identical to the proof of Theorem 2 at Section 2.4, and we just sketch it.

Sketch of the proof of Proposition 3.7. By reducing O, we may assume that it is a small disk centered at q and contained in U_q. Let $D \subset O$ be a disk whose iterates $f^i(D)$, $i \in \mathbb{N}$, are pairwise disjoint and all contained in O. We denote by $\hat{D} = \pi_q(D)$ the projection of D on T_q. Notice that π_q induces a diffeomorphism from D to $\hat{D}$. Let h be a diffeomorphism of S coinciding with the identity map on the complement of D. We denote by $\hat{h}$ the diffeomorphism of T_q that is the identity map on the complement of $\hat{D}$ and is $\pi_q \times h \times \pi_q^{-1}$ on $\hat{D}$. Let $g_h = f \circ h$, and let Y_h be the vector field on $S \setminus (\alpha_f \cup \omega_f)$ that coincides with Y in the complement of $\bigcup_{i>0} f^i(D)$ and with $(f^i \circ h)_*(Y)$ on $f^i(Q)$ for $i > 0$. The following lemma is the analogue of Lemma 2.4.

Lemma 3.8. *With the notation above, one has:*

- *g_h belongs to $\mathcal{D}_1$ and agrees with f in a neighborhood of $Fix(f)$;*
- *g_h is the time one map of the flow of the vector field Y_h;*
- *$(\psi_{f,q} \circ \hat{h}^{-1})_*(\hat{Y}_h) = \hat{X}_q$, so that one can choose $\psi_{g_h,q} = \psi_{f,q} \circ \hat{h}^{-1}$.*

Proof. The unique difficulty here is to show that g_h is the time one map of Y_h. For that, let H be the diffeomorphism of $S \setminus (\alpha_f \cup \omega_f)$ that is $f^i \circ h \circ f^{-i}$ on $f^i(D)$ for $i > 0$ and the identity map in the complement of $\bigcup_{i>0} f^i(D)$. Note that g_h is conjugate to f by H and $Y_h = H_*(Y)$. □

Conversely, for any disk $\hat{D} \subset T_q$ with small diameter, the connected components of $\pi_q^{-1}(\hat{D})$ are diffeomorphic to $\hat{D}$; we denote by D_i the component that is contained in $B^i(O)$ but not in $B^{i+1}(O)$. For any diffeomorphism $\hat{h}$ of T_q, with support in the disk $\hat{D}$, we denote by $\mathcal{L}_i h$ the lift of $\hat{h}$ that is supported in D_i.

Let now consider a diffeomorphism $\psi_{f,q}$ of T_q associated to f and Y. Since $\psi_{f,q}$ is isotopic to the identity map, the fragmentation lemma (Theorem 6) allows us to write $\psi_{f,q}$ as the composition $\psi_{f,q} = \hat{h}_k \circ \cdots \circ \hat{h}_1$ of finitely many diffeomorphisms $\hat{h}_i$ arbitrarily C^1-close to the identity map and each with support in an arbitrarily small disk. We then build the lifts $h_i = \mathcal{L}_{2i}(\hat{h}_i)$ whose supports are pairwise disjoint. Let $g = f \circ h_1 \circ \cdots \circ h_k$. Then g belongs to $\mathcal{D}_1$ and agrees with f in the complement of O and on $f^{2k+3}(O)$. Furthermore, since the diffeomorphisms $\hat{h}_i$ can be chosen close to the identity, g belongs to $\mathcal{U}$. Finally, applying inductively Lemma 3.8, we see that g is the time one map of a vector field $\tilde{Y}$ that coincides with the linear vector field X_q in a neighborhood of q. Hence it may be extended smoothly at q, ending the proof. $\square$

3.4 End of the proof of theorem 3

Let $\mathcal{D}_2 \subset \mathcal{D}_1$ be the subset of diffeomorphisms that are the time one map of the flow of a smooth vector field. The previous sections proved that $\mathcal{D}_2$ is dense in $\mathcal{D}_1$, hence in the open subset $\mathcal{O}$ of $\mathrm{Diff}^1(M)$. In order to prove Theorem 3, we will first compute $C(f)$ for $f \in \mathcal{D}_2$.

