Uniqueness of Morava K-theory

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What does this mean?

Question

What is Morava K-theory?

Answer

A particular cohomology theory, or spectrum.

Question

What do I mean by uniqueness?

Answer

Uniqueness of the multiplication, or A_∞ structure.

Cohomology

In algebraic topology, we often study spaces using cohomology:

$$n \mapsto H^n(X; A).$$

Theorem

Cohomology with coefficients in A is represented:

$$H^n(X; A) = [X, K(A, n)].$$

Here $[X, Y]$ denotes the set of homotopy classes of basepoint-preserving maps, and $K(A, n)$ is a space with only one homotopy group.

Generalized cohomology

If we fix a space E_n, we can consider

$$E^n(X) = [X, E_n].$$

Then

- $E^n(X)$ is homotopy invariant.
- A map $f : X \to Y$ induces a map $E^n(Y) \to E^n(X)$.

To get a cohomology theory, we should have a family $E = \{E_n\}_{n \in \mathbb{Z}}$ of spaces.
The suspension isomorphism

$$E^n(X) \cong E^{n+1}(\Sigma X)$$

implies that

$$[X, E_n] \cong [\Sigma X, E_{n+1}] = [X, \Omega E_{n+1}],$$

so we need $E_n \simeq \Omega E_{n+1}$.

Spectra

Definition

A prespectrum E is a sequence of spaces $\{E_n\}$ *with structure maps*

$$\Sigma E_n \to E_{n+1}.$$

Definition

A spectrum is a prespectrum E where each adjoint map

$$E_n \to \Omega E_{n+1}$$

is a homeomorphism.

Given a prespectrum E we can spectrify to get a spectrum, essentially by replacing E_n with $colim_k \Omega^k E_{n+k}$.
We have a one-to-one correspondence between cohomology theories and spectra.

Examples

Example

The sphere spectrum S is the spectrification of $n \mapsto S^n$.

Example

The spectrum HA representing $H^*(-; A)$ is given by

$$n \mapsto K(A, n).$$

Example

The spectrum representing complex K-theory is given by

$$n \mapsto \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even} \\ U & \text{if } n \text{ is odd} \end{cases}$$

This cohomology theory is defined in terms of complex vector bundles, and this description is a consequence of Bott periodicity.

Examples

Example

The spectrum representing real K-theory has

$$n \mapsto \begin{cases} \mathbb{Z} \times BO & \text{if } n \equiv 0 \text{ modulo } 8 \\ U/O & \text{if } n \equiv 1 \text{ modulo } 8 \\ Sp/U & \text{if } n \equiv 2 \text{ modulo } 8 \\ Sp & \text{if } n \equiv 3 \text{ modulo } 8 \\ \mathbb{Z} \times Bsp & \text{if } n \equiv 4 \text{ modulo } 8 \\ U/Sp & \text{if } n \equiv 5 \text{ modulo } 8 \\ O/U & \text{if } n \equiv 6 \text{ modulo } 8 \\ O & \text{if } n \equiv 7 \text{ modulo } 8 \end{cases}$$

A comparison

Spectra behave much like abelian groups:

Abelian groups	**Spectra**
Tensor product	Smash product
Integers $\mathbb{Z}$	Sphere spectrum S
Ring	S-algebra
Commutative ring	Commutative S-algebra

S-algebras

Having an S-algebra structure on E gives us extra structure. For example:

1. $E^*(X)$ becomes a graded ring.
2. We can talk about E-module spectra.
3. We can do almost anything we can do with (differential graded) R-modules for a ring R.

Goal

Given E, classify the S-algebra structures on E.

Example

If R is a (commutative) ring then HR is a (commutative) S-algebra

Example

The complex K-theory spectrum KU is a commutative S-algebra. The ring structure comes from tensor product of vector bundles.

Morava K-theory

The Morava K-theory spectrum $K(n)$ depends on a prime p and a positive integer n. The coefficient ring is

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$$

with $|v_n| = 2p^n - 2$.

Note that $K(n)_*$ is a graded field. These are essentially the only graded fields besides Hk for a field k. Nice things happen:

1. Duality between homology and cohomology:

$$K(n)^*(X) \cong Hom_{K(n)_*}(K(n)_*(X), K(n)_*).$$

2. Künneth isomorphism:

$$K(n)^*(X \times Y) \cong K(n)^*(X) \otimes_{K(n)_*} K(n)^*(Y).$$

The first Morava K-theory

Consider the complex K-theory spectrum KU. We can localize at a prime p, and we have a splitting

$$KU_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} L,$$

where L is the Adams summand, $L_* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$.
Then

$$K(1) = L/p,$$

so $K(1)$ is summand of mod p complex K-theory.

