

Research Statement

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My research lies in the crossroads of representation theory, geometry and number theory. I would like to describe in some detail, four projects that I have been thinking about in the last few years. The first three are related and have origin in my thesis. One central theme in these projects is a loose analogy with the theory of Weyl groups. This analogy motivates many of our definitions and indicates directions of proofs.

- (1) (see [Ba1], [Ba5], [Ba2]) The reflection group of the complex Lorentzian Leech lattice and its connection to the bimonster ¹, via complex hyperbolic geometry, as conjectured by Daniel Allcock.
- (2) ([Ba6], under preparation) Some meromorphic automorphic forms of type $U(1, n)$ related to the hyperbolic reflection group studied in (1), analytic continuation of similar forms of type $O(1, n)$ and their relation to Borcherds automorphic forms.
- (3) (see [Ba4]) Analogies between Weyl groups and unitary reflection groups and relevance of these analogies in characterizing the diagrams for unitary reflection groups.
- (4) (see [Ba3]) Study of a class of finite topological spaces that satisfy a Poincare duality theorem, aided by a new definition of orientation.

As I have tried to point out, these projects indicate many diverse exciting research directions: e.g. the monster manifold, $K(\pi, 1)$ problem for complex hyperplane complement, automorphic form on Complex hyperbolic space and higher category theory, just to name a few.

1. COMPLEX HYPERBOLIC REFLECTION GROUPS

1.1. The bimonster: Given a graph T , let $\mathcal{A}(T)$ denote the *Artin group*² of T . Let $\text{Cox}(T, n)$ be the quotient of $\mathcal{A}(T)$ by the relations $x^n = 1$ for all $x \in T$. Let Y_{555} (also called M_{666}) be the graph having the shape of a “Y” with 16 vertices (six in each hand including the central vertex). The Ivanov-Norton theorem ([Iv], [No]), conjectured earlier by Conway and Norton, gives a remarkably simple presentation for the bimonster as a quotient of $\text{Cox}(Y_{555}, 2)$, with one extra relation.

Let D be the incidence graph of the projective plane over \mathbb{F}_3 . The graph Y_{555} is a maximal sub-tree in D . Using Ivanov-Norton theorem, Conway et.al. showed (see [CSi], [CNS], [CP]) that the map from $\text{Cox}(Y_{555}, 2)$ to the bimonster extends to a surjection

$$(1.1) \quad \phi_2 : \text{Cox}(D, 2) \rightarrow M \wr 2.$$

The kernel is generated by some explicitly described simple relations called “deflating the 12-gons”. (see [CSi]).

1.2. A very special hyperbolic reflection group: Let L be the direct sum of the complex Leech lattice defined over $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$ and a hyperbolic cell. This is a Hermitian lattice of signature $(1, 13)$, whose underlying \mathbb{Z} -module is the unique even unimodular lattice of signature $(2, 26)$. Daniel Allcock showed in [Al1] that the complex reflection group of L

¹The bimonster, denoted by $M \wr 2$, is the wreath product of the sporadic simple group monster and $\mathbb{Z}/2\mathbb{Z}$.

² $\mathcal{A}(T)$ is the group generated by the set of vertices of T subject to the relations $xyx = yxy$ if $\{x, y\}$ is an edge of T and $xy = yx$ otherwise

has finite index in $\text{Aut}(L)$ and that there is a map ϕ from $\text{Cox}(Y_{555}, 3)$ to the reflection group of L . In [Ba1] we found that the map $\phi : \text{Cox}(Y_{555}, 3) \rightarrow \text{Aut}(L)$ extends to a map

$$(1.2) \quad \phi : \text{Cox}(D, 3) \rightarrow \text{Aut}(L).$$

This is surprising since it amounts to finding 26 vectors with specified inner products which fit together in the 14 dimensional lattice L . The main theorem of [Ba1] states that:

1.3. Theorem. *The map $\phi : \text{Cox}(D, 3) \rightarrow \text{Aut}(L)$ is onto.*

Put in another way, for each vertex r of D , there is a complex reflection $\phi(r)$ of order 3. These 26 reflections, called the *simple reflections*, braid or commute according to the edges of the graph D and generate $\text{Aut}(L)$. These 26 reflections do not form a minimal set of generators for $\text{Aut}(L)$. In-fact, in [Ba5] we show that 14 of these 26 reflections suffice to generate $\text{Aut}(L)$. However, our guiding intuition in this whole project is that the 26 node diagram behaves like the Coxeter-Dynkin diagram for L .

