HOMEWORK 4 SOLUTIONS, 20400 SECTION 51

14.2:1. $\phi(t) = f(2t, 3t) = e^{6t^2} + 4t^2 + 12t^2 = e^{6t^2} + 16t^2$. Thus $\phi'(t) = 12te^{6t^2} + 32t$ and $\phi''(0) = 44$. The Hessian $\nabla^2 f(0,0) = \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$, so with $h = (2,3)$ we get $\langle \nabla^2 f(0,0)h, h\rangle = \langle (13,6),(2,3)\rangle = 44$.

14.2:3. Define $\phi(t) = f(th)$. By the Directional Derivative Theorem, $\phi'(0) = 0$, and by Theorem 14.12, $\phi''(0) > 0$. Since f is $C2D$, $\phi''(t) > 0$ in some interval $(-r, r)$, so $\phi(t) > \phi(0)$ for $t \in (-r, r)$, which is to say $f(th) > f(0,0)$ in that interval.

14.2:4. In this case, r depends on h in a possibly discontinuous way, and it is possible that as $h \to h_*$, $r(h) \to 0$ (even though $r(h_*) \neq 0$).

14.2:5. This one I leave to you.

14.2:9. $A = \begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & -1 & 1/2 \\ 0 & 1/2 & -1 \end{bmatrix}$

14.3:1. I will only do part b. $\nabla g(x,y,z) = (2xe^{x^2-4y+y^2}, (-4+2y)e^{x^2-4y+y^2}, 2z)$, which is 0 when $(x,y,z) = (0,2,0)$. $\nabla^2 g(0,2,0) = \begin{bmatrix} 2e^{-4} & 0 & 0 \\ 0 & 2e^{-4} & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which is positive definite. Hence $(0,2,0)$ is a local minimizer, and $g(0,2,0) = e^{-4}$ is a local (and in fact a global) minimum.

14.3:3. First consider only the interior of K. Then $\nabla f(x,y) = ((1-x)e^{-x}\cos y, -xe^{-x}\sin y)$, which is never zero. (Only on the boundary.) Thus any max/min will occur on the boundary. Then we consider each of the four pieces of the boundary separately. If $x = -1$: consider $-e\cos y$, which has a min when $y = 0$, max when $y = \pm\pi$. This gives us candidates $f(-1,0) = -e$ for the min and $f(-1,-\pi) = f(-1,\pi) = e$ for the max. If $x = 1$: get candidate $f(1,0) = e^{-1}$ for the max and $f(1,-\pi) = f(1,\pi) = -e^{-1}$ for the max. The other two pieces of the boundary do not give any other candidates, so we see that $f(-1,0) = -e$ is the min and $f(-1,-\pi) = f(-1,\pi) = e$ is the max.

14.3:7. It follows from what we've done that $f(x+su) > f(x)$ for s sufficiently small and $f(x+tv) < f(x)$ for t sufficiently small, so with $r \leq min(s,t)$ the ball $B_r(x)$ will contain points y and z with $f(y) > f(x)$ and $f(y) < f(x)$.

14.3:8. $n = 2$, $f(x,y) = x^4 + y^4$ or $x^4 - y^4$ or $-x^4 - y^4$.

14.3:9. Same as the proof of Theorem 14.22, with $c/2$ replaced by c.

14.3:11. I will only do c). Let $f(x,y) = e^{x-y}$ Then $\nabla f(0,0) = (1,-1)$ and $\nabla^2 f(0,0) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, so $g(x,y) = 1 + x - y + 1/2(x^2+y^2) - xy$ is a second order approximation of f, and

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{e^{x-y} - 1 - x + y}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{1/2(x^2+y^2) - xy}{x^2+y^2} \\ &= 1/2 - \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}, \end{aligned}$$

and as we have seen this last limit does not exist (I didn't notice that when I assigned the problem).

Last problem:
a) We get $Av = v$ and $Aw = 3w$.
b) For any $u = (u_1, u_2)$, we can write $u = 1/2(u_1 - u_2)v + 1/2(u_1 + u_2)w$.
c) With $u = c_1 v + c_2 w$ we get $\langle Au, u\rangle = \langle c_1 v + 3c_2 w, c_1 v + c_2 w\rangle = 2c_1^2 + 6c_2^2$.
d) Given any $u \neq 0$, we want to show that $Q_A(u) > 0$. But $Q_A(u) = 2c_1^2 + 6c_2^2 = Q_B(c_1, c_2)$, so because B is positive definite this is positive and A is positive definite as well.
e) In general, given v and w with $Av = \lambda_1 v$ and $Aw = \lambda_2 w$ as in the problem, we get $(vA)w = \langle \lambda_1 v, w\rangle = \langle v_1, \lambda_2 w\rangle = v(Aw)$, which can only happen if $\langle v, w\rangle = 0$.

Then if we write $u = c_1 v + c_2 w$ we get $Q_A(u) = c_1^2\lambda_1 + c_2^2\lambda^2 > 0$ whenever $u \neq 0$.