Connection to stable homotopy groups

The ring $\pi_* S$ of stable homotopy groups of spheres is extremely complicated. There is a filtration of S where the n'th layer is essentially the $K(n)$-localization $L_{K(n)}S$ of the sphere spectrum.

Example

The spectrum $J = L_{K(1)}S$ gives the "image of J", an infinite family of elements in $\pi_* S$.

Main Theorem

Question

Is there an S-algebra structure on $K(n)$?

Theorem (Robinson 1989)

~~*Yes, the Morava K-theory spectrum $K(n)$ is an S-algebra in uncountably many different ways.*~~

Theorem (A)

Yes, the Morava K-theory spectrum $K(n)$ has an essentially unique S-algebra structure.

Compare this to:

Theorem (Goerss-Hopkins-Miller)

The Morava E-theory spectrum E_n has an essentially unique commutative S-algebra structure.

A_∞ structures

Let us return to spaces for a moment. Suppose $X = \Omega Y$ is a loop space.
Then we have a product

$$\phi_2 : X \times X \to X$$

given by concatenating loops.

Problem

ϕ_2 is not associative.

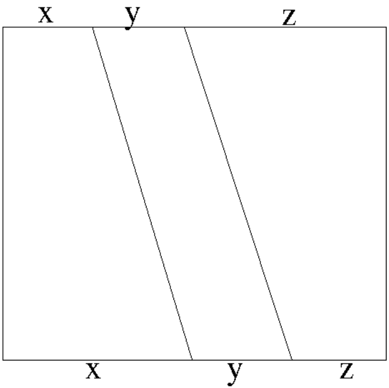
But $(xy)z$ and $x(yz)$ are homotopic, which is almost as good. We have a homotopy

$$\phi_3 : K_3 \times X^3 \to X.$$

Here $K_3 = [0, 1]$ is an interval:

A homotopy

The homotopy from $(xy)z$ to $x(yz)$ is given by the following picture:

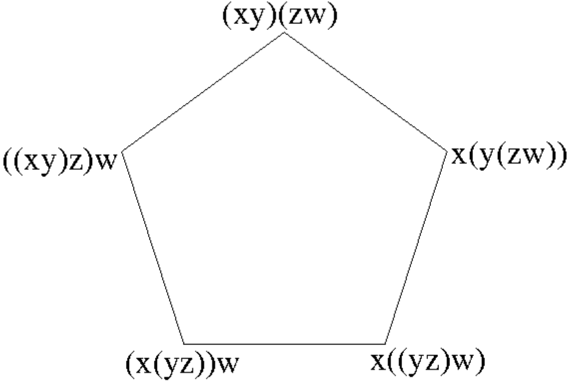


The Stasheff associahedra

What about multiplying 4 copies of X? We get a map

$$\phi_4 : K_4 \times X^4 \to X,$$

where K_4 is a pentagon:

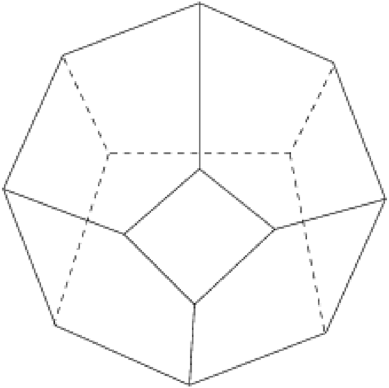


The Stasheff associahedra

With 5 copies of X we get

$$\phi_5 : K_5 \times X^5 \to X,$$

where K_5 is as follows:



The Stasheff associahedra

- The n'th associahedron is homeomorphic to an $(n-2)$-disk:

$$K_n \cong D^{n-2}$$

- The map $\phi_n : K_n \times X^n \to X$ is determined by $\phi_2, \ldots, \phi_{n-1}$ on $\partial K_n \times X^n$.
- The face poset of K_n is the poset of ways to parenthesize n variables.
- K_n contains $n+1$ copies of K_{n-1} on its boundary, corresponding to
  1. $x_1(x_2 \ldots x_n)$;
  2. $x_1 \ldots (x_i x_{i+1}) \ldots x_n$ for $1 \leq i \leq n-1$;
  3. $(x_1 \ldots x_{n-1})x_n$.

