HOMEWORK 5, 20400 SECTION 51

Solutions to selected exercises.

• Many of you had trouble with 15.1:7, so here's a solution, or rather two solutions. Solution 1: We can do this algebraically. A point in P looks like $(u, v, -u-v)$. Given $(x, y, z) \in \mathbb{R}^3$, let $g(u, v) = d((u, v, -u-v), (x, y, z))^2$ be the square of the distance from (x, y, z) to $(u, v, -u-v)$. Then

$$g(u, v) = x^2 + y^2 + z^2 - 2xu - 2yv + 2zu + 2zv + 2u^2 + 2v^2 + 2uv,$$

and $T(x, y, z)$ is the minimizer for $g(u, v)$. So we need $\nabla g(u, v) = (0, 0)$, or

$$\begin{aligned}\frac{dg}{du} &= -2x + 2z + 4u + 2v = 0;\\ \frac{dg}{dv} &= -2y + 2z + 2u + 4v = 0.\end{aligned}$$

Solving for u and v gives us $u = \frac{2}{3}x - \frac{1}{3}y - \frac{1}{3}z$ and $v = -\frac{1}{3}x + \frac{2}{3}y - \frac{1}{3}z$, so

$$T(x, y, z) = (\frac{2}{3}x - \frac{1}{3}y - \frac{1}{3}z, -\frac{1}{3}x + \frac{2}{3}y - \frac{1}{3}z, -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}z).$$

This shows that T is linear and that the associated matrix is

$$A = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

If you minimize the distance instead of the square of the distance you'll get the same thing, the algebra is just a bit more complicated.

Solution 2: Geometrically. The normal vector to the plane P is $n = \frac{1}{\sqrt{3}}(1, 1, 1)$, so if we write $(x, y, z) = v + cn$ where $c \in \mathbb{R}$ and $v \in P$ then v is the closest point to (x, y, z). We can find the coefficient c by taking the inner product with n. We find that $\langle (x, y, z), n\rangle = c$ because $\langle v, n\rangle = 0$. Thus

$$\begin{aligned}T(x, y, z) = v = (x, y, z) - \langle (x, y, z), n\rangle n &= (x, y, z) - \frac{1}{\sqrt{3}}(x + y + z)\frac{1}{\sqrt{3}}(1, 1, 1)\\ &= (x, y, z) - \frac{1}{3}(x + y + z, x + y + z, x + y + z)\\ &= (\frac{2}{3}x - \frac{1}{3}y - \frac{1}{3}z, -\frac{1}{3}x + \frac{2}{3}y - \frac{1}{3}z, -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}z)\end{aligned}$$

which again shows that T is linear and that the associated matrix is as above.

• We proved in class (or will prove) that if A is an $n \times n$ matrix with $\det A \neq 0$ and B is defined by $b_{ij} = \frac{1}{\det A}(-1)^{i+j} detA^{ji}$ then $(AB)_{ii} = 1$. Finish the proof that B is the inverse of A by showing that for $i \neq j$ we have $(AB)_{ij} = 0$. Hint: express $(AB)_{ij}$ as the determinant of some suitably chosen matrix C you know has determinant 0.

Solution: For $i \neq j$ we get $(AB)_{ij} = \frac{1}{\det A} \sum_{k=1}^{n} (-1)^{j+k} a_{ik} \det A^{jk}$. Let C be the matrix where the i'th and j'th rows are both the i-th row of A, otherwise each row of C is the same as the corresponding row of A. Then $\det C = 0$ since C has two identical rows, and $\det C = \sum_{k=1}^{n} (-1)^{j+k} c_{jk} \det C^{jk}$. Now $c_{jk} = a_{ik}$ and $C^{jk} = A^{jk}$, so $0 = \det C = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} \det A^{jk}$, and it follows that $(AB)_{ij} = 0$.

- We proved in class (or will prove) that we can calculate $\det A$ by cofactor expansion along any row, as $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A^{ij}$. Prove that $\det A = \det A^T$, and conclude that we can also calculate $\det A$ by cofactor expansion along any column as $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A^{ij}$.

For example, you can prove this by induction on n, the case $n = 1$ being trivial and $n = 2$ being easy to check. For $n \geq 3$, write

$$\det A = a_{11} \det A^{11} + \sum_{j=2}^{n} (-1)^{j+1} a_{1j} \det A^{1j}.$$

Now you want to compare this to

$$a_{11} \det A^{11} + \sum_{i=2}^{n} (-1)^{i+1} a_{i1} \det A^{i1}.$$

Notice that the first summand in each expression matches up. Now expand the rest of the summands in the first expression along the first column and the rest of the summands in the second expression along the first row.

Solution: By induction we know that we can calculate the determinant of an $(n-1) \times (n-1)$ matrix by cofactor expansion along any column. Thus we can write $\det A^{1j}$ as $\sum_{k=1}^{n-1} (-1)^{k+1} (A^{1j})_{k1} \det (A^{1j})^{k1}$. Let's write $A^{ij,kl}$ for A with row ij and column kl deleted. We find that $(A^{1j})_{k1} = a_{k+1,1}$ and $(A^{1j})^{k1} = A^{1k,1j}$, so $\det A^{1j} = \sum_{k=1}^{n-1} (-1)^{k+1} a_{k+1,1} \det A^{1k,1j}$. In total this gives

$$\det A = a_{11} \det A^{11} + \sum_{j=2}^{n} \sum_{k=1}^{n-1} (-1)^{j+k+2} a_{1j} a_{k+1,1} \det A^{1k,1j}.$$

Taking the cofactor expansin of the second expressin along the first column gives something similar, namely

$$\det A = a_{11} \det A^{11} + \sum_{i=2}^{n} \sum_{k=1}^{n-1} (-1)^{i+k+2} a_{i1} a_{1,k+1} \det A^{1i,1k}.$$

But these are just two different ways of writing the same sum, so they are equal and this proves that $\det A = \det A^T$.

- We proved in class (or will prove) that $\det A \neq 0$ if and only if the column vectors of A are linearly independent. Prove that $\det A \neq 0$ if and only if the row vectors of A are linearly independent. (A very simple proof is now possible.)

Solution: $\det A \neq 0$ is equivalent to $\det A^T \neq 0$ which is equivalent to the column vectors of A^T being linearly independent. But the column vectors of A^T are the row vectors of A.