

## PRODUCTS ON SPACES, REU 2007: LECTURE 5

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### 1. THE CLASSIFICATION OF NORMED DIVISION ALGEBRAS

We will use our understanding of Clifford algebras to prove the desired result about normed division algebras over  $\mathbb{R}$ : they exist only in dimensions 1, 2, 4 and 8.

Suppose  $K \cong \mathbb{R}^n$  is a normed division algebra. If  $a \in \text{Im}(K)$ , consider the “left multiplication by  $a$ ” map:

$$L_a: K \longrightarrow K, x \mapsto ax.$$

Then  $(L_a)^2(x) = a(ax) = (a^2)x = -\|a\|^2x$ . This should remind us of the universal property for Clifford algebras. We think of  $L_a$  as an  $n \times n$  matrix over  $\mathbb{R}$ . Then the linear map

$$\begin{aligned} L: \text{Im}(K) &\longrightarrow M_n(\mathbb{R}), \\ a &\longmapsto L_a \end{aligned}$$

satisfies  $L(a)^2 = -\|a\|^2$ . Therefore by the universal property of Clifford algebras,  $L$  uniquely extends to a ring map:

$$\text{Cliff}(\text{Im}(K)) \longrightarrow M_n(\mathbb{R}).$$

Thus  $\mathbb{R}^n$  has the structure of a  $\text{Cliff}(\text{Im}(K))$ -module via this map.

Recall (or take for granted) the fact that any finitely generated module  $M$  over  $M_n(F)$  for a field  $F$  is a direct sum of finitely many copies of  $F^n$  (where  $M_n(F)$  acts on  $F^n$  in the natural way that matrices do):

$$M \cong \bigoplus F^n.$$

Also, any finitely generated module  $M$  over  $M_n(F) \oplus M_n(F)$  looks like:

$$M \cong M_1 \oplus M_2,$$

where  $M_1$  and  $M_2$  are  $M_n(F)$ -modules with the first copy of  $M_n(F)$  acting on  $M_1$  and the second copy acting on  $M_2$ . This follows by projecting  $M$  onto  $M_1$  and  $M_2$  via the action of  $(I, 0)$  and  $(0, I)$ , respectively. This enables us to fill in our table of Clifford algebras with a list of all irreducible modules of each Clifford algebra:

$n$	$\text{Cliff}(n)$	Irr. modules over $\text{Cliff}(n)$
0	$\mathbb{R}$	$F_0 = \mathbb{R}$
1	$\mathbb{C}$	$F_1 = \mathbb{C}$
2	$\mathbb{H}$	$F_2 = \mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$	$F_3^+ = \mathbb{H}, F_3^- = \mathbb{H}$
4	$\mathbb{H}(2)$	$F_4 = \mathbb{H}^2$
5	$\mathbb{C}(4)$	$F_5 = \mathbb{C}^4$
6	$\mathbb{R}(8)$	$F_6 = \mathbb{R}^8$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$F_7^+ = \mathbb{R}^8, F_7^- = \mathbb{R}^8$
8	$\mathbb{R}(16)$	$F_8 = \mathbb{R}^{16}$

In our case, with the normed division algebra  $K \cong \mathbb{R}^n$ ,  $\text{Im}(K)$  has dimension  $n - 1$ , so  $\mathbb{R}^n$  is a  $\text{Cliff}(n - 1)$ -module. Examining the table, we see that this can only occur for  $n = 1, 2, 4$  or  $8$ . Since we know that  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  and  $\mathbb{O}$  exhibit normed division algebras in these dimensions, this concludes the proof of the classification of normed division algebras over  $\mathbb{R}$ .

## 2. SOME PROJECTIVE GEOMETRY

With an eye towards the construction of  $\mathbb{O}P^2$ , the octonionic projective plane, we'll first discuss the notions of projective geometry in general.

If  $K = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , we define projective  $n$ -space over  $K$  by:

$$KP^n = (K^{n+1} \setminus \{0\})/x \sim \lambda x \text{ for } \lambda \in K \setminus \{0\}.$$

Unfortunately, if  $K = \mathbb{O}$ ,  $x \sim \lambda x$  is *not* an equivalence relation. Transitivity fails because  $\mathbb{O}$  is not associative: if  $x \sim \lambda x \sim \mu(\lambda x)$ , then  $x \sim \mu(\lambda x)$  if and only if  $\mu(\lambda x) = (\mu\lambda)x$ , which does not hold in general. Nevertheless,  $\mathbb{O}P^1$  and  $\mathbb{O}P^2$  exist, while  $\mathbb{O}P^n$  does not exist for  $n \geq 3$ .