A_∞ structures

Definition

An A_∞ H-space is a space X with coherent maps

$$\phi_n : K_n \times X^n \to X.$$

An A_∞ ring spectrum is a spectrum A with coherent maps

$$\phi_n : (K_n)_+ \wedge A^{\wedge n} \to A.$$

Theorem

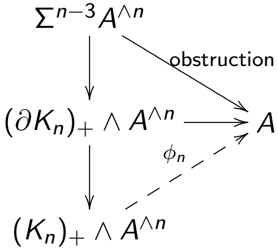
Any A_∞ ring spectrum A can be replaced by a weakly equivalent associative S-algebra.

Theorem (A, restated)

The moduli space of A_∞ structures on the Morava K-theory spectrum $K(n)$ is connected.

Obstruction theory

Can we build an A_∞ structure by induction?
ϕ_n is determined on $(\partial K_n)_+$, so we have the following extension problem:



A map to A is an A-cohomology class, so the obstruction lies in

$$[\Sigma^{n-3} A^{\wedge n}, A] \cong A^{3-n}(A^{\wedge n}).$$

The uniqueness obstruction lies in

$$A^{2-n}(A^{\wedge n}).$$

Question

How does the obstruction to an A_n structure change if we change the A_{n-1} structure by $f \in A^{3-n}(A^{\wedge n-1})$?

Answer

The obstruction changes by

$$a_1 f(a_2, \ldots, a_n) + \sum_{1 \le i \le n-1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n f(a_1, \ldots, a_{n-1}) a_n.$$

Each term corresponds to a copy of K_{n-1} on the boundary of K_n.

This is just like the formula for the differential in Hochschild cohomology.

The moduli space

Definition

Let $\mathcal{A}_n(A)$ be the category of A_n structures on A and let $B\mathcal{A}_n(A)$ be the geometric realization of the nerve of this category.

Remark

This is not really a category, because a map $f : X \to Y$ of A_n ring spectra is defined in terms of another family of polyhedra, encoding $f(x_1x_2) \simeq f(x_1)f(x_2)$ and higher homotopies.
We get an $(\infty, 1)$-category (aka quasi-category) instead of a category, but we can still take the geometric realization.

A tower of fibrations

Proposition

The maps

$$B\mathcal{A}_\infty(A) \to \ldots \to B\mathcal{A}_n(A) \to B\mathcal{A}_{n-1}(A) \to \ldots \to B\mathcal{A}_1(A)$$

form a tower of fibrations, where the fiber of $B\mathcal{A}_n(A) \to B\mathcal{A}_{n-1}(A)$ over a point is either:

- *Empty*

or

- *$Hom(\Sigma^{n-2}A^{\wedge n}, A)$.*

Note that an A_1 structure contains no information, so $B\mathcal{A}_1(A) = BAut(A)$ is the classifying space of the group of automorphisms of A.

A spectral sequence

Theorem (A)

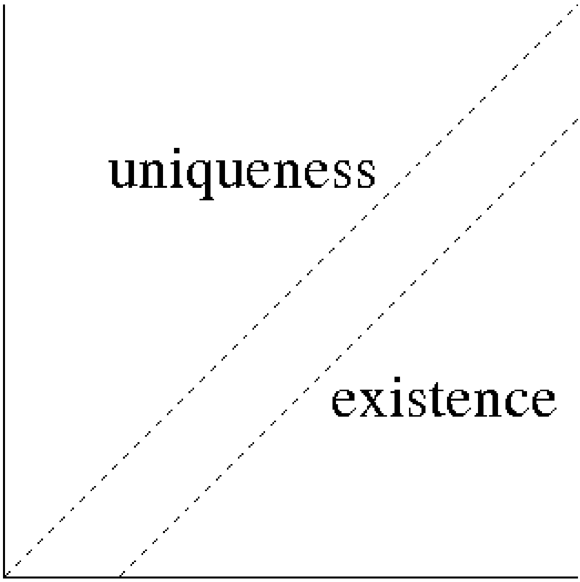
There is a spectral sequence $\{E_r^{s,t}\}$ *with* $E_1^{s,t}$ *defined for* $s \geq 0$ *and* $t - s \geq -1$ *converging to* $\pi_{t-s} B\mathcal{A}_\infty(A)$ *with the obstructions to* $B\mathcal{A}_\infty(A)$ *being nonempty on the subdiagonal* $t - s = -1$. *We have* $E_1^{0,-1} = \varnothing$, $E_1^{0,0} = 0$, $E_1^{0,1} \cong \pi_0 Aut(A)$ *and*

$$E_1^{s,t} \cong [\Sigma^{t-1} \bar{A}^{\wedge s+1}, A]$$

otherwise. Here $E_1^{s,t}$ *is a group for* $t - s \geq 1$, *a torsor over the corresponding group for* $t - s = 0$, *and a set for* $t - s = -1$.

