GOLDIE'S THEOREM

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ABSTRACT. This is an exposition of Goldie's theorem with a section on Ore localization and with an application to defining ranks for finitely generated modules over non–commutative noetherian rings.

1. INTRODUCTION

This paper was inspired by a problem of Lam discussed in section 4.1 and by the recent exposition [3] of Goldie's theorem. This contains a lot of interesting historical material and also gives a very concise proof of Goldie's theorem for the case of prime noetherian rings. I liked this version of the proof so much that I thought it worth while to write down a proof of the general case along the same lines. I have also included an exposition of Ore localization for completeness and in order to show the flatness properties of Ore's construction. All rings considered here will be associative with unit.

2. ORE LOCALIZATION

Let R be a ring, associative with unit but not necessarily commutative. Let S be a multiplicatively closed subset of R. The localization $R \to R[S^{-1}]$ is defined by the following co–universal property:

If $f : R \to A$ is a ring homomorphism sending S into the group of units of A then f factors uniquely through $R[S^{-1}]$.

I prefer to use the term co–universal for universal objects which map into something in order to be consistent with the standard terminology: cokernel, coproduct, etc. Standard properties of universal maps [6] show that $R \to R[S^{-1}]$ is unique up to a unique isomorphism: If $R \to B$ and $R \to C$ have the property we get unique maps $B \to C$ and $C \to B$ and the compositions must be the identity maps. The existence also follows by standard arguments of universal algebra. For example, let $R[S^{-1}]$ be the ring generated by the elements of R and a set $\bar{S} = \{\bar{s} \mid s \in S\}$ in 1–1 correspondence with S, the relations being those of R and $s\bar{s} = \bar{s}s = 1$ for all $s \in S$. This, of course, gives us no idea about the structure of $R[S^{-1}]$ or even whether it is just the zero ring. Ore's construction shows that under reasonable conditions on S, $R[S^{-1}]$ looks very much like the familiar construction which applies in the commutative case.

Definition 2.1. Let R and S be as above. I will say that a ring homomorphism $i : R \to A$ is a right Ore localization with respect to S if

(1) $i(S)$ is contained in the units of A.
(2) Each element of A has the form $i(r)i(s)^{-1}$ for $r \in R$ and $s \in S$.
(3) $i(r) = 0$ implies that $rs = 0$ for some $s \in S$.

Since $rs = 0$ for some $s \in S$ implies that $i(r) = 0$ by (1), we see that (3) just says that $\ker i$ is as small as possible.

Theorem 2.2. *Let S be a multiplicatively closed set of the ring R. Let $i : R \to A$ be a right Ore localization with respect to S. Then*

(1) *$i : R \to A$ is canonically isomorphic to the localization $R \to R[S^{-1}]$.*
(2) *A is flat as a left R–module.*
(3) *If I is a right ideal of A and $J = i^{-1}(I)$ then $J \otimes_R A \approx JA = I$.*

These results will follow from Ore's explicit construction of a right Ore localization as given in the next theorem.

Theorem 2.3. *Let S be a multiplicatively closed set of the ring R. Then there is a right Ore localization $i : R \to R_S$ with respect to S if and only if R and S satisfy the right Ore conditions:*

(1) *If $a \in R$ and $s \in S$ then there are $a_1 \in R$ and $s_1 \in S$ such that $as_1 = sa_1$.*
(2) *If $a \in R$ and $s \in S$ and $sa = 0$ then there is a $t \in S$ such that $at = 0$.*

Note that the second condition is automatic if S consists of regular elements (non–zero–divisors).

The "only if" part is quite easy. Suppose that we have $a \in R$ and $s \in S$ and $sa = 0$. Then $i(s)i(a) = 0$ but $i(s)$ is a unit so $i(a) = 0$. This implies that there is a $t \in S$ such that $at = 0$. If we have $a \in R$ and $s \in S$ write $i(s)^{-1}i(a) = i(b)i(t)^{-1}$ with $b \in R$ and $t \in S$. Then $i(at - sb) = 0$ so there is a $u \in S$ with $(at - sb)u = 0$. Let $s_1 = tu$ and $a_1 = bu$.

