REU CLASS VIGLEIK ANGELTVEIT

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Lecture 1

The main reference for much of the material in this course is [1].

We want to ask the following question: For which n can we equip $\mathbb{R}^n$ with a "reasonable" multiplication? The answer turns out to be that with a reasonable definition of "reasonable", only for $n = 1, 2, 4$ or 8.

First we need to explain what we mean by a "reasonable" multiplication. See Definition 3 and 5.

Definition 1. An *algebra* A is a vector space over $\mathbb{R}$ with a bilinear map

$$m : A \times A \to A$$

called multiplication, together with an element $1 \in A$ such that $m(1, a) = m(a, 1) = a$ for all $a \in A$.

Here bilinear means that for $a, b, c \in A$ and $r \in \mathbb{R}$ we have $m(ra, b) = m(a, rb) = r\,m(a, b)$, $m(a + b, c) = m(a, c) + m(b, c)$ and $m(a, b + c) = m(a, b) + m(a, c)$. We will abbreviate $m(a, b)$ by ab.

Remark 2. We do not assume associativity or commutativity, although associativity is often assumed in defining an algebra. So in general we could have $(ab)c \neq a(bc)$ and $ab \neq ba$. Hence this definition is less restrictive than the one given, for example, in Peter May's course.

We are especially interested in algebras without zero-divisors. So we make the following definition:

Definition 3. A *division algebra* is an algebra A such that if $a, b \in A$ with $ab = 0$ then either $a = 0$ or $b = 0$.

This excludes, among others, matrix algebras. We see that, for example, in $M_2(\mathbb{R})$, the ring of 2×2 matrices with coefficients in $\mathbb{R}$, if $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ we have $A^2 = 0$. It also excludes the familiar cross product on $\mathbb{R}^3$, since we note that $(1, 0, 0) \times (1, 0, 0) = (0, 0, 0)$.

We have the following hard theorem:

Theorem 4. *(Kervaire, Bott-Milnor, 1958) All division algebras have dimension* $1, 2, 4$ *or* 8.

We will (probably) not prove this theorem in this class, as the three proofs I know require either knowledge of K-theory, knowledge of secondary Steenrod operations, or knowledge of the Adams spectral sequence. The existence of a division algebra structure on $\mathbb{R}^n$ is equivalent to the existence of a *Hopf invariant one* map $S^{2n-1} \to S^n$. I might possibly be persuaded to talk about K-theory and the proof of the above theorem which uses K-theory following Adams, if there is sufficient interest.

Instead we will assume one more property of our "reasonable" multiplication, namely that it is normed.

Recall that a *normed* vector space is a vector space V with a norm (or absolute value)

$$\|\cdot\| : V \to \mathbb{R}$$

satisfying the following conditions:

- $\|v\| = 0$ if and only if $v = 0$.
- $\|rv\| = |r|\,\|v\|$ for all $r \in \mathbb{R}$ and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

For example, $\mathbb{R}^n$ is a normed vector space with

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

For future reference, note that this norm on $\mathbb{R}^n$ also gives the inner product on $\mathbb{R}^n$ via the formula

$$\langle v, w\rangle = \frac{1}{2}\big(\|v + w\|^2 - \|v\|^2 - \|w\|^2\big).$$

In general, every normed space does not give rise to an inner product space. An easy counterexample is given by the max norm on $\mathbb{R}^2$: $\|(x, y)\| = \max\{x, y\}$. Another counterexample is the Banach space L^1. However, it will turn out that the norms on the vector spaces we consider do give us inner products.

Definition 5. A *normed division algebra* is a division algebra A which is also a normed vector space and whose norm satisfies $\|ab\| = \|a\|\,\|b\|$ for all $a, b \in A$.

Remark 6. We could have defined a normed division algebra simply as an algebra A which is also a normed vector space, satisfying $\|ab\| = \|a\|\,\|b\|$ for all $a, b \in A$. Then, if $ab = 0$ we get $\|a\|\,\|b\| = 0$, so either $\|a\| = 0$ or $\|b\| = 0$, implying that either $a = 0$ or $b = 0$. That is, any normed algebra must be a division algebra.