There is a natural way to construct the lattice L out of the diagram D , analogous to the construction of root lattices from Dynkin diagrams (see 2.4-2.6 of [Ba5]). This construction makes clear the reason for the occurrence of $\text{Cox}(D, 3)$ in $\text{Aut}(L)$. Let

$$\Gamma := \mathbb{P} \text{Aut}(L)$$

be the automorphism group of L modulo scalars. Let $\mathbb{C}H^{13}$ be the 13 dimensional complex hyperbolic space³ on which Γ acts faithfully. Let $\mathcal{M} \subseteq \mathbb{C}H^{13}$ be the union of the mirrors of reflections in Γ . The construction of L from D implies that $\text{Aut}(D) \subseteq \Gamma$. So the finite group $\text{Aut}(D) \simeq 2.L_3(3)$ acts on $\mathbb{C}H^{13}$.

1.4. Theorem ([Ba1], prop. 6.1). *There is a unique point $\bar{\rho}$ (called the Weyl vector) in $\mathbb{C}H^{13}$ fixed by the finite group $\text{Aut}(D)$. The mirrors in $\mathbb{C}H^{13}$ fixed by the simple reflections, are precisely the mirrors that are closest to $\bar{\rho}$.*

Theorem 1.4 justifies our use of the Weyl group terminology. Our proof of theorem 1.3 also follows the analogy with Weyl groups. We define the height of a root as the hyperbolic distance of the corresponding mirror from the Weyl vector $\bar{\rho}$ and run a height reduction algorithm. Our proof depended on some computer verifications. Daniel Allcock has recently found a computer free proof of theorem 1.3 (see [Al3]).

To summarize, we find that the Artin group $\mathcal{A}(D)$ maps onto the finite group bimonster and the infinite reflection group Γ . The generators of $\mathcal{A}(D)$ map to elements of order 2 and 3 respectively. In [Ba1] we also show that the standard relators in the bimonster often have finite order in $\text{Aut}(L)$. For example, Conway's "deflation relation" holds in $\text{Aut}(L)$. Daniel Allcock has made a conjecture that relates $\Gamma = \mathbb{P} \text{Aut}(L)$ and the bimonster via complex hyperbolic geometry. In-fact the conjecture predates [Ba1], so the results in [Ba1] should be viewed as evidence towards it.

Consider the orbifold $(\mathbb{C}H^{13} \setminus \mathcal{M})/\Gamma$, which plays the role of the "braid space" in our example. Let G be the fundamental group of the orbifold $(\mathbb{C}H^{13} \setminus \mathcal{M})/\Gamma$. For each vertex $r \in D$ and the corresponding simple reflection $\phi(r)$, there is a canonical element in $g_r \in G$, the so called "generator of monodromy around the mirror of $\phi(r)$ ".

1.5. Conjecture (Allcock, see [Al2]). *There is a surjection from G to the bimonster defined by $g_r \mapsto \phi_2(r)$ (where ϕ_2 is defined in eq. (1.1)).*

³ $\mathbb{C}H^{13}$ consists of the set of complex lines of positive norm in the complex vector space $L \otimes_{\mathbb{R}} \mathbb{C}$.

By definition of G , there is a natural projection $\pi : G \rightarrow \Gamma$. This map from G to Γ and the conjectured map from G to bimonster are analogous to the maps from the braid group to the group of permutations. In [Al2], Daniel Allcock speculates about a possible moduli interpretation of the complex hyperbolic orbifold $\mathbb{C}H^{13}/\Gamma$, similar to such examples found by Deligne-Mostow (see [DM]) and by Allcock-Carleson-Toledo (see [ACT]).

In the conjectural picture described in [Al2], the orbifold $\mathbb{C}H^{13}/\Gamma$ classifies certain algebro-geometric objects. This space affords two (possibly ramified) covers, obtained by putting some sort of marking on these objects. The bimonster and Γ act on these two covers as monodromy groups. If this conjectural picture is correct, then it would provide a very satisfying explanation of the phenomenon observed in [Ba1]. Hopefully it would also provide a candidate for the monster manifold, whose existence is sought for in Hirzebruch's prize question (see [HBJ]). The orbifold $\mathbb{C}H^{13}/\Gamma$ contains several moduli spaces, including the largest Deligne-Mostow example of the space of 12 points on \mathbb{P}^1 .