Let $f \in \mathcal{D}_2$ be the time one map of a smooth vector field X, and consider $g \in C(f)$. By our assumptions on $\mathcal{D}_1$ (two different fixed points have different determinant), f and g have the same fixed points. Consider now any saddle p of f. The *unstable separatrices* are defined to be the connected components of $W^u(p) \setminus \{p\}$, and we denote them by $W^u_+(p)$, $W^u_-(p)$. Note that g preserves or exchanges the two separatrices of p.

Proposition 3.9. *Consider $f \in \mathcal{D}_2$. If $g \in C(f)$ preserves an unstable separatrix of a saddle $p \in \sigma_f$, then g belongs to the flow of X.*

As in the proof of Proposition 1.11 of Section 1.5, we deduce:

Corollary 3.10. *For $f \in \mathcal{D}_2$ the centralizer is either the flow of X (hence is isomorphic to $\mathbb{R}$) or is isomorphic to $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$.*

Every unstable separatrix of a saddle p is contained in the basin of a sink q of f. In this section, we endow each orbit space T_q with affine coordinates $(r, s) \in \mathbb{R}^2/\mathbb{Z}^2$ such that the vector field X projects to $\frac{\partial}{\partial s}$, and the vector field Z, whose expression in the local coordinates at q is $Z = 2\pi(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$, projects to $\frac{\partial}{\partial r}$. Note that the unstable separatrix of p is precisely one orbit of X and induces on T_q a circle $\{r\} \times \mathbb{R}/\mathbb{Z}$.

Proof of proposition 3.9. Assume that $g(W^u_+(p)) = W^u_+(p)$, for some saddle point p and let q be the sink of f whose basin contains this separatrix. The projection of the separatrix $W^u_+(p)$ on T_q will be denoted by $\{r_0\} \times \mathbb{R}/\mathbb{Z}$.

Since g commutes with f, it induces on T_q a diffeomorphism $g/_f$. By Lemma 2.3, g is locally the composition of an homothety and a rotation; hence the expression of $g/_f$ in the (r, s) coordinates is a translation: $(r, s) \mapsto (r + \alpha, s + \beta)$.

The fact that g leaves invariant $W^u_+(p)$ implies that $g/_f(\{r_0\} \times \mathbb{R}/\mathbb{Z}) = \{r_0 + \alpha\} \times \mathbb{R}/\mathbb{Z} = \{r_0\} \times \mathbb{R}/\mathbb{Z}$, so that $\alpha = 0$. In particular, this implies that $g/_f$ leaves invariant every orbit of $\frac{\partial}{\partial s}$. Consequently, g leaves invariant every X-orbit contained in the basin $W^s(q)$.

Let $q_0 = q$. Let q_1 be a source such that $W^u(q_1) \cap W^s(q_0) \neq \emptyset$. In the same way we endow the torus T_{q_1} with affine coordinates, and g induces on T_{q_1} a translation. The set $W^u(q_1) \cap W^s(q_0) \neq \emptyset$ is open and invariant by X; hence it contains an X-orbit, which is invariant by g. This orbit induces on T_{q_1} a circle of the form $\{r_1\} \times \mathbb{R}/\mathbb{Z}$, invariant by $g/_f$. This proves as before that g leaves invariant every X-orbit contained in $W^u(q_1)$.

Since S is connected and f is Morse-Smale, for every sink or source q there is a finite sequence $q_0, q_1, \ldots, q_n = q$ of alternating sources and sinks such that $W^u(q_i) \cap W^s(q_{i+1})$ or $W^s(q_i) \cap W^u(q_{i+1})$ is non empty for each $i = 0, \ldots, n-1$. The discussion above hence proves that g leaves invariant every X-orbit contained in the basin or a sink or of a source, hence leaves invariant every X-orbit.

