PRODUCTS ON SPACES, REU 2007: LECTURE 7

NOTES BY ANNA MARIE BOHMANN

1. Jordan Algebras and Trialities

Recall from last class that a Jordan algebra A is a commutative algebra satisfying

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2)$$

for all $a, b \in A$. A projection is an element $p \in A$ such that $p^2 = p$. We say that p has rank k if the longest chain

$$0 = p_0 < p_1 < \cdots < p_i = p$$

has $i = k$, where we define $p < q$ if $p \circ q = q \circ p = p$. A Jordan algebra is formally real if $a_1^2 + \cdots a_n^2 = 0$ implies $a_1 = \cdots = a_n = 0$ for all $a_i \in A$, and simple if it has no nontrivial ideals. Last time, we showed the following theorem.

Theorem 1.1. *If A is a formally real simple Jordan algebra, then $\mathbb{P}(A)$ is the set of rank one projections.*

In particular, $\mathbb{O}P^2 = \mathbb{P}(\mathfrak{h}_3(\mathbb{O}))$

In order to understand better what $\mathbb{O}P^2$ is, we will discuss trialities. First, we recall the definition of a duality.

Definition 1.2. A *duality* is a bilinear map $f : V_1 \times V_2 \to \mathbb{R}$ that is nondegenerate. That is, for $v_1 \neq 0$, $f(v_1, \cdot) : V_2 \to \mathbb{R}$ is nonzero as an element of V_2^*, and for $v_2 \neq 0$, $f(\cdot, v_2)$ is nonzero as an element of V_1^*.

Definition 1.3. A *triality* is a trilinear map

$$t : V_1 \times V_2 \times V_3 \to \mathbb{R}$$

that is nondegenerate. That is, $t(v_1, v_2, \cdot) : V_3 \to \mathbb{R}$ is nonzero for $v_1, v_2 \neq 0$, $t(v_1, \cdot, v_3)$ is nonzero for $v_1, v_3 \neq 0$ and $t(\cdot, v_2, v_3)$ is nonzero for $v_2, v_2 \neq 0$.

Dualizing, we get a map $m : V_1 \times V_2 \to V_3^*$. If we choose $e_1 \in V_1$ and $e_2 \in V_2$ nonzero, then we get maps $m(e_1, \cdot) : V_2 \to V_3^*$ and $m(\cdot, e_2) : V_1 \to V_3^*$, and saying that t is nondegenerate means that these maps become isomorphisms. We can thus identify V_1, V_2 and V_3 with a single vector space V and we get a multiplication $m : V \times V \to V$. This makes V is a division algebra. Similarly, a division algebra gives a triality via dualization.

Definition 1.4. A *normed triality* is a trilinear map

$$t : V_1 \times V_2 \times V_3 \to \mathbb{R}$$

of inner product spaces satisfying

$$|t(v_1, v_2, v_2)| \leq \|v_1\| \|v_2\| \|v_3\|$$

and such that given v_1, v_2, there exists $v_3 \neq 0$ with equality. Similarly given v_2, v_3 or v_1, v_3, there exist $v_1 \neq 0$ and $v_2 \neq 0$ such that equality holds.

A normed triality is equivalent to a normed division algebra in the same way a triality is equivalent to a division algebra.

2. Clifford Algebras, Trialities and $\mathbb{O}P^2$

We can make normed trialities out of Clifford algebras.

Let $\text{Cliff}_0(n) \subset \text{Cliff}(n)$ be the even subalgebra of $\text{Cliff}(n) = \text{Cliff}_0(n) \oplus \text{Cliff}_1(n)$.

Lemma 2.1. *The map* $\phi : \text{Cliff}(n-1) \to \text{Cliff}_0(n)$ *defined by* $\phi(e_i) = e_i e_n$ *for* $1 \le i \le n-1$ *is an isomorphism.*

Proof. Use a binomial coefficient argument to show that both sides have dimension $2^n - 1$. □

Recall that $\text{Cliff}(7) = \mathbb{R}[8] \oplus \mathbb{R}[8]$ and $\text{Cliff}(8) = \mathbb{R}[16]$. We have an irreducible $\text{Cliff}(8)$-module $P_8 \cong \mathbb{R}^{16}$. Let $V_8 = \mathbb{R}[8] \subset \text{Cliff}(8)$. We then get a map

$$V_8 \times P_8 \to \text{Cliff}(8) \times P_8 \to P_8$$

via the action of $\text{Cliff}(8)$ on P_8.

