REU CLASS BY VIGLEIK ANGELTVEIT: LECTURE 4

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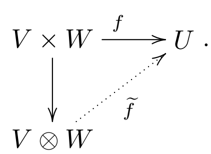
1. Clifford Algebras

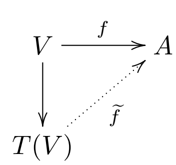
Given vectors spaces V and W over $\mathbb{R}$, we form their *tensor product* $V \otimes W$. Recall that we have two ways to describe $V \otimes W$, either with a universal property or as an explicit construction. The universal property for the tensor product $V \otimes W$ is that there exists a bilinear map $V \times W \to V \otimes W$, and given any bilinear map $f : V \times W \to U$, there exists a unique linear map $\widetilde{f} : V \otimes W \to U$ making the following diagram commutes:



On the other hand, the explicit construction of $V \otimes W$ gives us the following: If V has basis $\{v_1, \ldots, v_n\}$ and W has basis $\{w_1, \ldots, w_m\}$, then $V \otimes W$ has basis $\{v_i \otimes w_j\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. In particular, $\dim V \otimes W = mn$.

Now, given a vector space V, we can form an associative algebra, called the tensor algebra of V.

Definition 1.1. The *tensor algebra* $T(V)$ of a vector space V is an associative algebra satisfying the following universal property: There exists a linear map $V \to T(V)$ and given a map $f : V \to A$ for an associative algebra A, there exists a unique algebra map $\widetilde{f}$ so that the diagram



commutes.

We can construct $T(V)$, which shows that there indeed exists something satisfying the universal property of the definition. Explicitly, let

$$T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \otimes (V \otimes V \otimes V) \oplus \cdots ,$$

or more concisely, $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$. We note that any element in $T(V)$ must live in $V^{\otimes n}$ for some n, and give $T(V)$ an algebra structure by defining

$$(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n.$$

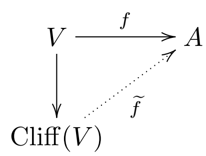
We use the tensor algebra to make the following definition.

Definition 1.2. Given a vector space V with an inner product, the *Clifford algebra* $\mathrm{Cliff}(V)$ is the quotient of the tensor algebra

$$T(V)/(v \otimes v \sim -\|v\|^2)$$

where $\|v\|^2 = \langle v, v\rangle$.

This definition satisifies the universal property that for all associative algebras A and maps $f : V \to A$ such that $f(v)^2 = -\|v\|^2$, there exists a unique algebra map $\widetilde{f} : \mathrm{Cliff}(V) \to A$ making the diagram



commute.

The defining relations $v \otimes v = -\|v\|^2$ allow us to derive a second relation that also holds in the Clifford algebra. Recall that for all $v, w \in V$, we have

$$\langle v, w\rangle = \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2).$$

This equation allows us to calculate

$$\begin{aligned}(v + w)^2 &= v \otimes v + v \otimes w + w \otimes v + w \otimes w \\ &= -\|v + w\|^2 \\ &= -\|v\|^2 - \|w\|^2 - 2\langle v, w\rangle\end{aligned}$$

and so we have

$$v \otimes w + w \otimes v = -2\langle v, w\rangle.$$

This relation gives an equivalent way to define $\mathrm{Cliff}(V)$. For ease of notation, we write $\mathrm{Cliff}(n) = \mathrm{Cliff}(\mathbb{R}^n)$.

Lemma 1.3. *If V has dimension n, then $\mathrm{Cliff}(V)$ has dimension 2^n. If V has basis $\{v_1, \dots, v_n\}$, then $\mathrm{Cliff}(V)$ has basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 0 \le k \le n,\ i_1 < \cdots < i_k\}$.*

As an example, note that if $n = 2$, $\mathrm{Cliff}(V)$ has basis $\{1, v_1, v_2, v_1 \otimes v_2\}$.

Proof. Any element in $T(V)$ is a linear combination of $x = v_{i_1} \otimes \cdots \otimes v_{i_k}$, without conditions on k or the i_j's. We show, by induction on k, that any such x is a linear combination of the claimed basis in $\mathrm{Cliff}(V)$. The base case is trivial.