Later in the class we will prove the following theorem:

Theorem 7. *(Hurwitz, 1898) The only normed division algebras are* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *and* $\mathbb{O}$.

These have dimension $1, 2, 4$ and 8. This is a weaker version of Theorem 4. It will take us some time to get to the proof. Here $\mathbb{C}$ is the complex numbers, $\mathbb{H}$ is the *quaternions* and $\mathbb{O}$ is the *octonions.*

Complex numbers. Recall that $\mathbb{C}$ is defined as pairs (a, b) of real numbers with

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

We usually write (a, b) as $a + bi$, and then we can recover the multiplication from $i^2 = -1$. This is clearly a normed division algebra, with $\|a + bi\| = \sqrt{a^2 + b^2}$.

$\mathbb{C}$ has another feature which we will find useful, namely *complex conjugation.*

Definition 8. A $*$-*algebra* is an algebra A equipped with a *conjugation*. A conjugation is a linear map

$$\begin{aligned} * : \quad & A \to A \\ & a \mapsto a^* \end{aligned}$$

such that $(a^*)^* = a$ and $(ab)^* = b^* a^*$.

For $\mathbb{C}$, the conjugation is given by $(a + bi)^* = a - bi$. As it turns out, any nonzero complex number has a (unique) inverse, which we can express in terms of the conjugation $*$, namely $z^{-1} = \frac{z^*}{\|z\|^2}$, as you can easily check.

Quaternions. The quaternions $\mathbb{H}$ is defined as quadruplets (a, b, c, d) of real numbers with

$$\begin{aligned} &(a, b, c, d) \cdot (e, f, g, h) = \\ &(ae - bf - cg - dh, af + be + ch - dg, ag + cd + bg - cf, ah + de + bg - cf). \end{aligned}$$

We write a quaternion as $a + bi + cj + dk$, and then

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

The quaternions were invented by Hamilton in 1843. Hamilton had for some time been searching for a way to multiply triplets, before suddenly realizing he needed 4-tuples. He went on to carve Equation 1 into the stone of the Brougham Bridge.

$\mathbb{H}$ is associative, meaning $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{H}$, but not commutative. For example, $ij = k$ while $ji = -k$, as you should check. The usual norm on $\mathbb{R}^4$ makes $\mathbb{H}$ into a normed division algebra. $\mathbb{H}$ also has a conjugation, given by $(a + bi + cj + dk)^* = a - bi - cj - dk$, and we can use the following lemma to check that $\mathbb{H}$ is a normed division algebra.

Lemma 9. *A nonzero quaternion q has inverse $\frac{q^*}{\|q\|^2}$.*

Proof. Check that $qq^* = \|q\|^2$. □

There is an alternative way to define $\mathbb{H}$, namely as pairs (a, b) of complex numbers. Then the multiplication is given by

$$(a, b) \cdot (c, d) = (ac - db^*, a^* d + cb) \tag{2}$$

and the conjugation is given by $(a, b)^* = (a^*, -b)$. Written like this, going from $\mathbb{R}$ to $\mathbb{C}$ is exactly like going from $\mathbb{C}$ to $\mathbb{H}$. We can then repeat this construction, at least one more time.

Octonions. The *octonions* $\mathbb{O}$ is the set of pairs (a, b) of quaternions, with multiplication as in Equation 2. We will write down a multiplication table for $\mathbb{O}$ later, to see that $\mathbb{O}$ is indeed nonassociative.

This makes $\mathbb{O}$ into a normed division algebra. Any nonzero octonion x has an inverse $x^{-1} = \frac{x^*}{\|x\|^2}$.

So why can't we keep going, to produce normed division algebras of dimension $16, 32, \ldots$? Every time we perform this construction, the new algebra is less nice. We start with $\mathbb{R}$, which is associative and commutative, and satisfies $a^* = a$ for all $a \in \mathbb{R}$. When we go from $\mathbb{R}$ to $\mathbb{C}$, we lose the property $a^* = a$ for all a. When we go from $\mathbb{C}$ to $\mathbb{H}$, we lose the property that $ab = ba$ for all a, b. When we go from $\mathbb{H}$ to $\mathbb{O}$, we lose the property that $(ab)c = a(bc)$ for all a, b, c. When we go from $\mathbb{O}$ to the *sedenions*, we lose the property that $ab = 0$ implies $a = 0$ or $b = 0$.