This shows that for every point $x \in S \setminus Fix(f)$, there exists $t(x) \in \mathbb{R}$ such that $g(x) = X^{t(x)}(x)$. Furthermore, the continuous map $x \mapsto t(x)$ is locally constant in the punctured neighborhoods of the sinks and of the sources; hence (using the fact that g and X commute with f) the map t is constant on every basin of a sink or a source. Since t takes the same value on any two intersecting basins, it follows that $t(x)$ is constant on the complement of the fixed points. Thus g belongs to the flow of X. $\square$

In order to conclude the proof of Theorem 3, it remains to show that there is a dense subset $\mathcal{D} \subset \mathcal{D}_2$ of diffeomorphisms such that, for each $f \in \mathcal{D}$, $C(f)$ is precisely the flow of the corresponding vector field. We first note that for the existence of an extra symmetry, it is necessary that for any saddle $p \in \sigma_f$, the two unstable separatrices belong to the basin of a same sink q. Assuming that this topological condition is satisfied, the proof is very similar to the argument on the circle: we will exhibit a new invariant of differentiable conjugacy which vanishes if the diffeomorphism has an extra symmetry. We will then show that arbitrarily small perturbations allow us to modify this invariant.

We first define precisely this invariant. For every sink $q \in \omega_f$, we consider the orbit space T_q of $W^s(q) \setminus \{q\}$, with its structure of an affine torus. For every saddle p whose unstable manifold has a unique sink q in the ω-limit set, let $\{r_+(p)\} \times S^1$ and $\{r_-(p)\} \times S^1$ be the projections of the separatrices $W^u_+(p)$ and $W^u_-(p)$ on the affine torus T_q. We consider the distance $|r_+ - r_-| \in [0, \frac{1}{2}]$ between the two points $r_+, r_- \in S^1 = \mathbb{R}/\mathbb{Z}$.

With these notations, for every saddle p in σ_f we define

$$\delta^u(p) = \begin{cases} \frac{1}{2} - |r_+(p) - r_-(p)| \text{ if } W^u_+(p), W^u_-(p) \text{ have the same } \omega\text{-limit set;} \\ \\ +\infty \text{ otherwise.} \end{cases}$$

As we'll explain in the proof of Lemma 3.11, the number $\delta^u(p)$ measures whether the projections of the two separatrices on an affine torus can be exchanged by a rotation.

The invariant of the dynamics we will work with is defined by

$$\delta^u(f) = \sup_{p \in \sigma_f} \delta^u(p).$$

We conclude the proof of Theorem 3 with the following two lemmas.

Lemma 3.11. *Let f be in $\mathcal{D}_2$. If its centralizer $C(f)$ is isomorphic to $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$, then $\delta^u(f) = 0$*

Proof. Let g be the element of order 2; it exchanges the unstable separatrices of every saddle p of f. In particular, $W^u_+(p)$ and $W^u_-(p)$ are contained in the basin of the same sink q. Moreover, g projects on T_q to the translation by $(1/2, 0)$. This implies that we must have $\delta^u(p) = 0$ and we are done. □

Lemma 3.12. *For f in a dense subset $\mathcal{D} \subset \mathcal{D}_2$ we have $\delta^u(f) \neq 0$.*

Proof. Let $f \in \mathcal{D}_2$ be such that $\delta^u(f) = 0$ and let X be the flow associated with f. The unstable separatrices of any saddle p are contained in the basin of a sink q. By an arbitrarily small perturbation of X with support in the complement of a neighborhood of the fixed points, one can change the projection in T_q of any of the two separatrices. Then the time one map of the perturbed flow is a diffeomorphism $g \in \mathcal{D}_2$ arbitrarily close to f and such that $\delta^u(g) \neq 0$. □

4 Huge centralizers in dimension larger than 3

4.1 Reduction to the existence of periodic islands

Theorem 4 is a consequence of the next result:

Theorem 8. *Let M be a compact manifold of dimension $d \geq 3$. Then there is a non-empty open subset $\mathcal{O} \subset \mathit{Diff}^1(M)$ and a dense subset $\mathcal{D}_0 \subset \mathcal{O}$ such that any diffeomorphism $f \in \mathcal{D}_0$ has a periodic point x such that $Df^n(x) = Id \in GL(T_xM)$ where n is the period of x.*

Before explaining the proof of Theorem 8 we explain here why it implies Theorem 4.

