PRODUCTS ON SPACES, REU 2007: LECTURE 8

NOTES BY ANNA MARIE BOHMANN

Harry– you missed the proof of Artin's theorem, and we're not putting it in the notes.

1. Vector Bundles

Today we will discuss K-theory, a tool for studying vector bundles that has connections with our previous work on division algebras. Informally, a *vector bundle* is a map of topological spaces $p : E \to X$ such that $p^{-1}(x) \cong \mathbb{R}^n$.

Definition 1.1. An *n-dimensional vector bundle* over a space X is a map $p : E \to X$ such that for all $x \in X$, there is a vector space structure on $p^{-1}(x)$ and there exists an open set $U \subset X$ containing x and a map $h : p^{-1}(U) \to U \times \mathbb{R}^n$ taking $p^{-1}(y)$ to $\{y\} \times \mathbb{R}^n$ by a linear isomorphism for each $y \in U$.

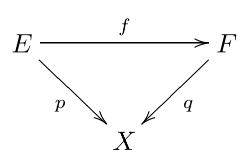
A good reference for vector bundles is Hatcher's unfinished book *K-theory and vector bundles*. It is available on his website.

Example 1. For any space X, the map $X \times \mathbb{R}^n \to X$ by projection onto the first factor makes $X \times \mathbb{R}^n$ a vector bundle. We call this a *trivial* vector bundle.

Example 2. If M is an n-manifold, then we can make the tangent bundle $T_M \to M$ where $p^{-1}(m)$ is the tangent space at m. This is an n-dimensional vector bundle.

Example 3. The Möbius bundle over S^1. Over each point on the circle, we place a copy of $\mathbb{R}$ so that is twists as we go around the circle. Basically, this is a Möbius band with each vertical bit replaced with a copy of $\mathbb{R}$.

Definition 1.2. An *isomorphism of vector bundles* is a commutative diagram



such that $f : p^{-1}(x) \to q^{-1}(x)$ is an isomorphism of vector spaces.

The subject of K-theory considers the question of how many vector bundles one can have over a given space, up to isomorphism. There are two important ways of creating new vector bundles from given ones which we now discuss.

Given vector bundles $E \xrightarrow{p} X$ and $F \xrightarrow{q} X$, we can make a new vector bundle

$$E \oplus F = \{(e, f) \mid e \in E, f \in F, p(e) = q(f)\}$$

Then $p \oplus q : E \oplus T \to X$ given by $(e, f) \mapsto p(e) = q(f)$ makes $E \oplus F$ a vector bundle. Note that for each $x \in X$, we have $(p \oplus q)^{-1}(x) \cong p^{-1}(x) \oplus q^{-1}(x)$. This is called the direct sum of the vector bundles E and F.

Similarly, we can form a vector bundle $E \otimes F$ with $(p \otimes q)^{-1}(x) = p^{-1}(x) \otimes q^{-1}(x)$, called the tensor product of E and F.

Example 4. Let $M \subset \mathbb{R}^N$ an n-manifold. The *normal bundle* $\nu_M \to M$ is the set

$$\nu_M = \{(m, \nu) \mid m \in M,\ \nu \in \mathbb{R}^N \text{ perpendicular to the tangent bundle at } m\}.$$

This is also a vector bundle over M. Furthermore, $T_M \oplus \nu_M \cong M \times \mathbb{R}^N$ is the trivial bundle.

Example 5. Let $E \to S^1$ be the Möbius bundle. Then $E \oplus E \to S^1$ is the trivial 2-dimensional bundle. Proof by pretty pictures.

Example 6. With E the Möbius bundle, $E \otimes E \to S^1$ is the trivial 1-dimensional vector bundle. This is because tensoring essentially "adds the twists" so that by the time you've come back around the circle, you've twisted back to where you started.

