1. Normed Division Algebras are alternative

Recall that a normed division algebra $K$ is alternative if every subalgebra generated by two elements is associative. We will show that every normed division algebra $K$ is alternative. Recall that for normed division algebras, the norm comes from an inner product.

Throughout, we will work with elements of $K$, a fixed normed division algebra.

**Lemma 1.1.** $\langle x_1 y, x_2 y \rangle = \langle x_1, x_2 \rangle \|y\|^2$.

**Proof.** We know that:

$$\|x_1 y + x_2 y\|^2 = \|x_1 + x_2\|^2 \|y\|^2.$$  

On the left hand side, we have:

$$\|x_1 y + x_2 y\|^2 = \langle x_1 y + x_2 y, x_1 y + x_2 y \rangle = \langle x_1 y, x_1 y \rangle + 2\langle x_1 y, x_2 y \rangle + \langle x_2 y, x_2 y \rangle = \|x_1 y\|^2 + 2\langle x_1 y, x_2 y \rangle + \|x_2 y\|^2.$$  

On the other hand, on the right hand side we have:

$$\|x_1 + x_2\|^2 \|y\|^2 = (\|x_1\|^2 + 2\langle x_1, x_2 \rangle + \|x_2\|^2) \|y\|^2.$$  

Cancelling yields:

$$\langle x_1 y, x_2 y \rangle = \langle x_1, x_2 \rangle \|y\|^2.$$

**Lemma 1.2.** $\langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle = 2\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$.

**Proof.** By the previous lemma,

$$\langle x_1 (y_1 + y_2), x_2 (y_1 + y_2) \rangle = \langle x_1, x_2 \rangle \langle y_1 + y_2, y_1 + y_2 \rangle.$$  

Expanding these expressions via bilinearity, we will have four terms on either side; cancelling the terms that appear on both sides, the result follows.

**Proposition 1.3.** The following equalities hold (where 1 is meant to denote 1 as a vector in $K$, not a scalar):

1. $x^2 - 2\langle x, 1 \rangle x + \langle x, x \rangle 1 = 0$,
2. $xy + yx - 2\langle x, 1 \rangle y - 2\langle y, 1 \rangle x + 2\langle x, y \rangle 1 = 0$.

**Proof.** For (1), take $(-, y)$ of the left hand side and apply the previous lemma:

$$\langle x^2, y \rangle - 2\langle x, 1 \rangle \langle x, y \rangle + \langle x, x \rangle \langle 1, y \rangle = 0.$$  

For (2), replace $x$ with $x + y$ in (1). Cancelling yields the result.
Recall that $\text{Re}(K) = \text{span}(1)$, the linear span of $1 \in K$, and $\text{Im}(K) = \text{Re}(K)^\perp = \{x \in K : \langle x, 1 \rangle = 0\}$, the orthogonal complement of $\text{Re}(K)$. Thus we have the following:

**Corollary 1.4.** If $x, y \in \text{Im}(K)$ and $\langle x, y \rangle = 0$, then $xy = -yx$.

The normed division algebras we have considered so far have further structure: they are $*$-algebras. Thus we give our normed division algebra $K$ a $*$-algebra structure by defining:

$$x^* = \text{Re}(x) - \text{Im}(x).$$

Equivalently, $x^* = -x + 2\langle x, 1 \rangle 1$.

**Proposition 1.5.** This gives $K$ the structure of a $*$-algebra.

**Proof.** It is clear that $(a^*)^* = a$, so we need to prove that $(ab)^* = b^*a^*$. This is just a calculation using our previous results:

$$b^*a^* = (-b + 2\langle b, 1 \rangle 1)(-a + 2\langle a, 1 \rangle 1)$$
$$= ba - 2\langle b, 1 \rangle a - 2\langle a, 1 \rangle b + 4\langle a, 1 \rangle \langle b, a \rangle$$
$$= -ab - 2\langle a, b \rangle 1 + 2\langle a, 1 \rangle \langle b, a \rangle$$
$$= -ab - 2\langle a, b \rangle 1 + 2\langle ab, 1 \rangle 1 + 2\langle a, b \rangle 1 = (ab)^*.$$

□

**Lemma 1.6.** The following identities hold:

1. $\langle xy, z \rangle = \langle y, x^* z \rangle$
2. $\langle xy, z \rangle = \langle x, zy^* \rangle$
3. $\langle xy, z^* \rangle = \langle yz, x^* \rangle$.

**Proof.** We’ll prove (1) here. The other proofs are nearly identical. Calculate that:

$$\langle y, x^* z \rangle = \langle y, (-x + 2\langle x, 1 \rangle)z \rangle$$
$$= -\langle y, xz \rangle + 2\langle x, 1 \rangle \langle y, z \rangle$$
$$= -\langle y, xz \rangle + \langle xy, z \rangle + \langle xz, y \rangle$$
$$= \langle xy, z \rangle.$$

□

Another lemma:

**Lemma 1.7.** The following identities hold:

1. $x(x^* y) = \|x\|^2 y$
2. $(xy^*)y = \|y\|^2 z$
3. $x(y^* z) + y(x^* z) = \langle x, y \rangle z$
4. $(xy^*)z + (xz^*)y = \langle y, z \rangle x$.

The proof is left to the avid reader.