Corollary 4.1. *Let M be a compact manifold of dimension $d \geq 3$. Then there is a non-empty open subset $\mathcal{O} \subset \mathit{Diff}^1(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that any diffeomorphism $f \in \mathcal{D}$ has the following property:*

there is an embedded ball $D_f \subset M$ of dimension d and an integer $n > 0$ such that $f^i(D_f) \cap D_f = \emptyset$ for $i \in \{1, \dots, n-1\}$ and the restriction of f^n to D_f is the identity map.

Proof. Given any f_0 in the set $\mathcal{D}_0$ given by Theorem 8, given any C^1-neighborhood $\mathcal{U}$ of f_0 and given any neighborhood V of the orbit $Orb(x, f_0)$, there exists $f \in \mathcal{U}$ that agrees with f in the complement of V, and such that f^n is the identity map in a neighborhood of x. □

End of the proof of Theorem 4. Consider f in the set $\mathcal{D}$ constructed in Corollary 4.1, and let D be a periodic ball of period n such that f^n coincides with the identity map on D (hence on the f-orbit of D). Let $\varphi\colon \mathbb{R}^d \to M$ be a smooth embedding such that $\varphi(\mathbb{D}^d) = D$. To any diffeomorphism $h \in \mathrm{Diff}^1(\mathbb{R}^d, \mathbb{R}^d \setminus \mathbb{D}^d)$ we associate $h_0\colon M \to M$, the diffeomorphism equal to the identity map in the complement of D and equal to $\varphi h \varphi^{-1}$ on D. For each $i \in \mathbb{Z}$ we set $h_i = f^i \circ h_0 \circ f^{-i}$; notice that h_i is a diffeomorphism of M with support contained in $f^i(D)$, and $h_{i+n} = h_i$. We denote by $h_\varphi\colon M \to M$ the diffeomorphism of M that coincides with h_i on $f^i(D)$, for every $i \in \mathbb{Z}$, and with the identity map in the complement of $\bigcup_i f^i(D)$.

By construction, h_φ commutes with f. Then $h \mapsto h_\varphi$ is an injective homomorphism from $\mathrm{Diff}^1(\mathbb{R}^d, \mathbb{R}^d \setminus \mathbb{D}^d)$ to $C(f)$. □

4.2 Existence of periodic orbits tangent to the identity map

The open set $\mathcal{O}$ produced in Theorem 8 and Corollary 4.1 is analoguous to those built in [BD]. We start by recalling some notions.

Let f be a diffeomorphism, and let $x \in Per_{hyp}(f)$ a hyperbolic periodic point. Given another hyperbolic periodic point y of f, we say that x and y are *homoclinically related* and we write $x \sim y$ if the stable and the unstable manifolds of the orbit of x transversely intersect the unstable and the stable manifolds of the orbit of y, respectively. Let $\Sigma(x, f) = \{y \in Per_{hyp}(f), y \sim x\}$. The *homoclinic class* $H(x, f)$ is the closure $H(x, f) = \overline{\Sigma(x, f)}$.

A point $x \in M$ is *chain recurrent* if for every $\delta > 0$ there exists a δ-pseudo-orbit $x = x_0, x_1, \dots, x_k = x$. The *chain recurrence class* $C(x, f)$ of a chain recurrent point x is the set of points y such that, for every $\delta > 0$, there is a δ-pseudo orbit starting at x and ending at y and a δ-pseudo orbit starting at y and ending at x.

For any periodic point $y \in Per(f)$, let $\pi(y)$ be its period, and let

$$J_f(y) = \frac{1}{\pi(y)} \log |DetDf^{\pi(y)}(y)|,$$

be the sum of the Lyapunov exponents of y.

Recall that an f-invariant set Λ admits a *dominated splitting* if there is an Df-invariant decomposition $TM|_\Lambda = E \oplus F$ of the tangent bundle TM over Λ as a direct sum of two invariant subbundles E and F such that:

- the dimension $dim(E(x))$ is independent on $x \in \Lambda$;
- the vectors in E are uniformly less expanded than the vectors in F; that is, there exists $N \in \mathbb{N}$ such that for any $x \in \Lambda$ and any non-zero vectors $u \in E(x)$ and $v \in F(x)$:

$$\frac{\|Df^N(u)\|}{\|u\|} < \frac{1}{2}\frac{\|Df^N(v)\|}{\|v\|}.$$

The bundles E and F of a dominated splitting are always continuous and extend continuously to a dominated splitting over the closure of Λ (elementary properties of dominated splitting are described in [BDV, Appendix B.1]). As a direct consequence, *if a set does not admit a dominated splitting, then the same holds for any dense subset of it.*