Suppose now that the Ore conditions are satisfied. Let $\mathcal{S}$ be the category with S as its set of objects and with $\mathrm{Hom}(s, t) = \{a \in R \mid sa = t\}$. We write $s \xrightarrow{a} t$. The composition $s \xrightarrow{a} t \xrightarrow{b} u$ is defined to be $s \xrightarrow{ab} u$. Note $sa = t$ and $tb = u$ so that $sab = u$. Let M be right R–module and define a functor $\widetilde{M}$ from $\mathcal{S}$ to abelian groups by $\widetilde{M}(s) = M$ and, if $s \xrightarrow{a} t$ then $\widetilde{M}(s \xrightarrow{a} t)$ is $M \to M$ given by $x \mapsto xa$. Define $M_S = \mathrm{colim}\, \widetilde{M}$.

Lemma 2.4. *The category $\mathcal{S}$ is filtered.*

Proof. There are two conditions to check

(1) For any two objects s and t of $\mathcal{S}$ there is an object u of $\mathcal{S}$ and maps $s \to u$, $t \to u$. Write $st_1 = ts_1$ with $s_1 \in S$ by the first Ore condition. Then $u = ts_1$ will do.
(2) If $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$ then there is a map $t \xrightarrow{c} u$ such that the compositions are equal. Since $t = sa = sb$ we have $s(a - b) = 0$. By the second Ore condition there is a $c \in S$ with $ac = bc$ and the required map is $t \xrightarrow{c} tc$.

$\square$

Corollary 2.5. *The functor $M \mapsto M_S$ is exact.*

This is immediate from the fact that filtered colimits of abelian groups are exact. [7, Theorem 2.6.15]. It is also easy to verify this directly from the following explicit description of M_S.

If $m \in \widetilde{M}(s)$ we write m/s for its image in M_S. The usual properties of filtered colimits show that $m/s = n/t$ if and only if there are a and b in R with $ma = nb$

and $sa = tb$ with $sa \in S$. We can define M_S directly as the set of fractions m/s with this equivalence relation. The following observation is often useful.

Corollary 2.6. *A finite number of elements $\mu_1 \ldots, \mu_n$ of M_S can be expressed with a common denominator as $\mu_i = m_i/s$ with s in S.*

This is immediate from Lemma 2.4 since the elements have representatives in a single $\tilde{M}(s)$.

So far M_S and R_S are just abelian groups. We define a map $M_S \times R_S \to M_S$ by $m/s \cdot a/t = ma_1/ts_1$ where $as_1 = sa_1$ with $s_1 \in S$. This definition is justified by the observation that if s, t and s_1 are units then $ms^{-1}at^{-1} = ma_1s_1^{-1}t^{-1}$.

Proposition 2.7. *This map $M_S \times R_S \to M_S$ is well defined, biadditive, and associative.*

We give the proof in several steps. The first shows that $m/s \cdot a/t = ma_1/ts_1$ is independent of the choice of a_1 and s_1.

Lemma 2.8. *If $aq = sb$ with $tq \in S$ and $as_1 = sa_1$ with $s_1 \in S$ then $ma_1/ts_1 = mb/tq$*

Proof. Let $s_1p = qu$ with $u \in S$. Since $as_1 = sa_1$ and $aq = sb$, we have $sa_1p = as_1p = aqu = sbu$. By the second Ore condition there is a $v \in S$ such that $a_1pv = buv$. Since $ts_1p = tqu \in S$ we have $ma_1/ts_1 = ma_1p/ts_1p = ma_1p/tqu = ma_1pv/tquv = mbuv/tquv = mb/tq$. $\square$

Lemma 2.9. *If $sq \in S$ and $s \in S$, there is a $p \in R$ such that $qp \in S$.*

Proof. Write $(sq)a = st$ with $t \in S$. The second Ore condition gives us a $u \in S$ with $(qa - t)u = 0$. Let $p = au$. $\square$

Corollary 2.10. *$m/s = 0$ in M_S if and only if $mt = 0$ for some $t \in S$.*

Proof. $m/s = 0$ if and only if there is a $q \in R$ with $sq \in S$ and $mq = 0$. Let $t = qp$ where p is given by the lemma. $\square$

Lemma 2.11. *$m/s \cdot a/t$ is well defined.*

Proof. It is sufficient to show that if sp and tq are in S then $m/s \cdot a/t = mp/sp \cdot aq/tq$. Let $spa_2 = aqs_2$ with s_2 in S. Then the right hand side is mpa_2/tqs_2. Since tq is in S so is tqs_2. By Lemma 2.9 we can find qs_2r in S so that $mpa_2/tqs_2 = mpa_2r/tqs_2r$. Since $spa_2r = aqs_2r$ we can choose $a_1 = pa_2r$ and $s_1 = qs_2r$ so that $sa_1 = as_1$ and $mpa_2r/tqs_2r = ma_1/ts_1$ which is the left hand side by Lemma 2.8. $\square$

