Finding a Jordan canonical form

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We are given a matrix

$$A = \begin{pmatrix} 5 & 1 & 1 & -5 \\ -6 & 0 & 1 & 3 \\ 7 & 2 & 4 & -11 \\ 4 & 1 & 1 & -4 \end{pmatrix}$$

regarded as an element in $M_4(\mathbb{C})$ and we want to put A on Jordan canonical form. This means finding an invertible matrix C such that

$$B = C^{-1}AC$$

is a block diagonal matrix, made up of Jordan blocks.

We first compute the characteristic polynomial for A. We find

$$\sigma(x) = \det(xI - A) = \ldots = (x-1)^3(x-2).$$

Hence we know that the minimal polynomial must be

$$m(x) = (x-1)^p(x-2)$$

for some $1 \leq p \leq 3$. We compute

$$(A - I)(A - 2I) = \ldots \neq 0$$

and

$$(A - I)^2(A - 2I) = \ldots = 0,$$

so in fact

$$m(x) = (x-1)^2(x-2).$$

Hence we know that the Jordan form of A has a 2×2 Jordan block $J_2(1)$, so

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

It remains to find C. To do this we need an especially nice basis for $\mathbb{C}^4$. We start by computing the idempotents E_1 and E_2. The partial fraction decomposition theorem from calculus tells us that we can write

$$\frac{1}{\sigma(x)} = \frac{a_1(x)}{(x-1)^3} + \frac{a_2}{x-2}.$$

We solve to find $a_1(x) = -1 + x - x^2$ and $a_2 = 1$. Hence

$$E_1 = (-I + A - A^2)(A - 2I) = \ldots = \begin{pmatrix} 4 & 0 & -3 & 3 \\ -1 & 1 & 1 & -1 \\ 7 & 0 & -6 & 7 \\ 3 & 0 & -3 & 4 \end{pmatrix}$$

and

$$E_2 = I(A - I)^3 = \ldots = \begin{pmatrix} -3 & 0 & 3 & -3 \\ 1 & 0 & -1 & 1 \\ -7 & 0 & 7 & -7 \\ -3 & 0 & 3 & -3 \end{pmatrix}$$

Note that E_1 has rank 3 and E_2 has rank 1, as expected. Hence $V_1 = E_1(\mathbb{C}^4)$ has dimension 3 and $V_2 = E_2(\mathbb{C}^4)$ has dimension 1. We can also check that $E_1^2 = E_1$ and $E_2^2 = E_2$.

To get a good basis, we pick some v_1 in V_1 with $(A - I)v_1 \neq 0$. I pick $v_1 = (3, -2, 6, 3)$, which is $E_1 w$ for $w = (0, -3, 1, 0)$. (One could argue that picking the first column of E_1 is more consistent.) Then $v_2 = (A - I)v_1 = (1, -1, 2, 1)$. We can check that v_1 is not an eigenvector for A, but v_2 is. We need to find one more eigenvector for A with eigenvalue 1; some linear algebra tells us that $v_3 = (-2, 6, -3, -1)$ will do the job.

We also pick an eigenvector (with eigenvalue 2) from V_2. The first column vector $v_4 = (-3, 1, -7, -3)$ will do just fine. Hence the matrix

$$C = \begin{pmatrix} 1 & 3 & -2 & -3 \\ -1 & -2 & 6 & 1 \\ 2 & 6 & -3 & -7 \\ 1 & 3 & -1 & -3 \end{pmatrix}$$

with column vectors $\{v_2, v_1, v_3, v_4\}$ will do the job of putting A on Jordan canonical form.