Let $x = v_{i_1} \otimes \cdots \otimes v_{i_k}$. If $v_{i_j} = v_{i_{j+1}}$ for some j, that is, we have a repeated vector, we use the relation $v_{i_j} \otimes v_{i_j} = -\|v_{i_j}\|^2$ to write x with $k-2$ tensor factors. If $i_j > i_{j+1}$ for some j, we use relation $v \otimes w = -w \otimes v - 2\langle v, w\rangle$ to switch v_{i_j} and $v_{i_{j+1}}$. This gives us an extra term with $k-2$ tensor factors.

Linear independence of our claimed basis is left as an exercise. □

The previous lemma allows us to make the following calculations.

- $\mathrm{Cliff}(0) = \mathbb{R}$, with basis $\{1\}$
- $\mathrm{Cliff}(1) \cong \mathbb{C}$, with basis $\{1, e_1\}$ identified with the basis $\{1, i\}$. Here e_1 is the standard basis vector for $\mathbb{R}$.
- $\mathrm{Cliff}(2) \cong \mathbb{H}$, with basis $\{1, e_1, e_2, e_1 \otimes e_2\} \leftrightarrow \{1, i, j, k\}$. We calculate $(e_1 \otimes e_2)^2 = e_1 \otimes e_2 \otimes e_1 \otimes e_2 = -e_1 \otimes e_1 \otimes e_2 \otimes e_2 = -(-1)(-1) = -1$

- $\mathrm{Cliff}(3) \cong \mathbb{H} \otimes \mathbb{H}$. Exercise. Note that $\mathrm{Cliff}(3)$ is associative, hence it can't be the octonians.

2. Grading and Periodicity

We can make $T(V)$ into a $\mathbb{Z}$-graded alegra by saying that $v_1 \otimes \cdots \otimes v_k$ has degree k. Note that two elements of degrees m and n multiply to give an element of degree $m+n$. In $\mathrm{Cliff}(V)$, this grading scheme doesn't quite work because the relation $v \otimes v = -\|v\|^2$ does not respect the grading. But we can make $\mathrm{Cliff}(V)$ into a $\mathbb{Z}/2$-graded algebra by setting $\mathrm{Cliff}^0(V) \subseteq \mathrm{Cliff}(V)$ to be the even part of $\mathrm{Cliff}(V)$, and $\mathrm{Cliff}^1(V)$ the odd part. Multiplication preserves even and oddness, so this makes

$$\mathrm{Cliff}(V) = \mathrm{Cliff}^0(V) \oplus \mathrm{Cliff}^1(V)$$

a $\mathbb{Z}/2$-graded algebra.

Definition 2.1. Given $\mathbb{Z}$ or $\mathbb{Z}/2$ graded algebras A and B, we make $A \otimes B$ into a graded algebra by setting

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} a_1 a_2 \otimes b_1 b_2$$

Proposition 2.2. *$\mathrm{Cliff}(V \oplus W) \cong \mathrm{Cliff}(V) \otimes \mathrm{Cliff}(W)$ as $\mathbb{Z}/2$-graded algebras.*

Proof. Define $f : V \oplus W \to \mathrm{Cliff}(V) \otimes \mathrm{Cliff}(W)$ by $f(v, w) = v \otimes 1 + 1 \otimes w$. Then

$$\begin{aligned} f(v,w)^2 &= (v \otimes 1 + 1 \otimes w)^2 \\ &= (v \otimes 1)(v \otimes 1) + (v \otimes 1)(1 \otimes w) + (1 \otimes w)(v \otimes 1) + (1 \otimes w)(1 \otimes w) \\ &= v^2 \otimes 1 + v \otimes w - v \otimes w + 1 \otimes w^2 \\ &= -\|v\|^2 - \|w\|^2 = -\|(v,w)\|^2. \end{aligned}$$

By the universal property of $\mathrm{Cliff}(V \oplus W)$, we thus see that f extends to a map $\mathrm{Cliff}(V \oplus W) \to \mathrm{Cliff}(V) \otimes \mathrm{Cliff}(W)$. One can check that this is an isomorphism. □

Corollary 2.3. *$\mathrm{Cliff}(n) = \mathrm{Cliff}(1)^{\otimes n} = \mathbb{C}^{\otimes n}$, as $\mathbb{Z}/2$ graded algebras.*

We now define another $\mathbb{Z}/2$-graded algebra derived from the tensor algebra, $\overline{\mathrm{Cliff}}(V)$. We will use this as a tool in proving the periodicity result that is the goal of this section.