Lemma 2.12. *The product is associative i.e. $m/s \cdot (a/t \cdot b/u) = (m/s \cdot a/t) \cdot b/u$.*

Proof. Let $sa_1 = as_1$ so the right hand side is $ma_1/ts_1 \cdot b/u$. Let $ts_1c = bv$ with v in S. Then the right hand side is ma_1c/uv. Let $b_1 = s_1c$. Then the left hand side is $m/s \cdot ab_1/uv = ma_1c/uv1$ using $ab_11 = as_1c = sa_1c$ $\square$

Remark 2.13. Note that $m/s \cdot 1/1 = m/s$ so $1/1$ acts as the unit. Also, for a in R, $1/1 \cdot a/1 = a/1$.

Lemma 2.14. *Given $a_1, \ldots, a_n$ in R and s in S we can find $a'_1, \ldots, a'_n$ in R and s_1 in S with $sa'_i = a_i s_1$ for $i = 1, \ldots, n$.*

Proof. Write $sa''_i = a_i t_i$ with t_i in S. By Lemma 2.4 the t_i all map into a single object s_1 of the category $\mathcal{S}$ so we can write $s_1 = t_i c_i$. Let $a'_i = a''_i c_i$. $\square$

Lemma 2.15. *The product $m/s \cdot a/t$ is biadditive.*

Proof. This is clear for m. For a let $a = a_1 + a_2$. By the previous lemma write $a_i s_1 = sa'_i$. Then $as_1 = sa'$ where $a' = a'_1 + a'_2$ so $ma'_1/ts_1 + ma'_2/ts_1 = ma'/ts_1$. $\square$

It now follows from the case $M = R$ that R_S is a ring with unit $1/1$ and, for any right R–module M, we see that M_S is a right R_S–module. The map $i : R \to R_S$ defined by $i(r) = r/1$ is easily seen to be a ring homomorphism and the map $i : M \to M_S$ by $i(m) = m/1$ is an R–homomorphism.

Theorem 2.16. *Let R and S satisfy the right Ore conditions. Then $i : R \to R_S$ is a right Ore localization of R with respect to S and is canonically isomorphic to the localization $R \to R[S^{-1}]$.*
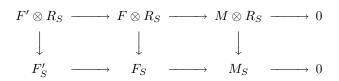
Proof. If $s \in S$, then $s/1 \cdot 1/s = 1/s \cdot s/1 = 1$ so $i(s) = s/1$ is a unit of R_S. Each element of R_S has the form $r/s = r/1 \cdot 1/s = i(r)i(s)^{-1}$ and $i(r) = r/1 = 0$ implies $rs = 0$ for some s in S by Corollary 2.10. Therefore R_S is a right Ore localization. For the last statement it is sufficient to show that $i : R \to R_S$ has the couniversal property characteristic of $i : R \to R[S^{-1}]$. Let $f : R \to A$ send S to the units of A. We must show that f factors uniquely as $R \xrightarrow{i} R_S \xrightarrow{g} A$. Now $r/s = r/1 \cdot 1/s = i(r)i(s)^{-1}$ so if g exists we have $g(r/s) = gi(r)gi(s)^{-1} = f(r)f(s)^{-1}$ showing that g is unique. For the existence, define g by this formula. To show it is well defined we must show that $f(r)f(s)^{-1} = f(rq)f(sq)^{-1}$ if sq lies in S but $f(rq)f(sq)^{-1} = f(r)f(s)^{-1}f(s)f(q)f(sq)^{-1} = f(r)f(s)^{-1}f(sq)f(sq)^{-1} = f(r)f(s)^{-1}$ as required. Clearly g is additive. It preserves products since if $sb_1 = bs_1$ then $g(a/s \cdot b/t) = g(ab_1/ts_1) = f(ab_1)f(ts_1)^{-1} = f(a)f(s)^{-1}f(s)f(b_1)f(ts_1)^{-1} = f(a)f(s)^{-1}f(sb_1)f(ts_1)^{-1} = f(a)f(s)^{-1}f(bs_1)f(ts_1)^{-1} = g(a/s)g(bs_1/ts_1) = g(a/s)g(b/t)$. This gives the required factorization since $gi(r) = g(r/1) = f(r)f(1)^{-1} = f(r)$ $\square$

