PRODUCTS ON SPACES, REU 2007: LECTURE 9

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1. Connections between K-theory and Clifford Algebras

Recall that we can add vector bundles via fibre-wise direct sum. This makes $\mathrm{Vect}(X)$, the collection of vector bundles (up to isomorphism) over a space X, a monoid under $\oplus$. We then define $KO(X) = \mathrm{Gr}(\mathrm{Vect}(X), \oplus)$, the group completion of the monoid $\mathrm{Vect}(X)$. By considering fibrewise tensor product or vector bundles, $KO(X)$ becomes a ring with multiplication given by $\otimes$.

Similarly, let Mod_R be the set of finitely generated R-modules up to isomorphism over a ring R. This is a monoid under $\oplus$, so we can group complete via the Grothendieck construction. The resulting group is called $K(R)$, the *algebraic K-theory* of the ring R.

Let $M_n = \mathrm{Gr}(\mathrm{Mod}_{\mathrm{Cliff}(n)}, \oplus)$, for each Clifford algebra $\mathrm{Cliff}(n)$. By consider the irreducible modules over each Clifford algebra, we can determine the structure of M_n for each n. Let $i_n \colon \mathrm{Cliff}(n-1) \to \mathrm{Cliff}(n)$ be the canonical inclusion. This induces a map:

$$i_n^* \colon M_n \to M_{n-1}.$$

Let $A_n = \mathrm{coker}(i_n^* \colon M_n \to M_{n-1})$. First consider the case of $n = 1$. If $M \cong \mathbb{C}^k$ is a $\mathbb{C}$-module, then $i_1^* M \cong \mathbb{R}^{2n}$, considered as an $\mathbb{R}$-module. Thus the image of i_1^* is the even dimensional R-modules, or in other words, i_1^* is multiplication by 2 on $\mathbb{Z}$. Therefore $A_1 = \mathrm{coker}(i_1^*) = \mathbb{Z}/2\mathbb{Z}$. We can determine the other A_n similarly, and thus fill out our table:

n	$\mathrm{Cliff}(n)$	Irr. modules over $\mathrm{Cliff}(n)$	A_n	$\widetilde{KO}(S^n)$
0	$\mathbb{R}$	$F_0 = \mathbb{R}$		
1	$\mathbb{C}$	$F_1 = \mathbb{C}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{H}$	$F_2 = \mathbb{H}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{H} \oplus \mathbb{H}$	$F_3^+ = \mathbb{H}, F_3^- = \mathbb{H}$	0	0
4	$\mathbb{H}(2)$	$F_4 = \mathbb{H}^2$	$\mathbb{Z}$	$\mathbb{Z}$
5	$\mathbb{C}(4)$	$F_5 = \mathbb{C}^4$	0	0
6	$\mathbb{R}(8)$	$F_6 = \mathbb{R}^8$	0	0
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$F_7^+ = \mathbb{R}^8, F_7^- = \mathbb{R}^8$	0	0
8	$\mathbb{R}(16)$	$F_8 = \mathbb{R}^{16}$	$\mathbb{Z}$	$\mathbb{Z}$

The repeating pattern of the groups A_n is precisely the pattern of the (reduced) K-theory of the spheres S^n, for $n = 1, \ldots, 8$, as indicated in the table. Bott periodicity is the statement that $\widetilde{KO}(S^n)$ exhibits eightfold periodicity just like the A_n.

If $A \subset X$ is a "nice" subspace of a space X, and we have a vector bundle $p \colon E \to X$ such that the restriction of E to A is trivial, then we get a vector bundle $\tilde{p} \colon \widetilde{E} \to X/A$ over the quotient space. Here $\tilde{p}^{-1}(x) = p^{-1}(x)$ is $x \notin A$, while $\tilde{p}^{-1}(*) = V$, where $* \in X/A$ is the basepoint (the image of A under the quotient map $X \to X/A$), and $p^{-1}(a) = V$ for all $a \in A$. Define $KO(V, A)$ to be the ring of dimension zero virtual vector bundles over X trivial on A. Here a virtual vector bundle over (X, A) is a pair of bundles $E \to X$ and $E' \to X$ whose restrictions to A are isomorphic, and the dimension of a virtual vector bundles is the difference $\dim(E) - \dim(E')$. The process of forming the induced bundle over the quotient gives an isomorphism:

$$KO(X, A) \cong \widetilde{KO}(X/A).$$

In particular, we have:

$$KO(D^n, S^{n-1}) \cong \widetilde{KO}(S^n).$$

Starting with a $\mathrm{Cliff}(n-1)$-module M_0, which we can consider as a $\mathrm{Cliff}_0(n)$-module, we can extend to a full $\mathrm{Cliff}(n)$-module $M = M_0 \oplus M_1$ with the indicated $\mathbb{Z}/2\mathbb{Z}$-grading, where $M_0 \cong M_1$ as $\mathrm{Cliff}(n-1)$-modules. For $x \in S^{n-1} \subset \mathbb{R}^n \subset \mathrm{Cliff}(n)$, multiplication by x gives an isomorphism

$$\mu_x \colon M_0 \xrightarrow{\cong} M_1.$$

The two trivial bundles $E_0 = D^n \times M_0 \to D^n$, $E_1 = D^n \times M_1 \to D^n$ along with the isomorphism $\mu \colon E_0 \mid_{S^{n-1}} \cong E_1 \mid_{S^{n-1}}$ define an element of $KO(D^n, S^{n-1}) \cong \widetilde{KO}(S^n)$. This process yields a map

$$f \colon M_{n-1} \longrightarrow \widetilde{KO}(S^n).$$

If the $\mathrm{Cliff}(n-1)$-module M_0 had the structure of a $\mathrm{Cliff}(n)$-module, then we can extend the map $\mu \colon E_0 \mid_{S^{n-1}} \to E_1 \mid_{S^{n-1}}$ to a map $\mu \colon E_0 \to E_1$ by:

$$\mu(x, e) = (x, (x + \sqrt{1 - \|x\|^2} e_n)e).$$

Therefore,

$$f(\mathrm{im}\, i^*_{n-1}) = 0 \subset KO(D^n, S^{n-1}),$$

so we get an induced map:

$$\overline{f} \colon A_n \longrightarrow KO(D^n, S^{n-1}) \cong \widetilde{KO}(S^n).$$

Theorem 1.1 (Atiyah-Bott-Shapiro)**.** *The map $\overline{f}$ is an isomorphism.*