Definition 2.4. $\overline{\mathrm{Cliff}}(V) = T(V)/(v \otimes v \sim \|v\|^2)$.

Note that $\overline{\mathrm{Cliff}}(V)$ satisfies a universal property similar to that of $\mathrm{Cliff}(V)$. Calculations similiar to those for $\mathrm{Cliff}(V)$ give us the following results.

- $\overline{\mathrm{Cliff}}(0) \cong \mathbb{R}$
- $\overline{\mathrm{Cliff}}(1) \cong \mathbb{R} \oplus \mathbb{R}$
- $\overline{\mathrm{Cliff}}(2) \cong M_2(\mathbb{R})$
- $\overline{\mathrm{Cliff}}(3) \cong M_2(\mathbb{C})$

We also have a proposition relating $\mathrm{Cliff}(n)$ and $\overline{\mathrm{Cliff}}(n)$.

Proposition 2.5. *As ungraded algebras,*

i. $\mathrm{Cliff}(n) \otimes \overline{\mathrm{Cliff}}(2) \cong \overline{\mathrm{Cliff}}(n+2)$
ii. $\overline{\mathrm{Cliff}}(n) \otimes \mathrm{Cliff}(2) \cong \mathrm{Cliff}(n+2)$

Proof. We prove *i.*; the proof of *ii.* is similar. Define $f : \mathbb{R}^{n+2} \to \text{Cliff}(n) \otimes \overline{\text{Cliff}}(2)$ by

$$f(e_i) = \begin{cases} 1 \otimes e_i & \text{if } i = 1, 2 \\ f(e_i) = e_{i-2} \otimes (e_1 \otimes e_2) & \text{if } 3 \leq i \leq n+2 \end{cases}.$$

We calculate $f(e_i)^2$ for all i. If $i = 1, 2$, $f(e_i)^2 = (1 \otimes e_i)^2 = 1 \otimes e_1^2 = 1$. If $3 \leq i \leq n+2$,

$$\begin{aligned} f(e_i)^2 &= (e_{i-2} \otimes (e_1 \otimes e_2))^2 \\ &= e_{i-2}^2 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_2 \\ &= -(e_{i-2}^{\otimes} e_1 \otimes e_1 \otimes e_2 \otimes e_2) \\ &= -(-1 \otimes 1) = 1. \end{aligned}$$

Hence in either case, $f(e_i)^2 = 1$, so we can apply the universal property of $\overline{\text{Cliff}}(n+2)$ to get a map $\overline{\text{Cliff}}(n+2) \to \text{Cliff}(n) \otimes \overline{\text{Cliff}}(2)$. One can show this map is an isomorphism. □

Lemma 2.6. *Let* $F = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$, *and write* $F(n)$ *for* $M_n(F)$. *Then*

i. $F(n) \cong \mathbb{R}(n) \otimes F$
ii. $\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(mn)$
iii. $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$
iv. $\mathbb{H} \otimes \mathbb{C} \cong \mathbb{C}(2)$
v. $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)$

The proof is just straight forward calculations. We can use these results to make a table that gives $\text{Cliff}(n)$ and $\overline{\text{Cliff}}(n)$ for $0 \leq n \leq 8$.

n	$\text{Cliff}(n)$	$\overline{\text{Cliff}}(n)$
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$

Moreover, after $n = 8$, we get a sort of periodicity in $\text{Cliff}(n)$. Applying Proposition 2.5 twice, we find that for any n,

$$\begin{aligned} \text{Cliff}(n+8) &\cong \text{Cliff}(n) \otimes \text{Cliff}(2)^{\otimes 2} \otimes \overline{\text{Cliff}}(2)^{\otimes 2} \\ &\cong \text{Cliff}(n) \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\ &\cong \text{Cliff}(n) \otimes \mathbb{R}(4) \otimes \mathbb{R}(4) \\ &\cong \text{Cliff}(n) \otimes \mathbb{R}(16). \end{aligned}$$

This means that we can calculate $\text{Cliff}(n)$ for any n from this table just by multiplying matrix dimensions by 16. Next time, we will use this result to prove that the only normed division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.