Corollary 2.17. *If $f : R \to A$ is a right Ore localization with respect to S, the map $g : R_S \to A$ given by the couniversal property is an isomorphism.*

Proof. The map is onto since any element of A has the form $f(r)f(s)^{-1} = g(r/s)$ It is injective since $g(r/s) = 0$ implies $f(r) = 0$. Therefore $rt = 0$ for some t in S and this implies that $r/s = 0$. $\square$

Proposition 2.18. *If M is a right R–module then $M \otimes_R R_S \xrightarrow{\approx} M_S$ by the map sending $m \otimes r/s$ to $m/1 \cdot r/s = mr/s$.*

Proof. This is clear if $M = R$. Since both sides preserve direct sums, the result holds when M is free. By applying the map to a resolution $F' \to F \to M \to 0$ we get a diagram

$$
\begin{array}{ccccccc}
F' \otimes R_S & \longrightarrow & F \otimes R_S & \longrightarrow & M \otimes R_S & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F'_S & \longrightarrow & F_S & \longrightarrow & M_S & \longrightarrow & 0
\end{array}
$$

and the 5–lemma shows that the vertical map on the right is an isomorphism. $\square$

Corollary 2.19. *R_S is flat as a left R–module.*

This is clear from Proposition 2.18 since $M \mapsto M_S$ is exact.

Lemma 2.20. *If L is an R_S–submodule of M_S and $N = i^{-1}(L) \subseteq M$ then $L = N_S$. In particular, if I is a right ideal of R_S and $J = i^{-1}(I)$ then $I = J \otimes_R R_S = i(J)R_S = J_S$.*

Proof. Since $M \mapsto M_S$ is exact, $N \hookrightarrow M$ implies $N_S \hookrightarrow M_S$ but $N_S = i(N)R_S = L$ since it clearly lies in L and if $m/s \in L$ then $m/s \cdot s/1 \in L$ so $m \in N$ and $m/s \in N_S$. $\square$

The assertions of Theorem 2.2 are implied by the above results.

3. GOLDIE'S THEOREM

Goldie's Theorem characterizes rings whose Ore localization with respect to the set of regular elements (non–zero–divisors) is semi–simple artinian. If T is a subset of a ring R the right annihilator of T is the ideal $r(T) = \{a \in R \mid Ta = 0\}$ and the left annihilator of T is the ideal $\ell(T) = \{a \in R \mid aT = 0\}$. Clearly $r(T)$ is a right ideal of R and $\ell(T)$ is a left ideal.

Definition 3.1. A ring R is called a right Goldie ring if it satisfies the following two conditions.

(1) R satisfies the ascending chain condition on right annihilators.
(2) R does not contain an infinite direct sum of non-zero right ideals.

In particular, right noetherian ring is a right Goldie ring.

A ring is called semiprime if it has no nilpotent 2–sided ideals and is called prime if for 2–sided ideals I and J, $IJ = 0$ implies $I = 0$ or $J = 0$.

Theorem 3.2 (Goldie). *Let R be a ring and let S be the set of regular elements of R. Then R has a right Ore localization R_S which is semisimple artinian if and only if R is a semiprime right Goldie ring and R_S is simple artinian if and only if R is a prime right Goldie ring.*

I will give a proof based on the accounts in [3]. Some of the proofs are also adapted from [4]. We first prove the "only if" part. Since an artinian ring is noetherian it is a Goldie ring. We consider the three conditions separately.

Lemma 3.3. *Let R be a subring of A. If A satisfies the ascending chain condition on right annihilators so does R.*

Proof. If X is a subset of R write $r_R(X)$ for its right annihilator in R and $r_A(X)$ for its right annihilator in A. Suppose that we have subsets X_n of R such that $r_R(X_1) \subseteq r_R(X_2) \subseteq r_R(X_3) \subseteq \ldots$ Let $Y_n = X_n \cup X_{n+1} \cup X_{n+2} \cup \ldots$. Then $r_R(Y_n) = r_R(X_n)$. Since $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$ we get an ascending chain $r_A(Y_1) \subseteq r_A(Y_2) \subseteq r_A(Y_3) \subseteq \ldots$ which stops by the hypothesis. Since $r_R(Y) = r_A(Y) \cap R$, the original ascending chain also stops. $\square$