Complex matrices. It turns out that we can represent a quaternion $a + bi + cj + dk$ as a 2×2 matrix $\left[\begin{smallmatrix} a+bi & c+di \\ -c+di & a-bi \end{smallmatrix}\right]$ of complex numbers. This way the quaternion multiplication corresponds to matrix multiplication in $M_2(\mathbb{C})$. To be precise, we can define a map

$$\phi : \mathbb{H} \to M_2(\mathbb{C})$$

this way, and ϕ is a map of rings, meaning that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$. (You should check this.) Note that

$$\det \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix} = a^2 + b^2 + c^2 + d^2 = \|a + bi + cj + dk\|^2.$$

Now consider

$$S^3 = \{x \in \mathbb{R}^4 = \mathbb{H} : \|x\| = 1\}.$$

Because the multiplication on $\mathbb{H}$ is normed, the Hermitian multiplication induces a map

$$m : S^3 \times S^3 \to S^3.$$

Moreover, since the multiplication on $\mathbb{H}$ is associative, this makes S^3 into a group. Note that because $x^{-1} = \frac{x^*}{\|x\|^2}$ for any $x \in \mathbb{H}$, $x^{-1} = x^*$ for $x \in S^3$.

Definition 10. For $A \in M_n(\mathbb{C})$, let $A^* = \bar{A}^t$ be the conjugate transpose. Then $(A^*)^* = A$ and $(AB)^* = B^*A^*$, so this makes $M_n(\mathbb{C})$ into a $*$-algebra.

Definition 11. The *unitary group* $U(n) \subset M_n(\mathbb{C})$ is the group of matrices U with $U^*U = UU^* = I_n$. The *special unitary group* $SU(n) \subset U(n)$ is the subgroup of $U(n)$ consisting of determinant 1 matrices. Note that since determinant is multiplicative, $SU(n)$ is indeed a subgroup.

With $\phi : \mathbb{H} \to M_2(\mathbb{C})$ as above, note that $\phi(x^*) = (\phi(x))^*$. Thus for $x \in S^3$, we have $I_n = \phi(1) = \phi(x^*x) = \phi(x)^*\phi(x)$. Also for $x \in S^3$, we see that $\det(\phi(x)) = 1$. These two observations imply that if $x \in S^3$ then $\phi(x) \in SU(2)$. Thus we get a map

$$\phi : S^3 \to SU(2)$$

of groups.

Theorem 12. *The map $\phi : S^3 \to SU(2)$ is an isomorphism of groups.*

Proof. The map ϕ is clearly injective, so it is enough to show that ϕ is also surjective. Let $U = \left[\begin{smallmatrix} a+bi & c+di \\ e+fi & g+hi \end{smallmatrix}\right]$ be an element of $SU(2)$. Then we know that $U^{-1} = U^*$. By Cramer's rule, we also know that

$$U^{-1} = \frac{1}{\det U} \begin{bmatrix} g+hi & -c-di \\ -e-fi & a+bi \end{bmatrix} = \begin{bmatrix} g+hi & -c-di \\ -e-fi & a+bi \end{bmatrix},$$

so $U^{-1} = U^*$ gives us

$$\begin{bmatrix} g+hi & -c-di \\ -e-fi & a+bi \end{bmatrix} = \begin{bmatrix} a-bi & e-fi \\ c-di & g-hi \end{bmatrix}.$$

Thus $g = a$, $h = -b$, $e = -c$ and $f = d$, so $U = \left[\begin{smallmatrix} a+bi & c+di \\ -c+di & a-bi \end{smallmatrix}\right]$. Then $U = \phi(a + bi + cj + dk)$, so this shows that ϕ is surjective. $\square$

References

[1] John C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39(2):145–205 (electronic), 2002.