PRODUCTS ON SPACES, REU 2007: LECTURE 6

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1. Projective Spaces and Jordan Algebras

Let K be an associative normed division algebra. Then we can define projective n-space in the standard way:

$$KP^n = (K^{n+1} \setminus \{0\})/x \sim \lambda x \text{ for } \lambda \in K \setminus \{0\} .$$

A hyperplane $H \subset KP^n$ of dimension k is the image of a $(k+1)$-dimensional vector subspace of K^{n+1} in KP^n. We may identify a hyperplane H with the projection map $P_H \colon K^{n+1} \to K^{n+1}$ with image H.

If $\{v_1, \ldots, v_{k+1}\}$ is an orthonormal basis for H, let:

$$V = \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_{k+1} \\ | & \cdots & | \end{bmatrix},$$

an $(n+1) \times (k+1)$ matrix. Then P_H is represented as a linear map by the matrix VV^*. Notice that $V^*V = I_{k+1}$, so:

$$(VV^*)^2 = VV^*VV^* = VI_{k+1}V^* = VV^*.$$

Thus VV^* is a projection, as required. Also, by diagonalizing, $\operatorname{tr}(VV^*) = k+1$.

Points of KP^n are one-dimensional subspaces L of K^{n+1}, which we may take to be spanned by a single vector $\overline{x} = (x_0, \ldots, x_n)$ with $\|\overline{x}\| = 1$. Thus using the associated matrix VV^* of the projection P_L, we can describe KP^n as:

$$KP^n \cong \left\{ \text{matrices of the form } \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_0^* & \cdots & x_n^* \end{bmatrix} = \begin{bmatrix} x_0x_0^* & \cdots & x_0x_n^* \\ \vdots & & \vdots \\ x_nx_0^* & \cdots & x_nx_n^* \end{bmatrix}, \right.$$
$$\left. \text{with } \|(x_0, \ldots, x_n)\| = 1 \right\}$$

Notice the matrices VV^* satisfy $(VV^*)^* = VV^*$.

An $n \times n$ matrix comprised of $A \in \mathrm{M}_n(K)$ such that $A^* = A$ is called a Hermitian matrix. Let $\mathscr{H}_n(K)$ denote the set of Hermitian matrices. Unfortunately, $\mathscr{H}_n(K)$ is not closed under matrix multiplication, so does not comprise an algebra over K with this product structure. We thus introduce the following product:

$$A \circ B = \tfrac{1}{2}(AB + BA).$$

Then for $A, B \in \mathscr{H}_n(K)$,

$$(A \circ B)^* = \tfrac{1}{2}(B^*A^* + A^*B^*) = \tfrac{1}{2}(BA + AB) = A \circ B.$$

Therefore $\mathscr{H}_n(K)$ is an algebra over K under this multiplication.

Definition 1.1. A *Jordan algebra* is a commutative (but not necessarily associative) algebra $(A, \circ)$ satisfying:

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2.$$

Lemma 1.2. *If $(A, \cdot)$ is an alternative algebra, then $(A, \circ)$ with $a \circ b = \frac{1}{2}(ab + ba)$ is a Jordan algebra.*

Proof. Notice that $(A, \circ)$ is commutative by the definition of $\circ$. For any $a, b \in A$,

$$a \circ (b \circ a^2) = a \circ \tfrac{1}{2}(ba^2 + a^2b) = \tfrac{1}{4}(a^3b + a^2ba + aba^2 + ba^3) = (a \circ b) \circ a^2.$$

Hence $(A, \circ)$ is Jordan. $\square$

Definition 1.3. A Jordan algebra A is *formally real* if:

$$a_1^2 + \cdots + a_n^2 = 0 \quad \text{implies that:} \quad a_1 = \cdots = a_n = 0.$$

Recall that an ideal B of a commutative algebra $(A, \circ)$ is a vector subspace strongly closed under mutplication:

$$\text{if } a \in A \text{ and } b \in B, \text{ then } a \circ b \in B.$$

A is *simple* if its only ideals are $\{0\}$ and A.

Theorem 1.4 (Jordan, von Neumann, Wigner 1934)**.** *Every formally real Jordan algebra is a direct sum of simple formally real Jordan algebras. Furthermore, the simple formally real Jordan algebras are determined up to isomorphism by the following list:*

- $\mathscr{H}_n(\mathbb{R})$ *for all* $n \geq 1$, *with multplication* $a \circ b = \frac{1}{2}(ab + ba)$
- $\mathscr{H}_n(\mathbb{C})$
- $\mathscr{H}_n(\mathbb{H})$
- $\mathbb{R}^n \oplus \mathbb{R}$, *with multplication* $(v, \alpha) \circ (w, \beta) = (\alpha w + \beta v, \langle v, w \rangle + \alpha\beta)$
- $\mathscr{H}_3(\mathbb{O})$

If A is a simple formally real Jordan algebra, we can define a projective space $\mathbb{P}(A)$ from it as follows. First of all, a *projection* in A is an element $p \in A$ with $p^2 = p$. For projections p, q, we say that $p < q$ if $p \neq q$ and:

$$p \circ q = q \circ p = p.$$

We also say that p has rank k if the maximal length of chains of projections

$$0 = p_0 < p_1 < \cdots < p_d = p$$

is $d = k$. Then define $\mathbb{P}(A)$ to be the set of rank one projections in A. A dimension k hyperplane in $\mathbb{P}(A)$ is defined to be a rank $k+1$ projection in A. One can then show that $\mathbb{P}(A)$ satisfies the axioms for a projetive space.

2. Construction of $\mathbb{OP}^1$

Any rank one projection in $\mathscr{H}_2(\mathbb{O})$ is given by:

$$A = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x^* & y^* \end{bmatrix} = \begin{bmatrix} \|x\|^2 & xy^* \\ x^*y & \|y\|^2 \end{bmatrix},$$

where $(x, y) \in \mathbb{O}^2$ and $\|(x, y)\| = \|x\|^2 + \|y\|^2 = 1$. Given $(x, y) \in \mathbb{O}^2 \setminus \{0\}$, we can normalize (x, y) to get a rank one projection, i.e. a point of $\mathbb{OP}^1$; call this point $[x, y]$. Notice that $[x, y] \neq [\lambda x, \lambda y]$ in general! However, if $y \neq 0$, then

$[x, y] = [xy^{-1}, 1]$ and if $x \neq 0$, then $[x, y] = [1, yx^{-1}]$. This yields two charts on $\mathbb{OP}^1$ which look like $\mathbb{O} \cong \mathbb{R}^8$ that glue together to yield a homeomorphism $\mathbb{OP}^1 \cong S^8$.

The careful reader will note that we have not discussed the topology of $\mathbb{OP}^1$, so this treatment is not really complete. However, by giving $\mathscr{H}_2(\mathbb{O})$ an appropriate function space topology (such as the compact-open topology) and letting $\mathbb{OP}^1$ inherit the quotient topology from $\mathscr{H}_2(\mathbb{O})$, these worries may be avoided. We would rather give the general algebraic idea of how to construct projective space over the Octonions then worry about all of the fussy details.