Lemma 3.4. *Let R be a subring of A such that A is flat as a left R–module. If A does not contain an infinite direct sum of non-zero right ideals then the same is true of R.*

Proof. If I is a right ideal of R the flatness implies that $I \otimes_R A \to A$ is injective so $I \otimes_R A = IA$. If $I = \bigoplus I_\alpha$ it follows that $IA = \bigoplus I_\alpha A$. This implies that $I_\alpha A = 0$ for almost all α. Therefore the same is true of I_α. $\square$

Corollary 3.5. *If $R \subseteq A$ and A is flat as a left R–module then R is a right Goldie ring if A is.*

In particular, this applies if A is the right Ore localization $A = R_S$ with respect to the set S of regular elements of R, assuming that this exists.

The following lemma completes the proof of the "only if" part of the theorem.

Lemma 3.6. *Let S be the set of regular elements of R and assume that the right Ore localization $A = R_S$ exists. Then R is semiprime if A is semisimple artinian and R is prime if A is simple artinian.*

Proof. Suppose I is a 2–sided ideal of R. Then I_S is a right ideal of A so AI_S is a 2–sided ideal of A and hence is generated by a central idempotent e. Write $e = \sum \alpha_i \beta_i$ where α_i lies in A and β_i lies in I_S. By Corollary 2.6 we can write $\beta_i = a_i/s$ with a_i in R and s in S. This shows that $es = se$ lies in AI so if $J \subset R$ and $IJ = 0$ then $seJ \subseteq AIJ = 0$. Since s is regular it follows that $eJ = 0$. Suppose first that $I^2 = 0$. Then $eI = 0$ but I lies in $AI_S = eA$ so e acts a 1 on I. Therefore $I = 0$ showing that R is semiprime. If A is simple and $I \neq 0$ then the 2–sided ideal AI_S must be A so we can take $e = 1$. Therefore $IJ = 0$ implies $J = eJ = 0$ showing that R is prime. $\square$

Remark 3.7. Surprisingly, the proof of Goldie's theorem only uses the ascending chain condition on right annihilators $r(x)$ of elements. Therefore if a semiprime ring having no infinite sum of right ideals satisfies the ascending chain condition on right annihilators of elements it satisfies the ascending chain condition on all right annihilators.

The following lemmas assume that R satisfies the following conditions.

Hypothesis.
- R *is a semiprime ring.*
- R *satisfies the ascending chain condition on right annihilators $r(x)$ of elements.*
- R *does not contain an infinite direct sum of non-zero right ideals.*

Lemma 3.8. *Under the hypothesis, all nil left and right ideals of R are 0.*

Proof. Suppose I is a nil left ideal. If $I \neq 0$ choose $a \in I$, $a \neq 0$ with $r(a)$ maximal subject to these conditions. If $r \in R$ then $ra \in I$ is nilpotent. Let $(ra)^{k+1} = 0$ with the least k. We claim $k = 0$. If not then $(ra)^k$ lies in I and is non–zero. Since $r(a) \subseteq r((ra)^k)$ the maximality of $r(a)$ forces $r(a) = r((ra)^k)$ but $ra \in r((ra)^k)$ so $ra \in r(a)$. This shows that $ara = 0$ for all $r \in r$ so $aRa = 0$. Therefore $(RaR)^2 = 0$ and hence $RaR = 0$ since R is semiprime, showing that $a = 0$.

Now if J is a nil right ideal and $a \in J$ then aR is nil. This implies that the left ideal Ra is also nil since for $r \in R$ $(ra)^{n+1} = 0$ if $(ar)^n = 0$. Therefore it follows that $a = 0$ by what has just been proved. $\square$

Lemma 3.9. *Under the hypothesis, if $J \neq 0$ is a left or right ideal there is a non–zero element x in J such that $r(x) = r(x^2)$ and therefore $r(x) \cap xR = 0$.*

Proof. By Lemma 3.8 there is an element $z \in J$ which is not nilpotent. The chain $r(z) \subseteq r(z^2) \subseteq \ldots$ is constant from $r(z^n)$ on and we let $x = z^n$. If $y \in r(x) \cap xR$ then $y = xa$ for some a and $xy = x^2 a = 0$ so $a \in r(x^2) = r(x)$ showing that $y = xa = 0$. $\square$

A right ideal I of R is called essential if it meet all non–zero right ideals non–trivially i.e. $J \neq 0$ implies $I \cap J \neq 0$.