**Proposition 1.8.** The following equalities hold (and are known as the Moufang Identities):

1. $(ax)(ya) = a((xy)a)$
2. $a(x(ay)) = (a(xa))y$
3. $x(a(ya)) = ((xa)y)a$
Proof. For (1), take $\langle - , z \rangle$ of either side with any $z \in K$. The left side is then:

$$\langle (ax)(ya), z \rangle = 2\langle y, x^* a^* \rangle \langle a, z \rangle - \langle yz, (x^*a^*)a \rangle$$

$$= 2\langle xy, a^* \rangle \langle a, z \rangle - \|a\|^2 \langle yz, x^* \rangle.$$ 

The right hand side is:

$$\langle a((xy)a), z \rangle = 2\langle xy, a^* \rangle \langle a, z \rangle - \langle (xy)z, a^*a \rangle$$

$$= 2\langle xy, a^* \rangle \langle a, z \rangle - \|a\|^2 \langle xy, z^* \rangle$$

Thus equality (1) holds. The proofs of the other identities are similar. □

Setting $x = 1$ or $y = 1$ in each equality yields the equations that determine that:

**Corollary 1.9.** $K$ is alternative.

2. **The Cayley-Dickson Construction**

First we introduce some new terminology:

**Definition 2.1.** A $\ast$-algebra $A$ is real if $a^* = a$ for all $a \in A$. $A$ is nicely-normed if $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$ for all nonzero $a \in A$.

If $A$ is nicely normed, then $\|a\| = \sqrt{aa^*}$ is a norm on $A$. On the other hand, normed division algebras are nicely normed since $aa^* = a^*a = \|a\|^2$ in that case. If $A$ is a nicely normed $\ast$-algebra, define:

$$\text{Re}(a) = \frac{a + a^*}{2}, \quad \text{Im}(a) = \frac{a - a^*}{2}.$$

**Lemma 2.2.** If $a \in A$ is nonzero, then $a^{-1} = \frac{a^*}{\|a\|^2}$.

**Theorem 2.3.** Let $A$ be a $\ast$-algebra. Then $A$ is a normed division algebra if and only if $A$ is nicely normed and alternative.

**Proof.** We have already shown that normed division algebras are nicely normed and alternative. Suppose that $A$ is nicely normed and alternative. We need to show that $\|ab\| = \|a\|\|b\|$. First of all $a, b, a^*, b^*$ are all in the subalgebra generated by $\text{Im}(a), \text{Im}(b)$, hence they commute under multiplication with each other. It follows that:

$$\|ab\|^2 = (ab)(ab)^* = (ab)(b^*a^*) = a(bb^*)a^* = \|b\|^2(aa^*) = \|a\|^2\|b\|^2.$$

Hence $A$ is a normed division algebra. □

The Cayley-Dickson construction generalizes the construction we used to get the complex numbers from the real numbers, the quaternions from the complex numbers, etc. Given a $\ast$-algebra $A$, we define a new $\ast$-algebra $A'$ whose elements are pairs $(a,b)$ with $a,b \in A$ and multiplication and $\ast$ defined by:

$$(a,b)(c,d) = (ac - db^*, a^*d + cb), \quad (a,b)^* = (a^*, -b).$$

**Lemma 2.4.** Under this definition, $A'$ is a $\ast$-algebra.
Proof. An easy calculation shows that \(( (a, b)(c, d))^* = (c, d)^*(a, b)^* \). □

Lemma 2.5. If \( A \) is nicely normed then \( A' \) is nicely normed.

Proof. Again, this follows by a direct calculation:
\[
(a, b) + (a, b)^* = (a, b) + (a^*, -b) = (a + a^*, 0) \in \mathbb{R},
\]
while if \((a, b) \neq (0, 0)\),
\[
(a, b)(a, b)^* = (a, b)(a^*, -b) = (aa^* + bb^*, 0) > 0.
\]
□

Proposition 2.6. Given any \( * \)-algebra \( A \), the resulting \( * \)-algebra \( A' \) is never real.

Proposition 2.7. \( A \) is real (and thus commutative) if and only if \( A' \) is commutative and associative.

To prove these propositions here would merely deprive the reader of the pleasure of doing so.

Proposition 2.8. \( A \) is commutative and associative if and only if \( A' \) is associative.

Proof. For one direction, we have the following calculations:
\[
((a, b)(c, d))(e, f) = (ac - db^*, a^*d + cb)(e, f)
\]
\[
= ((ac - db^*)e - f(ac - db^*)^*, (ac - db^*)^*f + e(a^*d + cb))
\]
\[
= ((ac)e - (db^*)e - f(c^*a^*) - f(bd^*), (c^*a^*)f - (bd^*)f + e(a^*d) + e(cb)),
\]
while:
\[
(a, b)((c, d)(e, f)) = (a(ce - b(fd^*) - (c^* f)b^* - (cd)b^*, a^*c^* f) + a^*(ed) + (ce)b - (fd^*)b).
\]
These two expressions are equal for all elements \( a, b, c, d, e, f \in A \) if and only if \( A \) is commutative and associative. □

A final proposition:

Proposition 2.9. \( A \) is associative and nicely normed if and only if \( A' \) is alternative and nicely normed.

Proof. Use the following equality:
\[
(a, b)(a, b) = (a^2 - bb^*, a^*b + ab) = (a^2 - \|b\|^2, 2Re(a)b).
\]
□

The upshot of this series of propositions is that they apply to yield:
\[
\mathbb{R} \text{ is real} \implies \mathbb{C} \text{ is commutative} \implies \mathbb{H} \text{ is associative} \implies \mathbb{O} \text{ is alternative}
\]
Hence we may conclude that the octonions are a normed division algebra, without ever doing explicit calculations in the octonions themselves.