PRODUCTS ON SPACES, REU 2007: LECTURE 2

NOTES BY JOHN LIND

1. More on the Quaternions

To clear up something from the previous lecture, it was claimed that any norm $\|\cdot\|$ on a vector space V gives rise to an inner product via the formula:

$$\langle v, w\rangle = \tfrac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2).$$

This is not true in general. For example, the norm $\|(x,y)\| = \max(|x|,|y|)$ on $\mathbb{R}^2$ gives rise to a pairing which is not bilinear. However, the following is true:

Theorem 1.1. *$\|\cdot\|$ gives rise to an inner product if and only if:*

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

More to the point for our context is the following result:

Theorem 1.2. *If K is a normed division algebra (finite dimensional), then the norm comes from an inner product.*

Proof. Identify K with $\mathbb{R}^n$. Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be the subgroup preserving the norm. Let S^{n-1} be the unit sphere for this norm. If $x \in S^{n-1}$, i.e. $\|x\| = 1$, then $L_x\colon K \to K$, $L_x(v) = xv$ preserves the norm. Thus G maps $1 \in S^{n-1}$ to all of S^{n-1}. G is compact, so we may find an inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ which is preserved by G as follows. Pick an arbitrary inner product $\{\cdot,\cdot\}$ on $\mathbb{R}^n$, then define:

$$\langle v, w\rangle = \int_G \{gv, gw\}\, dg.$$

The integration is taken with respect to the Haar measure on G. If G were finite, taken with the discrete topology, this would just be:

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}\{gv, gw\}.$$

Define a new norm provisionally by $|x|' = \sqrt{\langle x, x\rangle}$, then scale this norm so that $|1|' = 1$. Let $\tilde{S}^{n-1} \subset \mathbb{R}^n$ be the unit sphere for $|\cdot|'$, i.e. $\tilde{S}^{n-1} = \{x \in \mathbb{R}^n : |x|' = 1\}$. Again, G maps 1 to all the points of $\tilde{S}^{n-1}$, so $S^{n-1} \subset \tilde{S}^{n-1}$. This forces $S^{n-1} = \tilde{S}^{n-1}$, and so $|\cdot|' = \|\cdot\|$. Hence our original norm comes from the G-invariant inner product $\langle\cdot,\cdot\rangle$. $\square$

Recall from last time that we have an isomorphism of groups:

$$S^3 = \{x \in \mathbb{H} \colon \|x\| = 1\} \cong \mathrm{SU}(2).$$

Any $x \in \mathbb{H}$ acts on $\mathbb{H} = \mathbb{C}^2$ by left multiplication. We can write down the matrix for this map; since multiplication in $\mathbb{H}$ is defined to be:

$$(a,b)\cdot(c,d) = (ac - db^*, a^*d + cb),$$

we have:

$$(a,b)\cdot(1,0)=(a,b),\qquad (a,b)(0,1)=(-b^*,a^*).$$

Hence mutiplication by (a,b) in matrix form is $\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$.

There is an inner product on $\mathbb{C}^n$ given by:

$$\langle (z_1,\dots,z_n),(w_1,\dots,w_n)\rangle = z_1^*w_1+\dots+z_n^*w_n.$$

If $s\in S^3=\mathbb{H}^\times$ is a unit quaternion, then $L_x\colon \mathbb{C}^2\to\mathbb{C}^2$, left multiplication by x, preserves this inner product. Furthermore, $A\in \mathrm{M}_2(\mathbb{C})$ preserves this inner product if and only if $A\in \mathrm{U}(2)$:

$$\langle Az,Aw\rangle=(Az)^*Aw=z^*A^*Aw,$$

and so $\langle Az,Aw\rangle=\langle z,w\rangle$ if and only if $A^*A=1$. This gives a map $\phi\colon S^3\to \mathrm{U}(2)$, which factors through $\mathrm{SU}(2)$, since $\det\phi(x)=1$, for $x\in S^3$.