Now Theorem 8 follows from

Proposition 4.2. *Let M be a compact manifold of dimension $dim(M) \geq 3$. There is a non empty open subset $\mathcal{O} \subset \mathrm{Diff}^1(M)$ and a continuous function $f \in \mathcal{O} \mapsto x_f \in M$ such that, for every $f \in \mathcal{O}$:*

- *x_f is a hyperbolic periodic saddle point of f with $J_f(x) > 0$;*
- *there exists $y_f \in \Sigma(x_f, f)$ such that $J_f(y_f) < 0$;*
- *the chain recurrent class $C(x_f, f)$ does not admit a dominated splitting.*

We now deduce Theorem 8 from Proposition 4.2

Proof of Theorem 8. We just repeat briefly here the proof given in [BD]. Fix a diffeomorphism $f \in \mathcal{O}$, and a neighborhood $\mathcal{U} \subset \mathcal{O}$ of f. We will prove that $\mathcal{U}$ contains a diffeomorphism g having a periodic orbit whose derivative at the period is the identity.

According to [BC], for C^1-generic diffeomorphisms, the chain recurrent class of every periodic orbit is equal to its homoclinic class. Hence there exists $f_0 \in \mathcal{U}$ such that $C(x_{f_0}, f_0) = H(x_{f_0}, f_0)$.

For every $\varepsilon > 0$ we consider the set

$$\Sigma_\varepsilon(f_0) = \{y \in \Sigma(x_{f_0}, f_0), |J_{f_0}(y)| < \varepsilon\}.$$

Any two points in $\Sigma(x_{f_0}, f_0)$ are homoclinically related. As a consequence, given a finite set $X \subset \Sigma(x_{f_0}, f)$, there is a hyperbolic basic set of f_0 containing X. From this fact and from the hypotheses $J_{f_0}(x_{f_0}) > 0$ and $J_{f_0}(y_{f_0}) > 0$, we deduce that, for every $\varepsilon > 0$, the set $\Sigma_\varepsilon(f_0)$ satifies the two following properties:

1. the set $\Sigma_\varepsilon(f_0)$ is dense in $\Sigma(x_{f_0}, f_0)$, and hence in $H(x_{f_0}, f_0) = C(x_{f_0}, f_0)$; it follows that $\Sigma_\varepsilon(f_0)$ does not admit a dominated splitting;

2. the set $\Sigma_\varepsilon(f_0)$ admits *transitions* as defined in [BDP]. This is implied by the fact that, given any finite subset $X \subset \Sigma_\varepsilon(f_0)$, there is a hyperbolic basic set K_X containing X whose periodic orbits are contained in $\Sigma_\varepsilon(f_0)$: $K_X \cap Per(f_0) \subset \Sigma_\varepsilon(f_0)$.

Since $\Sigma_\varepsilon(f_0)$ admits transitions and does not admit a dominated splitting, [BDP] implies that, for every $\delta > 0$, there is a periodic point $x \in \Sigma_\varepsilon(f_0)$ and a δ-small perturbation g_0 of f_0 agreeing with f_0 on the orbit of x (and in the complement of an arbitrarily small neighborhood of the orbit of x) such that $Dg_0^{\pi(x)}(x)$ is an homothety, where $\pi(x)$ is the period of x. Notice that, for $\delta > 0$ small enough, we have $|J_{g_0}(x)| < 2\varepsilon$.

As ε can be chosen arbitrarily small, for $\varepsilon << \delta$ there is a δ-small perturbation g of g_0 coinciding with g_0 and f on the orbit of x (and in the complement of an arbitrarily small neighborhood of the orbit of x) such that $Dg^{\pi(x)}(x)$ is the identity map. For δ small enough, g belongs to $\mathcal{U}$ concluding the proof. $\square$

It remains to explain how one can build the open set $\mathcal{O}$ announced in Proposition 4.2. Our construction (as in [BD]) is based on the coexistence in a single homoclinic class of periodic orbits having complex eigenvalues of any rank. Let us explain this notion.