Example 7. If $\mathbb{K}$ is a normed division algebra, $\mathbb{K} \cong \mathbb{R}^n$, then we form the *canonical bundle*

$$E \to \mathbb{K}P^1 \cong S^1$$

by

$$E = \{(L, v) \mid L \in \mathbb{K}P^1 \text{ (thought of as line in } \mathbb{K}^2) \text{ and } v \in L\}.$$

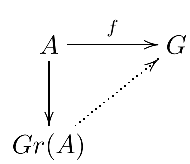
Then $p^{-1}(L) \cong \mathbb{K} \cong \mathbb{R}^n$, so this is an n-dimensional vector bundle.

This example gives the connection between our previous work on normed division algebras and K-theory.

2. K-theory and the Grothendieck Construction

Definition 2.1. Let $\mathrm{Vect}^n(X)$ be the collection of n-dimensional vector bundles over X, and let $\mathrm{Vect}(X)$ be the collection of isomorphism classes of vector bundles over X. Note that if X is not connected, we should allow the dimension of a vector bundle to be different over different connected components of X. Then $\mathrm{Vect}(X)$ is an associative and commutative semiring under $\oplus$ and $\otimes$. Recall that a semiring is a ring with no additive inverses.

Definition 2.2. Given a commutative monoid A (that is, a group without inverses), we can construct a group $Gr(A)$ called the *Grothendieck construction* on A such that the following universal property holds: For any group G and any map $f : A \to G$, there exists a unique map $\widetilde{f} : Gr(A) \to G$ making the diagram



commute.

Explicitly, we construct $Gr(A)$ as $Gr(A) = A \times A / \sim$ where we set $(a_1, b_1) \sim (a_2, b_2)$ if there exists $k \in A$ such that $a_1 + b_2 + k = a_2 + b_1 + k$. The intuition is to think of (a, b) as $a - b$, so we've just thrown in additive inverses for all the elements. Note that addition in $Gr(A)$ is componentwise.

Example 8. $Gr(\mathbb{N}) \cong \mathbb{Z}$.

Definition 2.3. The real K-theory of a space X is $KO(X) = Gr(\text{Vect}(X), \oplus)$, the Grothendieck construction on the monoid $\text{Vect}(X)$ under direct sum.

Using the multiplication on $\text{Vect}(X)$, we get a multiplication on $KO(X)$ by setting

$$(E_1, F_1)(E_2, F_2) = (E_1 \otimes E_2 + F_1 \otimes F_2, E_1 \otimes F_2 + E_2 \otimes F_1).$$

This makes $KO(X)$ into a commutative ring.

Similarly, we can define the *complex K-theory* $K(X)$ of a space by using complex vector spaces. The following theorem holds.

Theorem 2.4 (Bott periodicity)**.**

$$KO(S^{n+8}) \cong KO(S^n)$$
$$K(S^{n+2}) \cong K(S^n)$$

Pick $x_0 \in X$. Then we get a map $\epsilon : KO(X) \to \mathbb{Z}$ by $(E, F) \mapsto \dim p^{-1}(x_0) - \dim q^{-1}(x_0)$. Then define $\widetilde{KO}(X) = \ker(\epsilon)$.

Calculations that we will not do give us the following table:

n	1	2	3	4	5	6	7	8
$\widetilde{KO}(S^n)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
generated by canonical bundle over	$\mathbb{R}P^1$ (Möbius bundle)	$\mathbb{C}P^1$		$\mathbb{H}P^1$				$\mathbb{O}P^1$

To elucidate, note that Example 7 tells us that the canonical bundle over $\mathbb{O}P^1$, for instance, gives us a vector bundle over S^8; the claim here is that this bundle is nontrivial, and every vector bundle over S^8 is isomorphic to an n-fold sum of this bundle for some n so that this canonical bundle generates a copy of $\mathbb{Z}$ in $\text{Vect}(S^n)$. Note that in Examples 5 and 6 we have already in essence seen that the Möbius bundle generates a copy of $\mathbb{Z}/2$ in $\widetilde{KO}(S^1)$.