Here $\bar{A}$ is the cofiber of the unit map $S \to A$.

The moduli space spectral sequence



The moduli space spectral sequence and *THH*

Theorem (A)

Let $\{\tilde{E}_r^{s,t}\}$ denote the potential spectral sequence converging to topological Hochschild cohomology of A. Then

$$E_1^{s,t} \cong \tilde{E}_1^{s+1,t-1}$$

for $s \geq 1$ and the d_1 differentials agree.

Theorem

The E_2 term $\tilde{E}_2^{*,*}$ for the spectral sequence converging to $THH^*(K(n))$ is given by

$$K(n)_*[q_0, \ldots, q_{n-1}]$$

where $|q_i| = (1, -2p^i + 1)$. In particular this E_2 term is concentrated in even total degree so there can be no differentials.

The term $E_1^{0,1}$

An A_1 structure is no data, so

$$E_1^{0,1} = \pi_1 BAut(K(n)) = [K(n), K(n)]^{\times}.$$

We have

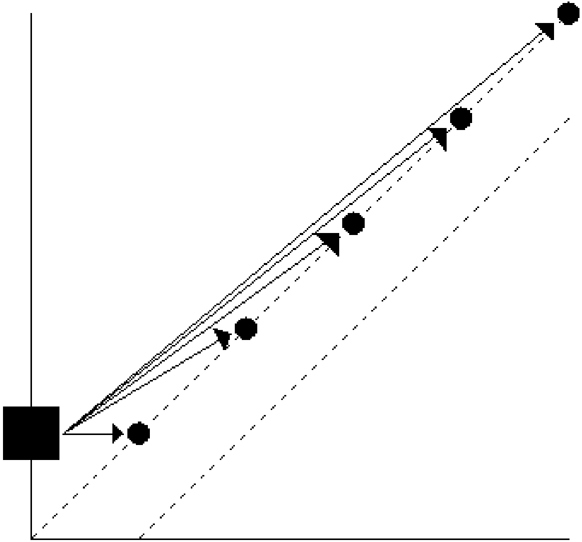
$$[K(n), K(n)] = Hom_{K(n)_*}(K(n)_* K(n), K(n)_*)$$

and $K(n)_* K(n)$ is big, so this is a big group.

$$K(n)_* K(n) \cong \Lambda(\alpha_0, \ldots, \alpha_{n-1}) \otimes$$
$$K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i).$$

The moduli space spectral sequence for $K(n)$

The group $E_1^{0,1}$ kills everything on the diagonal.



Non-uniqueness in other situations

Theorem (A)

If $n > 1$ or $p > 2$ the moduli space of A_∞ structures on $E_n/\mathfrak{m}$ is not connected. In fact, THH of $E_n/\mathfrak{m}$ varies over the moduli space or A_∞ structures. There exists an A_∞ structure on $E_n/\mathfrak{m}$ such that

$$THH(E_n/\mathfrak{m}) \simeq E_n.$$

Theorem (A)

For each $1 \leq i \leq p-1$ there exists an A_∞ structure on KU/p such that $THH(KU/p)$ is a degree i extension of $KU_p^\wedge$.

Out of time??

I expect to be out of time by now. If there is still time left, let's go on...

The first obstruction

Suppose $n = 1$. Then the first obstruction is $v_1 q_0^p$, where q_0 comes from the mod p Bockstein.

Theorem (Dugger-Shipley)

The n'th k-invariant for a connective S-algebra A lies in

$$THH^{n+2}(P_{n-1}A; H\pi_n A).$$

Now we try to build $P_{2p-2}k(1)$, which has an $\mathbb{F}_p$ in degree 0 and $2p-2$. The first k-invariant lies in

$$THH^{2p}(H\mathbb{F}_p; H\mathbb{F}_p).$$

There is a spectral sequence

$$E_2 = Ext_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow THH^*(H\mathbb{F}_p; H\mathbb{F}_p).$$

The mod p Bockstein τ_0 gives a class $(\sigma\tau_0)^p$ in the E_2 term of this spectral sequence, but there is a differential

$$d_{p-1}(\sigma\xi_1) = (\sigma\tau_0)^p$$

in this spectral sequence.