Next, we'll consider an action of $\mathbb{H}$ on $\mathbb{R}^3$. Define a map $\psi\colon \mathbb{H}\to \mathrm{M}_3(\mathbb{R})$ by:

$$\psi(a+bi+cj+dk)=\begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2ad+2bc & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2ab+2cd & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

It is a calculational exercise to show that this map is multiplicative, but not additive. We would like a better understanding of ψ, since this formula is not very enlightening. To this end, set:

$$\mathrm{Re}(a+bi+cj+dk)=a,$$
$$\mathrm{Im}(a+bi+cj+dk)=bi+cj+dk.$$

Embed $\mathbb{R}^3$ in $\mathbb{H}$ by $\sigma(v_1,v_2,v_3)=iv_1+jv_2+kv_3$.

Exercise. Let $v\times w$ denote the cross product of vectors in $\mathbb{R}^3$. Then:

(1) $\mathrm{Re}(xy)=\mathrm{Re}(yx)$ for all $x,y\in\mathbb{H}$.
(2) $v\times w=\mathrm{Im}(vw)$ for all $v,w\in\mathbb{R}^3=\mathrm{Im}(\mathbb{H})$.

Notice that this implies that $\mathrm{Re}(xvx^*)=0$ for all $x\in\mathbb{H}$, $v\in\mathbb{R}^3$. To see this, note that:

$$\mathrm{Re}(xvx^*)=\mathrm{Re}(x^*xv)=\|x\|^2\mathrm{Re}(v)=0.$$

Hence $xvx^*\in\mathrm{Im}(\mathbb{H})=\mathbb{R}^3$. For each $x\in\mathbb{H}$, define $T_x\colon\mathbb{R}^3\to\mathbb{R}^3$ by $T_x(v)=xvx^*$. This is linear in v, hence can be written as a matrix. A calculation shows that it is precisely the matrix for ψ above.

Lemma 1.3. *$\psi\colon\mathbb{H}\to\mathrm{M}_3(\mathbb{R})$ is multiplicative.*

Proof. We need to equate T_{xy} and the composite $T_x\circ T_y$:

$$(T_x\circ T_y)(v)=T_x(yvy^*)=xyvy^*x^*=T_{xy}(v).$$

□

Notice that $\det(\psi(x))=\|x\|^6$ and that $\psi(x^*)=\psi(x)^t$. If $x\in S^3$, then $\det(\psi(x))=1$, and $\psi(x)^t\psi(x)=I_3$. Thus the image of ψ lies in $\mathrm{SO}(3)$, the special orthogonal group, consisting of matrices A with $A^tA=I$ and $\det(A)=1$.

Theorem 1.4. *The map $\psi\colon S^3\to\mathrm{SO}(3)$ is surjective with kernel $\{\pm1\}$.*

Proof. It's clear that $\psi(x) = \psi(-x)$. Next, using the matrix form for ψ, check that $\psi^{-1}(I) = \{\pm 1\}$. Alternatively, since ψ on the unit sphere is conjugation (as $x^* = x^{-1}$), we have $\psi(x) = I$ if and only if x commutes with all elements of S^3 under multiplication, hence is real and so must be 1 or -1. To see that ψ is surjective, recall that elements of SO(3) are orientation preserving rotations of $\mathbb{R}^3$, so SO(3) is generated by matrices of the form:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos(\theta) \end{bmatrix},$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus it suffices to hit all matrices of the above form. This is achieved by the following calculation:

$$\begin{aligned} \psi(\cos(\tfrac{\theta}{2}) - i\sin(\tfrac{\theta}{2})) &= \begin{bmatrix} \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 0 & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$

Similarly, $\psi(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}) = R_y(\theta)$ and $\psi(\cos\frac{\theta}{2} - k\sin\frac{\theta}{2}) = R_z(\theta)$. □