Lemma 3.10. *Under the hypothesis, if $r(s) = 0$ then sR is essential.*

Proof. We must show that if $a \neq 0$ then $sR \cap aR \neq 0$. Suppose that $sR \cap aR = 0$. We claim that $\sum_{n=0}^{\infty} s^n aR$ is a direct sum. Suppose $\sum_{n=p}^{q} s^n ar_n = 0$ with $s^p ar_p \neq 0$. Since $r(s) = 0$ we get $\sum_{n=p}^{q} s^{n-p} ar_n = 0$ leading to the contradiction that $ar_p \in sR \cap aR = 0$. Since R has no infinite direct sum of ideals, this is impossible. $\square$

Corollary 3.11. *Under the hypothesis, if $r(s) = 0$ then $\ell(s) = 0$ so s is regular.*

Proof. Suppose $\ell(s) \neq 0$. Let $x \in \ell(s)$ be as in Lemma 3.9. Since sR is essential, $sR \cap xR \neq 0$ so $sa = xb \neq 0$ for some a and b but $x \in \ell(s)$ so $x^2 b = xsa = 0$ and therefore $xb = 0$ contrary to our assumption. $\square$

Lemma 3.12. *Assume the hypothesis. Let $a_1, a_2, \ldots, a_n \in R$ satisfy $a_i \in r(a_j)$ for all $j < i$ and $r(a_i) \cap a_i R = 0$ for all i. Then $\sum_{i=1}^{n} a_i R$ is a direct sum. Therefore there is no infinite sequence with the given properties.*

Proof. The result is trivial for $n = 1$. By induction $I = \sum_{i=2}^{n} a_i R$ is a direct sum. Since $I \subseteq r(a_1)$ and $r(a_1) \cap a_1 R = 0$, $a_1 R + I$ is also a direct sum. $\square$

Lemma 3.13. *Under the hypothesis, if I is an essential right ideal then I contains a regular element.*

Proof. Let $a_1, a_2, \ldots, a_n \in I$ be a maximal sequence satisfying the conditions of Lemma 3.12. Then $J = r(a_1) \cap r(a_2) \cap \cdots \cap r(a_n) \cap I = 0$ otherwise, by Lemma 3.9 we could find $a_{n+1} \in J$ satisfying $r(a_{n+1}) \cap a_{n+1} R = 0$ and extend our sequence. Since $\sum_{i=1}^{n} a_i R$ is a direct sum, we see that if $s = \sum_{1}^{n} a_i$ then $r(s) = \bigcap_{1}^{n} r(a_i)$. Therefore $r(s) \cap I = J = 0$. Since I is essential this implies that $r(s) = 0$ so s is regular by Corollary 3.11. $\square$

Lemma 3.14. *Under the hypothesis, R satisfies the Ore conditions with respect to the set S of regular elements.*

Proof. The second condition is automatically satisfied when S is the set of regular elements. For the first condition, let $a \in R$ and $s \in S$ be given. Let $I = \{r \in R \mid ar \in sR\}$, a right ideal. We claim I is essential. Let $J \neq 0$ be a right ideal. If $aJ = 0$ then $J \subseteq I$. Suppose $aJ \neq 0$. Since sR is essential by Lemma 3.10 $aJ \cap sR \neq 0$. Let $j \in J$ with $aj \neq 0$ and $aj \in sR$. The definition of I shows that $j \in I$ so $I \cap J \neq 0$ showing that I is essential. By Lemma 3.13, I contains a regular element s_1 and $as_1 = sa_1$ for some a_1 by the definition of I. $\square$

Lemma 3.15. *Under the hypothesis, if I is any right ideal of R then $I \oplus J$ is essential for some right ideal J*

Proof. Look for non–zero right ideals J_i such that $I \oplus J_1 \oplus J_2 \oplus \cdots \oplus J_n$ is direct. Since there is no infinite direct sum of right ideals, such a sequence has to stop say at J_n. Let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_n$. Then $I \oplus J$ is essential otherwise if $(I \oplus J) \cap J' = 0$ with $J' \neq 0$ we can extend our sequence by letting $J_{n+1} = J'$. $\square$