Philosophically, what is a projective space? In the Euclidean plane, two lines almost always intersect in a point, but occasionally they do not intersect, precisely when they are parallel. This suggests that Euclidean space needs to be completed in some sense, so that the following propositions hold without qualification:

- Given a pair of distinct lines, there exists a unique point on both.
- Given a pair of distinct points, there exists a unique line containing both points.
- There exists four points such that no three are on the same line.
- There exist four lines such that no three contain the same point.

We take these four propositions as axioms for the projective plane. To construct the projective plane, add a point at infinity for each "direction" in the plane. Then these four axioms hold by decreeing that each line in the plane intersects the point at infinity corresponding to the direction it points in. Notice the duality between points and lines in the axioms.

We can similarly give axioms for  $n$ -dimensional projective space, by specifying incidence relations between hyperplanes of appropriate dimension, but we will not do this explicitly here. We'll also take the following result for granted:

**Theorem 2.1.** *Any projective space of dimension  $\geq 3$  is  $K\mathbb{P}^n$  for some skew field  $K$ , i.e. an associative (but not necessarily commutative) ring with multiplicative inverses.*

Given  $KP^n$ , define:

$$E = \{(L, x) : L \in KP^n \text{ is a subspace of } K^{n+1} \text{ of dimension one, and } x \in L\}.$$

Notice that we have a natural map  $p: E \rightarrow KP^n$  give by  $(L, x) \mapsto L$ . Notice that  $p^{-1}(L) \cong K$  as a vector space. This is an example of a *vector bundle*, one the most powerful and pervasive ideas in modern mathematics. In general, a vector bundle is a map  $p: E \rightarrow B$ , with  $E$  the *total space* and  $B$  the *base space* such that for each  $b \in B$ ,  $p^{-1}(b)$  is a vector space of the same dimension, called the *fibre*. There is an additional technical requirement which states that  $E$  is a “twisted product” of the base space  $B$  with the fibre. The simplest, and most trivial example is a product bundle  $p: B \times F \rightarrow B$ ,  $(b, v) \mapsto b$ . There is a more general notion of a *fibre bundle* where the fibre  $F$  need only be a topological space, not a vector space. For more on fibre bundles, see the following references: Steenrod’s “The topology of Fibre Bundles”, Husemoller’s “Fibre Bundles”, and Milnor’s “Characteristic Classes”.

Returning to our discussion, set:

$$S(E) = \{(L, x) : L \in KP^n, x \in L, \|x\| = 1\}.$$

If  $K \cong \mathbb{R}^n$ , then  $\{x \in K : \|x\| = 1\} = S^{n-1}$ . Hence the fibre of the bundle  $S(E) \rightarrow KP^n$  is  $p^{-1}(L) \cong S^{n-1}$ . For this reason, such bundles are called *sphere bundles*. In particular, with  $KP^1 = S^1, S^2, S^4$  or  $S^8$ , corresponding to  $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$  or  $\mathbb{O}$ , we have sphere bundles:

$$\begin{aligned} S^0 &\longrightarrow S(E) \longrightarrow S^1 \\ S^1 &\longrightarrow S(E) \longrightarrow S^2 \\ S^3 &\longrightarrow S(E) \longrightarrow S^4 \\ S^7 &\longrightarrow S(E) \longrightarrow S^8. \end{aligned}$$

But in these cases,  $S(E)$  itself a sphere in  $K^2$ , so the above bundles *are entirely comprised of spheres*:

$$\begin{aligned} S^0 &\longrightarrow S^1 \xrightarrow{\iota} S^1 \\ S^1 &\longrightarrow S^3 \xrightarrow{\eta} S^2 \\ S^3 &\longrightarrow S^7 \xrightarrow{\nu} S^4 \\ S^7 &\longrightarrow S^{15} \xrightarrow{\sigma} S^8. \end{aligned}$$

The projections  $\iota, \eta, \nu$  and  $\sigma$  are called the *Hopf maps*, and the bundles are called *Hopf bundles*.

If we take points  $x \neq y$  of  $S^2$ , then  $\eta^{-1}(x)$  and  $\eta^{-1}(y)$  will be circles in  $S^3$  that are linked once, like the links comprising a metal chain. Similarly,  $\nu^{-1}(x)$  and  $\nu^{-1}(y)$  are two linked 3-spheres in  $S^7$ , and  $\sigma^{-1}(x)$  and  $\sigma^{-1}(y)$  are two linked 7-spheres in  $S^{15}$ .

We say that a map  $S^{2n-1} \rightarrow S^n$  with fiber  $S^k$  has Hopf invariant one if the inverse images of distinct points are “once-linked”  $k$ -spheres, as in the above examples. Of course, we have not gotten technical with the details, but we hope the reader will indulge him or herself to just grok the concept.

J.F. Adams proved the following result in the 1970’s to great acclaim:

**Theorem 2.2.** *There exists a Hopf invariant one map  $S^{2n-1} \rightarrow S^n$  only for  $n = 1, 2, 4$  or  $8$ .*