Let f be a diffeomorphism and x a periodic point of f of period $\pi(x)$. An *eigenvalue* of x is a (real or complex) eignevalue of the derivative $Df^{\pi(x)}(x)$. We say that x has a complex eigenvalue of rank $(i, i+1)$, for $i \in \{1, \dots, dim(M) - 1\}$, if there is a $Df^{\pi(x)}$-invariant splitting $T_xM = E \oplus F \oplus G$ such that:

- $dim(E) = i - 1$, $dim(F) = 2$, $dim(G) = dim(M) - i - 1$;

- the restriction of $Df^{\pi(x)}(x)$ to F has a pair of complex (non-real) conjugated eigenvalues; let λ be the modulus of these eigenvalues;

- the eigenvalues of the restrictions of $Df^{\pi(x)}(x)$ to E (resp. G) have modulus strictly less (resp. larger) than λ.

Lemma 4.3. *Let K be a f invariant set. Assume that K contains a periodic point x having a complex eigenvalue of rank $(i, i+1)$. Assume that $TM = E \oplus F$ is a dominated splitting on K. Then $dimE \neq i$.*

Proof. Just notice that $E(x)$ and $F(x)$ are $Df^{\pi(x)}(x)$-invariant, and that the eigenvalues corresponding to $E(x)$ are strictly less that those corresponding to $F(x)$. $\square$

Corollary 4.4. *Let K be an f-invariant set containing periodic points x_i having a complex eigenvalue of rank $(i, i+1)$ for every $i \in \{1, \ldots, dimM - 1\}$. Then K does not admit a dominated splitting.*

Hence Proposition 4.2 is a direct consequence of:

Proposition 4.5. *For any compact manifold M with $dimM \geq 3$, there is a non empty open subset $\mathcal{O} \subset Diff^1(M)$ and a continous map $f \in \mathcal{O} \mapsto x_f \in M$ such that, for every $f \in \mathcal{O}$:*

1. *the point x_f is a hyperbolic periodic point of stable index (dimension of the stable manifold) equal to 1;*
2. *$J_f(x_f) > 0$ and there is a hyperbolic periodic point y_f, homoclinically related to x_f, such that $J_f(y_f) < 0$*
3. *for every $i \in \{2, \ldots, dimM - 1\}$, there is a hyperbolic periodic point $x_{i,f}$ homoclinically related to x_f, and having a complex eigenvalue of rank $(i, i+1)$.*
4. *the chain recurrence class $C(x_f, f)$ contains a hyperbolic periodic point z_f of stable index equal to 2, and having a complex eigenvalue of index $(1, 2)$.*

Proof. The properties described in items (1)(2)(3) are open properties, and are easy to get by a local argument: we just describe how to obtain a finite number of periodic saddles of index 1 that are homoclinically related.

The property of item (4) comes from the notion of blender defined in [BD]. A blender Λ is:

- a (uniformly) hyperbolic compact set Λ that is also partially hyperbolic: there is an invariant dominated splitting $E^s \oplus E^u \oplus E^{uu}$ over Λ such that the dimension of E^u is equal to 1; here we assume that the dimension of the stable bundle E^s is 1, so that $dim(E^{uu}) = dimM - 2$. The partially hyperbolic structure extends to a neighborhood U of Λ; and
- an open region $V \subset U$ (called *the caracteristic region of the blender*), endowed with a cone field C^{uu} around a the bundle E^{uu};

with the following property. There exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$ and any ball $D^{uu} \subset U$ of dimension $dim(D^{uu}) = dim(E^{uu})$, tangent to the conefield C^{uu}, and crossing the region V, then D^{uu} meets the stable manifold of the continuation Λ_g of Λ for g. (See [BDV, Section6.2] for a more detailed discussion of the notion of blender, and references.)