We can now finish the proof of Goldie's theorem. By Lemma 3.14 R has a right Ore quotient $A = R_S$ where S is the set of regular elements. All that remains is to show that A is semisimple artinian and simple if R is prime. Let $i : R \to A$ be the canonical map sending r to $r/1$. Let $\mathfrak{a}$ be a right ideal of A and let $I = i^{-1}(\mathfrak{a})$. Then I is a right ideal of R and $\mathfrak{a} = I_S$. By Lemma 3.15 we can find a right ideal J of R such that $I \oplus J$ is essential. Let $\mathfrak{b} = J_S$. Since A is flat as a left R–module, $(I \oplus J)_S = I_S \oplus J_S = \mathfrak{a} \oplus \mathfrak{b}$. But $I \oplus J$ contains a regular element by Lemma 3.13 so $\mathfrak{a} \oplus \mathfrak{b} = (I \oplus J)_S = A$ showing that every right ideal of A is a direct summand of A. This implies that A is semisimple artinian by Lemma 3.17.

Suppose now that R is prime. Let $\mathfrak{a}$ be a 2–sided ideal of A and let $\mathfrak{a} = I_S$ as above where $I = i^{-1}(\mathfrak{a})$ is a 2–sided ideal of R. The next lemma shows that $\mathfrak{a} = I_S = A$ showing that A is simple.

Lemma 3.16. *Under the hypothesis, if R is prime every non–zero 2–sided ideal is essential as a right ideal and therefore contains a regular element.*

Proof. Let I be a non–zero 2–sided ideal and let J be a non–zero right ideal. Since R is prime the product of the non–zero 2–sided ideals AJ and I is non–zero. It follows that $JI \neq 0$ but $JI \subseteq I \cap J$. Therefore I is essential and so contains a regular element by Lemma 3.13. $\square$

The following lemma is a special case of [2, Ch.I,Th.4.2].

Lemma 3.17. *A ring A is semisimple artinian if and only if each right ideal of A is a direct summand of A.*

Proof. The only if part is well known and will not be used here. For the if part let I be the sum of all simple (i.e. minimal non–zero) right ideals of A. Then $I = A$ otherwise I would be contained in a maximal right ideal M. Since M is a direct summand we have $A = M \oplus J$ and $J \approx A/M$ which is simple so $J \subseteq I$, a contradiction. We can write $1 = a_1 + \cdots + a_n$ where each a_i lies in a simple ideal I_i. Therefore $A = 1A = I_1 + I_2 + \cdots + I_n$. This shows that A has finite length in the sense of the Jordan–Hölder Theorem and therefore A satisfies the descending chain conditiion on right ideals.

Suppose now that $J^2 = 0$ where J is a 2–sided ideal. Then $I_i J^2 = 0$ but $I_i J$ is either I_i or 0 by Schur's Lemma so $I_i J = 0$ for all i and therefore $J = \sum I_i J = 0$. $\square$

4. A RANK MAP FOR RIGHT NOETHERIAN RINGS

As an application of Goldie's theorem we show how to define a rank map for non–commutative noetherian rings. If M is a finitely generated module over a commutative noetherian ring R and if P is a minimal prime of R we can define the rank $\mathrm{rk}_P(M)$ of M at P to be the length of the localized module M_P over the artinian local ring R_P. This is additive on short exact sequences and so defines a homomorphism $\mathrm{rk} : G_0(R) \to \mathbb{Z}$. The following theorem extends these results to

the non–commutative case. If R is a right noetherian ring we let $G_0(R)$ be the Grothendieck group $K_0(\mathcal{M}(R))$ of the category $\mathcal{M}(R)$ of finitely generated right R–modules with respect to short exact sequences.

Theorem 4.1. *Let R be a right noetherian ring and let P be a minimal prime of R. Then there is a homomorphism $\rho_P : G_0(R) \to \mathbb{Z}$ with the following properties:*

(1) *If M is a finitely generated right module then $\rho_P(M) \geq 0$.*
(2) $\rho_P(R/P) > 0$.

This theorem was inspired by the following problem in [5].

Corollary 4.2. *Let R be a right noetherian ring and let M be a right R–module having a finite free resolution*

$$0 \to R^{e_n} \to \cdots \to R^{e_1} \to R^{e_0} \to M \to 0.$$

Then $\chi(M) = \sum_0^n (-1)^i e_i \geq 0$.

