PRODUCTS ON SPACES, REU 2007: LECTURE 3

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1. Normed Division Algebras are alternative

Recall that a normed division algebra K is *alternative* if every subalgebra generated by two elements is associative. We will show that every normed division algebra K is alternative. Recall that for normed division algebras, the norm comes from an inner product.

Throughout, we will work with elements of K, a fixed normed division algebra.

Lemma 1.1. $\langle x_1 y, x_2 y\rangle = \langle x_1, x_2\rangle \|y\|^2$.

Proof. We know that:

$$\|x_1 y + x_2 y\|^2 = \|x_1 + x_2\|^2 \|y\|^2.$$

On the left hand side, we have:

$$\begin{aligned}\|x_1 y + x_2 y\|^2 &= \langle x_1 y + x_2 y, x_1 y + x_2 y\rangle \\ &= \langle x_1 y, x_1 y\rangle + 2\langle x_1 y, x_2 y\rangle + \langle x_2 y, x_2 y\rangle \\ &= \|x_1 y\|^2 + 2\langle x_1 y, x_2 y\rangle + \|x_2 y\|^2.\end{aligned}$$

On the other hand, on the right hand side we have:

$$\|x_1 + x_2\|^2 \|y\|^2 = \big(\|x_1\|^2 + 2\langle x_1, x_2\rangle + \|x_2\|^2\big)\|y\|^2.$$

Cancelling yields:

$$\langle x_1 y, x_2 y\rangle = \langle x_1, x_2\rangle \|y\|^2.$$

□

Lemma 1.2. $\langle x_1 y_1, x_2 y_2\rangle + \langle x_1 y_2, x_2 y_1\rangle = 2\langle x_1, x_2\rangle\langle y_1, y_2\rangle$.

Proof. By the previous lemma,

$$\langle x_1(y_1 + y_2), x_2(y_1 + y_2)\rangle = \langle x_1, x_2\rangle\langle y_1 + y_2, y_1 + y_2\rangle.$$

Expanding these expressions via bilinearity, we will have four terms on either side; cancelling the terms that appear on both sides, the result follows. □

Proposition 1.3. *The following equalities hold (where* 1 *is meant to denote* 1 *as a vector in* K*, not a scalar):*

1. $x^2 - 2\langle x, 1\rangle x + \langle x, x\rangle 1 = 0$,
2. $xy + yx - 2\langle x, 1\rangle y - 2\langle y, 1\rangle x + 2\langle x, y\rangle 1 = 0$.

Proof. For (1), take $\langle -, y\rangle$ of the left hand side and apply the previous lemma:

$$\langle x^2, y\rangle - 2\langle x, 1\rangle\langle x, y\rangle + \langle x, x\rangle\langle 1, y\rangle = 0.$$

For (2), replace x with $x + y$ in (1). Cancelling yields the result. □

Recall that $\mathrm{Re}(K) = \mathrm{span}(1)$, the linear span of $1 \in K$, and $\mathrm{Im}(K) = \mathrm{Re}(K)^\perp = \{x \in K : \langle x, 1\rangle = 0\}$, the orthogonal complement of $\mathrm{Re}(K)$. Thus we have the following:

Corollary 1.4. *If $x, y \in \mathrm{Im}(K)$ and $\langle x, y\rangle = 0$, then $xy = -yx$.*

The normed division algebras we have considered so far have further structure: they are $*$-algebras. Thus we give our normed division algebra K a $*$-algebra structure by defining:

$$x^* = \mathrm{Re}(x) - \mathrm{Im}(x).$$

Equivalently, $x^* = -x + 2\langle x, 1\rangle 1$.

Proposition 1.5. *This gives K the structure of a $*$-algebra.*

Proof. It is clear that $(a^*)^* = a$, so we need to prove that $(ab)^* = b^*a^*$. This is just a calculation using our previous results:

$$\begin{aligned} b^*a^* &= (-b + 2\langle b, 1\rangle 1)(-a + 2\langle a, 1\rangle 1) \\ &= ba - 2\langle b, 1\rangle a - 2\langle a, 1\rangle b + 4\langle a, 1\rangle\langle b, 1\rangle \\ &= -ab - 2\langle a, b\rangle 1 + 4\langle a, 1\rangle\langle b, a\rangle \\ &= -ab - 2\langle a, b\rangle 1 + 2\langle ab, 1\rangle 1 + 2\langle a, b\rangle 1 = (ab)^*. \end{aligned}$$

□

Lemma 1.6. *The following identities hold:*

(1) $\langle xy, z\rangle = \langle y, x^*z\rangle$
(2) $\langle xy, z\rangle = \langle x, zy^*\rangle$
(3) $\langle xy, z^*\rangle = \langle yz, x^*\rangle$.

Proof. We'll prove (1) here. The other proofs are nearly identical. Calculate that:

$$\begin{aligned} \langle y, x^*z\rangle &= \langle y, (-x + 2\langle x, 1\rangle)z\rangle \\ &= -\langle y, xz\rangle + 2\langle x, 1\rangle\langle y, z\rangle \\ &= -\langle y, xz\rangle + \langle xy, z\rangle + \langle xz, y\rangle \\ &= \langle xy, z\rangle. \end{aligned}$$

□

Another lemma:

Lemma 1.7. *The following identities hold:*

(1) $x(x^*y) = \|x\|^2 y$
(2) $(xy^*)y = \|y\|^2 z$
(3) $x(y^*z) + y(x^*z) = \langle x, y\rangle z$
(4) $(xy^*)z + (xz^*)y = \langle y, z\rangle x$.

The proof is left to the avid reader.