Let f be a diffeomorphism having

- a blender Λ containing a periodic point x of index 1 and such that $J_f(x) > 0$,
- a hyperbolic periodic point z of index 2 such that:
  - the unstable manifold $W^u(z)$ crosses the caracteristic region of the blender, remaining tangent to the strong unstable cone field C^{uu},
  - the stable manifold $W^s(z)$ intersects transversely $W^u(x)$,
  - the stable eigenvalue of z is not real (hence z has a complex eigenvalue of rank $(1,2)$),
- for any $i \in \{2, ..., dimM - 1\}$, a hyperbolic periodic point x_i of index 1 homoclinically related to x and having a complex eigenvalue of rank $(i, i+1)$,
- a hyperbolic periodic point y homoclinically related to x and such that $J_f(y) < 0$.

All these properties are robust. Hence there is a small neighborhood $\mathcal{O}$ of f such that the continuations of the periodic points x, y, z, x_i and of the blender Λ are well-defined for every $g \in \mathcal{O}$ and satisfy all the properties above. We conclude by noting that, for every $g \in \mathcal{O}$, the continuation z_g of z belongs to the chain recurrence class of the continuation x_g. $\square$

5 Huge centralizers of symplectomorphisms

The aim of this section is to prove that the set of symplectomorphisms having a large centralizer is dense in $\mathcal{O} \subset \mathrm{Symp}^1_\omega(M)$, the open subset consisting of symplectomorphisms having a robustly totally elliptic periodic point. The argument is analogous to the idea of Theorem 4.

Proposition 5.1. *There is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that, for $f \in \mathcal{D}$ there is a ball $D \subset M$ and a integer $n > 0$ such that $D \cap f^i(D) = \emptyset$ for $i \in \{1, \dots, n-1\}$, and the restriction of f^n to D is the identity map.*

This proposition is a consequence of the two following classical lemmas, that can be easily obtained by considering generating functions.

Lemma 5.2 (Symplectic Franks Lemma). *Given a symplectomorphism f and a neighborhood $\mathcal{U} \subset Symp^1_\omega(M)$ of f, there exists $\varepsilon > 0$ with the following property. For every point $x \in M$, every neighborhood V of x, every symplectic linear isomorphism $A\colon T_xM \to T_{f(x)}M$ with $\|A - Df(x)\| < \varepsilon$, there exists $g \in \mathcal{U}$ such that:*

- *$g(x) = f(x)$ and g coincides with f in the complement of V;*

- $Dg(x) = A$.

Lemma 5.3 (Linearizing perturbation). *Given a symplectomorphism f, a neighborhood $\mathcal{U} \subset Symp^1_\omega(M)$ of f, a periodic point x of f of period n and a neighborhood V of the orbit of x, there exists $g \in \mathcal{U}$ such that:*

- *g coincides with f on the orbit of x and in the complement of V; in particular x is periodic for g;*
- *there is a neighborhood V_0 of x and a chart $\psi\colon V_0 \to \mathbb{R}^{2d}$, such that*
  - *$\psi(x) = 0 \in \mathbb{R}^{2d}$,*
  - *$\psi_*(\omega)$ is the canonical symplectic form on $\mathbb{R}^{2d}$,*
  - *the expression of g^n in this chart (i.e. the local symplectomorphism $\psi \circ g^n \circ \psi^{-1}$) is the symplectic linear map $D\psi(x) \circ Df^n(x) \circ D\psi(x)^{-1}$.*

Proof of Proposition 5.1. According to Lemmas 5.3 and 5.2 for every $f \in \mathcal{O}$, any neighborhood $\mathcal{U}$ of f and any robustly totally elliptic point x of f, there exists g in $\mathcal{U}$ agreeing with f on the orbit of x, and such that:

- $Dg^n(x)$ (where n is the period of x) is a totally elliptic matrix whose eigenvalues λ_i have the form $e^{\alpha_i 2i\pi}$ with $\alpha_i = \frac{p_i}{q_i}$, $p_i, q_i \in \mathbb{Z}$, $p_i \wedge q_i = 1$;
- there is a neighborhood V of x and a symplectic chart $\psi\colon V \to \mathbb{R}^{2d}$ such that $\psi(x) = 0 \in \mathbb{R}^{2d}$ and the expression of g^n in this chart is $Dg^n(x)$.