Corollary 1.5. *Precomposing with the isomorphism* $\mathrm{SU}(2) \cong S^3$, ψ *induces a* 2*-to-*1 *surjection with kernel* $\{\pm 1\}$*:*

$$\mathrm{SU}(2) \to \mathrm{SO}(3).$$

2. The Octonions

We define the *octonions* to be $\mathbb{O} = \mathbb{H}^2$ with multiplication:

$$(a, b) \cdot (c, d) = (ac - db^*, a^*d + cb).$$

We let $\mathbb{O}$ have basis $\{1, \ell\}$ over $\mathbb{H}$, which means that $\{1, i, j, k, \ell, i\ell, j\ell, k\ell\}$ is a basis for $\mathbb{O}$ over $\mathbb{R}$. With respect to this basis, the multiplication can be represented by the following multiplication table:

	i	j	k	ℓ	$i\ell$	$j\ell$	$k\ell$
i	-1	k	$-j$	$i\ell$	$-\ell$	$-k\ell$	$j\ell$
j	$-k$	-1	i	$j\ell$	$k\ell$	$-\ell$	$-i\ell$
k	j	$-i$	-1	$k\ell$	$-j\ell$	$-i\ell$	$-\ell$
ℓ	$-i\ell$	$-j\ell$	$-k\ell$	-1	i	j	k
$i\ell$	ℓ	$-k\ell$	$j\ell$	$-i$	-1	$-k$	j
$j\ell$	$k\ell$	ℓ	$-i\ell$	$-j$	k	-1	$-i$
$k\ell$	$-j\ell$	$i\ell$	ℓ	$-k$	$-j$	i	-1

Notice in particular that:

$$(ij)\ell = k\ell, \quad \text{while} \quad i(j\ell) = -k\ell.$$

Thus the octonions are *not associative*. Similarly to the quaternions, we let:

$$\mathrm{Re}(a+bi+cj+dk+a'\ell+b'(i\ell)+c'(j\ell)+d'(k\ell))=a,$$

$$\mathrm{Im}(a+bi+cj+dk+a'\ell+b'(i\ell)+c'(j\ell)+d'(k\ell))=$$

$$bi+cj+dk+a'\ell+b'(i\ell)+c'(j\ell)+d'(k\ell)$$

For $x \in \mathbb{O}$, set $x^* = \mathrm{Re}(x) - \mathrm{Im}(x)$.

Exercise. $x^{-1} = \dfrac{x^*}{\|x\|^2}$.

The octonions were invented independently by Graves and Cayley (1883, 1885). Because they are nonassociative, they are harder to work with. John Baez describes it like this:

> The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.

Definition 2.1. An algebra A is *power-associative* if the subalgebra $\langle x\rangle$ generated by any single element $x \in A$ is associative. A is *alternative* if the subalgebra generated by any two elements is associative.

For a power-associative algebra, x^n is well-defined independent of order. For an alternative algebra,

$$(a^2)b = a(ab); \tag{1}$$

$$(ab)a = a(ba); \tag{2}$$

$$(ba)a = b(a^2). \tag{3}$$

Proposition 2.2. *Any two of the conditions (1), (2), (3) imply the third.*

Proof. Define the associator $[\cdot,\cdot,\cdot]\colon A^3 \to A$ by:

$$[a,b,c] = (ab)c - a(bc).$$

Then Equation 1 says that $[a,a,b]=0$. This means that $[a+b,a+b,c]=0$. By linearity this means that $[a,b,c]=-[b,a,c]$. Thus Equation 1 says that the associater changes sign whenever we switch the first 2 variables. This is an if and only if: If the associator always changes sign whenever we switch the first 2 variables, then $[a,a,b]=-[a,a,b]=0$.

Similarly, Equation 2 says that the associator changes sing when we switch the first and the third variables, and Equation 3 says that the associator changes sign when we switch the last 2 variables. Assuming Equation 1 and 2, we can switch the last 2 variables by doing (13), then (12) and then (13). The other cases are similar. $\square$

We'll see next time that any normed division algebra is alternative.