Proposition 1.8. *The following equalities hold (and are known as the Moufang Identities):*

(1) $(ax)(ya) = a((xy)a)$
(2) $a(x(ay)) = (a(xa))y$
(3) $x(a(ya)) = ((xa)y)a$

Proof. For (1), take $\langle -, z\rangle$ of either side with any $z \in K$. The left side is then:

$$\begin{aligned}\langle (ax)(ya), z\rangle &= \langle ya, (x^*a^*)z\rangle \\ &= 2\langle y, x^*a^*\rangle\langle a, z\rangle - \langle yz, (x^*a^*)a\rangle \\ &= 2\langle xy, a^*\rangle\langle a, z\rangle - \|a\|^2\langle yz, x^*\rangle.\end{aligned}$$

The right hand side is:

$$\begin{aligned}\langle a((xy)a), z\rangle &= \langle (xy)a, a^*z\rangle \\ &= 2\langle xy, a^*\rangle\langle a, z\rangle - \langle (xy)z, a^*a\rangle \\ &= 2\langle xy, a^*\rangle\langle a, z\rangle - \|a\|^2\langle xy, z^*\rangle \\ &= 2\langle xy, a^*\rangle\langle a, z\rangle - \|a\|^2\langle yz, x^*\rangle.\end{aligned}$$

Thus equality (1) holds. The proofs of the other identities are similar. □

Setting $x = 1$ or $y = 1$ in each equality yields the equations that detemine that:

Corollary 1.9. *K is alternative.*

2. The Cayley-Dickson Construction

First we introduce some new terminology:

Definition 2.1. A $*$-algebra A is *real* if $a^* = a$ for all $a \in A$. A is *nicely-normed* if $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$ for all nonzero $a \in A$.

If A is nicely normed, then $\|a\| = \sqrt{aa^*}$ is a norm on A. On the other hand, normed division algebras are nicely normed since $aa^* = a^*a = \|a\|^2$ in that case. If A is a nicely normed $*$-algebra, define:

$$\mathrm{Re}(a) = \frac{a + a^*}{2}, \qquad \mathrm{Im}(a) = \frac{a - a^*}{2}.$$

Lemma 2.2. *If $a \in A$ is nonzero, then $a^{-1} = \dfrac{a^*}{\|a\|^2}$.*

Theorem 2.3. *Let A be a $*$-algebra. Then A is a normed division algebra if and only if A is nicely normed and alternative.*

Proof. We have already shown that normed division algebras are nicely normed and alternative. Suppose that A is nicely normed and alternative. We need to show that $\|ab\| = \|a\|\|b\|$. First of all a, b, a^*, b^* are all in the subalgebra generated by $\mathrm{Im}(a)$, $\mathrm{Im}(b)$, hence they commute under multiplication with each other. It follows that:

$$\|ab\|^2 = (ab)(ab)^* = (ab)(b^*a^*) = a(bb^*)a^* = \|b\|^2(aa^*) = \|a\|^2\|b\|^2.$$

Hence A is a normed division algebra. □

The Cayley-Dickson construction generalizes the construction we used to get the complex numbers from the real numbers, the quaternions from the complex numbers, etc. Given a $*$-algebra A, we define a new $*$-algebra A' whose elements are pairs (a, b) with $a, b \in A$ and multiplication and $*$ defined by:

$$(a, b)(c, d) = (ac - db^*, a^*d + cb), \qquad (a, b)^* = (a^*, -b).$$

Lemma 2.4. *Under this definition, A' is a $*$-algebra.*

Proof. An easy calculation shows that $\big((a,b)(c,d)\big)^* = (c,d)^*(a,b)^*$. □

Lemma 2.5. *If A is nicely normed then A' is nicely normed.*

Proof. Again, this follows by a direct calculation:

$$(a,b) + (a,b)^* = (a,b) + (a^*, -b) = (a + a^*, 0) \in \mathbb{R},$$

while if $(a,b) \neq (0,0)$,

$$(a,b)(a,b)^* = (a,b)(a^*, -b) = (aa^* + bb^*, 0) > 0.$$

□

Proposition 2.6. *Given any $*$-algebra A, the resulting $*$-algebra A' is never real.*

Proposition 2.7. *A is real (and thus commutative) if and only if A' is commutative and associative.*

To prove these propositions here would merely deprive the reader of the pleasure of doing so.

Proposition 2.8. *A is commutative and associative if and only if A' is associative.*

Proof. For one direction, we have the following calculations:

$$\begin{aligned}\big((a,b)(c,d)\big)(e,f) &= (ac - db^*, a^*d + cb)(e,f) \\ &= \big((ac - db^*)e - f(ac - db^*)^*, (ac - db^*)^*f + e(a^*d + cb)\big) \\ &= \big((ac)e - (db^*)e - f(c^*a^*) - f(bd^*), (c^*a^*)f - (bd^*)f + e(a^*d) + e(cb)\big),\end{aligned}$$

while:

$$(a,b)\big((c,d)(e,f)\big) = \big(a(ce) - b(fd^*) - (c^*f)b^* - (ed)b^*, a^*)c^*f) + a^*(ed) + (ce)b - (fd^*)b\big).$$

These two expressions are equal for all elements $a, b, c, d, e, f \in A$ if and only if A is commuative and associative. □

A final proposition:

Proposition 2.9. *A is associative and nicely normed if and only if A' is alternative and nicely normed.*

Proof. Use the following equality:

$$(a,b)(a,b) = (a^2 - bb^*, a^*b + ab) = (a^2 - \|b\|^2, 2\mathrm{Re}(a)b).$$

□

The upshot of this series of propositions is that they apply to yield:

$$\mathbb{R} \text{ is real} \implies \mathbb{C} \text{ is commutative} \implies \mathbb{H} \text{ is associative} \implies \mathbb{O} \text{ is alternative}$$

Hence we may conclude that the octonions are a normed division algebra, without ever doing explicit calculations in the octonions themselves.