In [Ba5], we took a step towards conjecture 1.5 by proving the following theorem:

1.6. Theorem. *By theorem 1.3, there is a surjection $\mathcal{A}(D) \rightarrow \Gamma$ (also denoted by ϕ). The map $\phi : \mathcal{A}(D) \rightarrow \Gamma$ lifts to a homomorphism $\psi : \mathcal{A}(D) \rightarrow G$ given by $\psi(r) = g_r$. So we have the following commutative diagram (with the bottom row exact):*

$$\begin{array}{ccccccc}
 & & \mathcal{A}(D) & & & & \\
 & & \downarrow \psi & \searrow \phi & & & \\
 1 & \longrightarrow & \pi_1(\mathbb{C}H^{13} \setminus \mathcal{M}) & \longrightarrow & G & \xrightarrow{\pi} & \Gamma \longrightarrow 1
 \end{array}$$

Theorem 1.6 answers a question raised in [Al2] and provides us with an action of $\text{Cox}(D, 2)$ on a (possibly ramified) cover of $(\mathbb{C}H^{13} \setminus \mathcal{M})/\Gamma$. To obtain an action of bimonster on this ramified cover, one has to show that the deflation relations hold and that the action is non-trivial. If, further, the map ψ is onto, then that proves conjecture 1.5. This discussion provides a lot of interesting questions for the future.

1.7. The quaternionic Lorentzian Leech lattice and other examples: In [Ba2] we found a quaternionic hyperbolic reflection group that illustrates the same analogy with Weyl groups, utilized in [Ba1]. We replace the ring $\mathbb{Z}[e^{2\pi i/3}]$ by a ring of quaternions (the Hurwitz integers), replace the complex Leech lattice by the quaternionic form of Leech lattice, replace the graph $D = \text{Inc}(\mathbb{P}^2(\mathbb{F}_3))$ by $\text{Inc}(\mathbb{P}^2(\mathbb{F}_2))$ and order 3 complex reflections by order 4 quaternionic reflections. With this setting, the story of [Ba1] repeats, that is, we have exact analogs of theorem 1.3 and 1.4. Few more such complex lattices, for which analogs of theorem 1.3 and 1.4 hold, are studied in chapter 4 of my thesis. It would be nice to have a conceptual explanation of the phenomenon observed in these examples.

2. AUTOMORPHIC FORMS ON HYPERBOLIC SPACES

2.1. The complex hyperbolic case: We maintain the notations of section 1. Automorphic forms of type $U(1, n)$ can be constructed by Borchers singular theta correspondence (see [Bo]). One starts with a modular form of singular weight with poles at cusps, takes the singular theta lift to obtain an automorphic form for $O(2, 2n)$ and then restricts this function to the complex hyperbolic space. In our example, this procedure yields automorphic forms defined on $\mathbb{C}H^{13}$, invariant under (a finite index subgroup of) Γ , with zeros and poles along

the mirrors (see [Al1]). In [Ba5] we give an elementary method to construct meromorphic automorphic forms on $\mathbb{C}H^{13}$ with the above properties, simply by writing down an infinite series: $E_m(z) = \sum_{r \in \Phi} \langle r, z \rangle^{-6m}$ (with m large enough). The sum here is on the set of all roots of L and z is a positive norm vector of $L \otimes_{\mathcal{E}} \mathbb{C}$. It would be interesting to understand the relation of these forms with the Borchers forms.

Let K be the complex hyperbolic line in $\mathbb{C}H^{13}$, fixed point-wise by the group $L_3(3)$ of diagram automorphisms (K is isomorphic to the real hyperbolic plane). If f is a meromorphic automorphic form invariant under Γ with zeroes and poles along the mirrors, then we show in [Ba5] that the restriction of f to K is a modular form of level thirteen. So we have an explicit method to obtain modular forms from automorphic forms on $\mathbb{C}H^{13}$. It would be interesting to understand how this is related to the (inverse of) the Borchers lift from modular forms of singular weight to automorphic forms of type $U(1, n)$.

The Borchers forms were used in [AF] to obtain explicit projective embedding of a complex hyperbolic orbifold that classify cubic surfaces. If the automorphic forms on $\mathbb{C}H^{13}$ mentioned above can be used to obtain a projective embedding of $\mathbb{C}H^{13}/\Gamma$, then that may help in finding a moduli interpretation of this orbifold.

2.2. The real hyperbolic case and analytic continuation: Let $II_{1,25}$ be the unique even unimodular \mathbb{Z} -lattice of signature $(1, 25)$. The functions $E_n(z)$ defined above are analogous to the Eisenstein series. In the same spirit, we can define a function $E(z, s)$, that is analogous to the real analytic Eisenstein series. The function $E(z, s)$ is defined on positive norm vectors of $II_{1,25} \otimes_{\mathbb{Z}} \mathbb{R}$ and is invariant under $\text{Aut}(II_{1,25})$. In [Ba6] (under preparation) we study the function $E(z, s)$, calculate its Fourier expansion at a cusp of Leech type, in terms of Gauss and Kloosterman sums. A theorem of Selberg from [Se] implies that the function has a meromorphic continuation beyond its abscissa of convergence. Prof. Borchers suggested that the residue of this function at the “first pole” might be related to automorphic forms of type $O(1, 25)$ obtained by singular theta lift.