Proof. We have $[M] = \sum_0^n (-1)^i [R^{e_i}] = \chi(M)[R]$ in $G_0(R)$ so $\rho_P(M) = \chi(M)\rho_P(R)$. Since $\rho_P(M) \geq 0$ and $\rho_P(R) \geq \rho_P(R/P) > 0$, the result follows. $\square$

We recall some well known facts about the primes associated to a module. As above, $r(X) = \{r \in R \mid Xr = 0\}$ will denote the right annihilator of X.

Let R be a right noetherian ring and let P be a prime ideal of R. I will say that a right R–module N is P–prime if it is non–zero and $r(L) = P$ for all non–zero submodules L of N.

Definition 4.3. *If M is a right R–module let $\mathrm{Ass}(M)$ be the set of primes P such that M contains an P–prime submodule.*

This agrees with the classical definition if R is commutative since in this case, R/P is P–prime for a prime ideal P and if x is a non–zero element of a P–prime module N then Rx is isomorphic to R/P since $r(x) = r(Rx)$ in the commutative case.

Theorem 4.4. *If R is a right noetherian ring and M is a non–zero right R–module then $\mathrm{Ass}(M) \neq \emptyset$.*

Proof. Let N be a non–zero submodule with $P = r(N)$ maximal. Then P is prime since if $IJ \subseteq P$ with $I \subset P$ and $J \subset P$ then $NI \neq 0$ and $J \subseteq r(NI)$ contradicting the maximality of P. If $L \subseteq N$ and $L \neq 0$ then $P \subseteq r(L)$ so $P = r(L)$ by the maximality of P. $\square$

Corollary 4.5. *Let R be a right noetherian ring and let M be a finitely generated right R–module. Then M has a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each M_i/M_{i-1} is annihilated by some prime ideal P_i of R.

Proof. If $M \neq 0$ it has a P_1–prime submodule M_1. If $M_1 \neq M$, the same argument on M/M_1 gives us a P_2–prime submodule M_2/M_1. Continuing in this way we get the required filtration. The process must stop since M is noetherian. $\square$

Corollary 4.6. *Let R be a right noetherian ring and let $\mathcal{M}'$ be the full subcategory of $\mathcal{M}(R)$ formed by modules which are annihilated by a prime ideal of R (depending on the module). Then $K_0(\mathcal{M}') \to K_0(\mathcal{M}(R)) = G_0(R)$ is an isomorphism, the inverse sending $[M]$ to $\sum[M_i/M_{i-1}]$ where $\{M_i\}$ is as in Corollary 4.5.*

Proof. This follows immediately by the usual Jordan–Hölder-Zassenhaus devissage argument [1, Ch. VIII,Theorem 3.3]. □

We can now prove Theorem 4.1. Let $\mathcal{M}'$ be as in Corollary 4.6 It will clearly suffice to prove Theorem 4.1 for $\mathcal{M}'$. The extension of ρ from $\mathcal{M}'$ to $\mathcal{M}(R)$ is given by $\rho(M) = \sum \rho(M_i/M_{i-1})$ so $\rho(M) \geq 0$ as required.

Let P be a minimal prime ideal of the right noetherian ring R. Let $A = R/P$. By Theorem 3.2, A has a right Ore quotient B which is simple artinian. We define a map $K_0(\mathcal{M}') \to K_0(B) = \mathbb{Z}$ as follows: Let $M \in \mathcal{M}'$. Then $MQ = 0$ for some prime ideal Q of R. If $Q \neq P$ we send M to 0. If $Q = P$, we send M to $[M \otimes_A B]$. This is well defined: If $MP = 0 = MQ$ with $Q \neq P$ then $M(P + Q) = 0$. Let $I = (P + Q)/P$. Since this is a non–zero 2–sided ideal of A it contains a regular element s by Lemma 3.16. It follows that $M \otimes_A B = 0$ since $m \otimes b = ms \otimes i(s)^{-1}b = 0$.

Suppose we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{M}'$. If $MQ = 0$ for $Q \neq P$, all three terms map to 0. If $MP = 0$ our sequence is an exact sequence over A so by Lemma 2.19, $0 \to M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B \to 0$ is exact. Therefore the map factors through $K_0(\mathcal{M}')$ giving us the required homomorphism ρ_P of Theorem 4.1. Clearly $\rho_P(M) \geq 0$ for $M \in \mathcal{M}'$. Since $R/P = A$ maps to $[B]$ we have $\rho_P(R/P) = n > 0$ where $B = \mathcal{M}_n(D)$ with D a division ring.

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