Let m be the smallest common multiple of the integers q_i. Then g^{nm} is the identity map in a neighborhood of x. It follows that g admits a periodic ball such that the return map is the identity map, concluding the proof. $\square$

End of the proof of Theorem 5. Now the proof of Theorem 5 is identical to the proof of Theorem 4: consider a diffeomorphism $f \in \mathcal{D}$ and a periodic ball D on which the first return map is the identity map. Shrinking D if necessary, we can assume that there is a symplectic chart $\psi\colon D \to \mathbb{R}^{2d}$ inducing a symplectomorphism from D to the standard ball $\mathbb{D}^{2d} \subset \mathbb{R}^{2d}$.

Let φ be a symplectic diffeomorphism of $\mathbb{R}^{2d}$ that is the identity map on the complement of $\mathbb{D}^{2d}$. Let g_0 be the diffeomorphism of M that is the identity map on the complement of D and that is $\psi^{-1}\varphi\psi$ on D, and let $g_n = f^n g_0 f^{-n}$. Then the diffeomophism g_φ coinciding with the identity map in the complement of the f-orbit of D and with g_i on $f^i(D)$ is a symplectomorphism of M commuting with f. $\square$

References

[AY] Afraimovich, V.; Young, T., *Mather invariants and smooth conjugacy on S^2*. J. Dynamical & Control Systems **3** (2000), 341–352.

[ABC] Arnaud, M.-C.; Bonatti, Ch.; Crovisier, S., *Dynamiques symplectiques génériques.* Ergod. Th. & Dynam. Sys. **25** (2005), 1401–1436.

[B] Banyaga, A., *On the structure of the group of equivariant diffeomorphisms.* Topology **16** (1977), no. 3, 279–283.

[BC] Bonatti, Ch.; Crovisier S., *Récurrence et généricité.* Invent. Math. **158** (2004), 33–104.

[BCW_1] Bonatti, Ch.; Crovisier S.; Wilkinson, A., *The centralizer of a C^1 generic diffeomorphism is trivial* (announcement). ArXiv:0705.0225. To appear in Electronic Research Announcements in Mathematical Sciences.

[BCW_2] Bonatti, Ch.; Crovisier S.; Wilkinson, A., *The centralizer of a C^1 generic diffeomorphism is trivial.* ArXiv:0804.1416.

[BCW_3] Bonatti, Ch.; Crovisier S.; Wilkinson, A., *C^1 genericity of trivial centralizers for conservative diffeomorphisms.* Journal of Modern Dynamics, vol 2 no 2, April 2008 p. 359-373.

[BD] Bonatti, Ch; Díaz, L.J., *On maximal transitive sets of generic diffeomorphisms.* Publ. Math. Inst. Hautes Études Sci. **96** (2002), 171–197.

[BDP] Bonatti, Ch; Díaz, L.J.; Pujals, E.R., *A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources.* Ann. of Math. **158** (2003), 355–418..

[BDV] Bonatti, Ch; Díaz, L.; Viana, M., *Dynamics beyond uniform hyperbolicity.* Encyclopedia of Math. Sci. **102**, Springer (2005).

[BGV] Bonatti, Ch.; Gourmelon, N.; Vivier, T., *Perturbations of the derivative along periodic orbits.* Ergod. Th. & Dynam. Sys. **26** (2006), 1307–1337.

[FF] Farb, B.; Franks, J., *Groups of homeomorphisms of one-manifolds. III. Nilpotent subgroups.* Ergod. Th. & Dynam. Sys. **23** (2003), 1467–1484.

[K] Kopell, N., *Commuting diffeomorphisms.* In *Global Analysis*, Proc. Sympos. Pure Math., Vol. XIV, AMS (1970) 165–184.

[M] Mather, J., *Commutators of diffeomorphisms.* Comm. Math. Helv. **48** (1973), 195–233.

[P] Palis, J, *Vector fields generate few diffeomorphisms.* Bull. Amer. Math. Soc. **80** (1974), 503–505.

[S_1] Smale, S., *Dynamics retrospective: great problems, attempts that failed.* Nonlinear science: the next decade (Los Alamos, NM, 1990). Phys. D **51** (1991), 267–273.

[S_2] Smale, S., *Mathematical problems for the next century.* Math. Intelligencer **20** (1998), 7–15.

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