3. DIAGRAMS FOR UNITARY REFLECTION GROUPS

Coxeter and Broue et.al defined a diagram D_G for each unitary reflection group G (see [Co], [BMR]). These are analogous to Dynkin diagrams. The vertices of D_G correspond to a minimal set of generators for G . But other than this, the definition of D_G is ad-hoc and case by case. It is remarkable that, inspite of the ad-hoc definition, the diagram D_G contains a lot of non-trivial invariant theoretic information about the group G .

3.1. Using the complex diagrams: While trying to generalize Conway’s “holy constructions” for Leech lattice (see [CS]) in order to identify Leech cusps in the lattice L (see section 1), we got started on a project to study the diagrams for unitary reflection groups (see [Ba4]). To illustrate the usefulness of the diagrams, we give a new proof of the classification of “Eisenstein root lattices”, using the affine diagrams with “balanced numbering”. The proof is similar to the familiar A-D-E classification and only uses diagram combinatorics. The classification of Eisenstein root lattices was used by Allcock in [Al3] to give a computer free proof of theorem 1.3.

3.2. Characterizing the complex diagrams: Next, we define a “Weyl vector” for an unitary reflection group. We show that Weyl vectors exist for all unitary reflection groups. Based on this notion of Weyl vector, we give an algorithm (see 3.8-3.9 of [Ba4]) to pick a

minimal set of complex reflections generating G . For Weyl groups, the algorithm converges in one step and immediately yields a set of simple reflections. By computer experiment, we find that the algorithm works when G is “primitive” and has a set of roots whose \mathbb{Z} -span forms a discrete subset of the ambient vector space. For the groups known as G_{29} , G_{31} , G_{33} and G_{34} , the diagrams produced by the algorithm are different from those in the literature. We should remark that these four groups are among a few exceptional cases, for which there seem to be some doubt in the literature as to whether the known diagrams are the “right ones”. Other than these exceptional cases, our algorithm produces the known diagrams.

The work done in [Ba4] indicates that the complex diagrams D_G probably have a geometric origin, but we are unable to obtain a definitive characterization of D_G for all G . To quote from [BMR], “This seem to be the beginning of a long journey... Moreover new problems now begin to emerge: how to characterize the diagram invariants like the degrees, co-degrees, zeta functions...”.

4. FINITE TOPOLOGICAL SPACES AND POINCARÉ DUALITY

Finite topological spaces, inspite of their simple combinatorial description, are topologically interesting enough to afford weak homotopy type of any finite simplicial complex (see [Mc]). The origin of the study of finite spaces may be found in the earliest works in algebraic topology, where one extracts simple combinatorial information from continuous spaces via triangulation and defines algebraic invariants out of the combinatorial data. The theory of simplicial sets is the modern incarnation of this combinatorial approach to topology.

4.1. Poincaré Duality for a class of finite spaces: While trying to better understand the classical proof of Poincaré duality via dual cell decomposition, we stumbled upon a class of finite spaces, that model the cell structure of a finite Euclidean polyhedral cell complex (see [Ba3]). We call them combinatorial cell complexes (or c.c.c for short). In the classical proof of Poincaré duality, one relates homology and cohomology by taking the dual of a cell complex. However, the cells of the dual cell complex of a simplicial complex need not be simplices. This is our point of departure from simplicial sets. We allow more general cell shapes so that the duality is built into the setup.

The key new definition is that of an orientation on a c.c.c. With this definition in place, we proceed to define homology/cohomology groups and develop enough algebraic topology to prove a Poincaré duality theorem for c.c.c’s satisfying suitable regularity conditions ([Ba3], th. 9.2). We also show that under these regularity assumption, the homology is functorial. If the c.c.c happen to come from a simplicial complex, then our homology is nothing but the simplicial homology.

4.2. further problems: The article [Ba3] studies only the combinatorial aspects of the theory of c.c.c’s. We need to study the topological aspects of this theory, for example the relationship between the homology groups of a c.c.c defined in [Ba3] and the singular homology of the corresponding cell complex.

On a different vein, we are only able to prove functoriality of homology groups by first showing that our homology groups are invariant under barycentric subdivision. The proof of this proposition involves much combinatorial technicality. As was suggested by Prof. Peter May, it would be nice to have a “shape category” so that (some variant of) a c.c.c becomes a presheaf (of sets) on this shape category. This would make the theory intrinsically functorial.

This also makes us wonder if the combinatorial study of shapes of cells might have some bearing on certain approaches to higher category theory, where the shapes of cells defining the higher morphisms encode much of the structure (e.g. see [St], [BD]). The material developed in [Ba3] has formal similarity with Goresky-Mcpherson’s intersection homology theory and graph homology (see [LV]). All these will be interesting avenues to explore.

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