Consider P_8 as a $\text{Cliff}_0(8)$-module. As a $\text{Cliff}_0(8)$-module, P_8 decomposes as

$$P_8 \cong S_8^+ \oplus S_8^-$$

where S_8^+ and S_8^- are irreducible $\text{Cliff}_0(8)$-modules. Hence our map becomes

$$V_8 \times (S_8^+ \oplus S_8^-) \to S_8^+ \oplus S_8^-.$$

Since Cliff_8 acts on P_8, we have established that the even part sends S_8^+ to itself and same with S_8^-. That is,

$$V_8 \times S_8^+ \to S_8^-$$
$$V_8 \times S_8^- \to S_8^+$$

We claim that S_8^+ and S_8^- are self-dual as modules over $\text{Cliff}_0(8)$, so dualizing either of the above maps, we get a triality

$$t : V_8 \times S_8^+ \times S_8^- \to \mathbb{R}$$

This gives us an alternate construction of the octonians $\mathbb{O}$, since we can identify each vector space V_8, S_8^+, S_8^- with $\mathbb{R}^8$. The same constructions works to rediscover $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ using other Clifford algebras.

Moreover, we also get a map $\mathfrak{h}_3(\mathbb{O}) \xrightarrow{\cong} \mathbb{R}^3 \times V_8 \times S_8^+ \times S_8^1$ via sending

$$\begin{bmatrix} \alpha & z^* & y^* \\ z & \beta & x \\ y & x^* & \gamma \end{bmatrix} \to ((\alpha, \beta, \gamma), x, y, z)$$

The point is that the Jordan product on $\mathfrak{h}_3(\mathbb{O})$ is defined using only the maps

$$V_8 \times S_8^+ \to S_8^-$$
$$V_8 \times S_8^- \to S_8^+$$
$$S_8^+ \times S_8^- \to V_8$$

and inner products. As an exercise, check that this map is a well-defined isomorphism; this also proves that $\mathfrak{h}_3(\mathbb{O})$ is a Jordan algebra.

Theorem 2.2. *Any rank* 1 *projection in* $\mathfrak{h}_3(\mathbb{O})$ *has the form*

$$p = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} u^* & v^* & w^* \end{bmatrix}$$

where $[u, v, w] = 0$ *and* $\|u\|^2 + \|v\|^2 + \|w\|^2 = 1$.

Proof. Suppose $A = \left[\begin{smallmatrix} \alpha & z^* & y^* \\ z & \beta & x \\ y & x^* & \gamma \end{smallmatrix}\right]$ has trace 1 and $A^2 = A$. Note that projections must have trace 1 by a diagonalizaton argument. Then

$$A^2 = \begin{bmatrix} \alpha^2 + \|z\|^2 + \|y\|^2 & \alpha z^* + z^*\beta + y^*x^* & \alpha y^* + z^*x + y^*\gamma \\ z\alpha + \beta z + xy & \|z\| + \beta^2 + \|x\| & zy^* + \beta x + x\gamma \\ y\alpha + x^*z + \gamma y & yz^* + x^*\beta + \gamma x^* & \|y\| + \|x\| + \gamma^2 \end{bmatrix}$$

so $z = (\alpha+\beta)z + xy$ and $\alpha + \beta = 1 - \gamma$, so $z = (1-\gamma)z + xy$. and $\gamma z = xy$, if $\gamma \neq 0$. This and some more work implies that all the entries in A are in a subalgebra generated by two elements. We know that such a subalgebra is an associative normed division algebra, and so must sit inside the quaternions, that is, must be isomorphic to a subalgebra of $\mathbb{H}$. The theorem follows by the corresponding result for $\mathbb{H}$. □

The upshot is that for associative vector spaces, we can take any element to get something in the projective space; here we must take elements with associator 0.

Given $(x, y, z) \in \mathbb{O}^3 \setminus \{0\}$ with $[x, y, z] = 0$, we can normalize (via multiplication by real number) to get an element in $[x, y, z] \in \mathbb{O}P^2$. If $z \neq 0$, we then see that $[x, y, z] = [xz^{-1}, yz^{-1}, 1]$. This shows that $\mathbb{O}P^2$ is a 16-manifold covered by the three open sets

$$\{[x, y, 1]\}$$
$$\{[x, 1, z]\}$$
$$\{[1, y, z]\}.$$

We won't worry about the smooth structure on $\mathbb{O}P^2$, but it is a smooth manifold.

Finally, note that a rank 2 projection in $\mathfrak{h}_3(\mathbb{O})$ looks like $1 - p$ where p is a rank 1 projection. This gives a duality between points and lines in $\mathbb{O}P^2$.