K-THEORY OF COHERENT RINGS

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ABSTRACT. We show that some basic results on the K-theory of noetherian rings can be extended to coherent rings.

1. INTRODUCTION

The main object of this paper is to show that $K_i(R[t]) = K_i(R)$ for coherent rings $R$ which are regular (every finitely presented module has finite projective dimension). This gives a partial answer to a question of O. Braeunling who asked when this result holds for non-noetherian rings $R$. His question, which was suggested by [7], was forwarded to me by T. Y. Lam. At the same time, C. Quitté sent me a copy of his book (with H. Lombardi) [8] which recommends coherent rings as a substitute for noetherian rings in constructive mathematics. This suggested the above result.

An old result of Gersten [4, Th. 3.1] shows that $K_i(R[t]) = K_i(R)$ if $R$ is regular and $R[x, y]$ is coherent. Here we show that it is sufficient to assume that only $R$ is coherent using results of Quillen [6] not available when Gersten’s paper was written. Recent work relating coherence properties to the vanishing of negative $K$-theory can be found in [1].

Most of this paper is expository since the proofs are modifications of standard proofs in Algebraic K-Theory. To avoid endless repetition, I will only consider the case of left modules. The results, of course, are also true for right modules with the obvious changes. The symbol $t$ in $R[t]$ and $R[t, t^{-1}]$ will always be an indeterminate.

2. COHERENT MODULES

For the readers convenience, we recall here the basic facts about coherent modules and rings. For a detailed and comprehensive account see [5] (for the commutative case).

Definition 2.1. Let $R$ be an associative ring. A left $R$-module $M$ is called pseudo-coherent if every map $R^n \to M$ with $n < \infty$ has a finitely generated kernel. In other words, every finitely generated submodule of $M$ is finitely presented. A coherent module is a finitely generated pseudo-coherent module. The ring $R$ is called coherent if it is coherent as a left $R$-module.

In [8] the terminology has been changed. The pseudo-coherent modules are called coherent and coherent modules are called finitely generated coherent modules. I will stick to the more familiar terminology here to avoid confusion with the usage in algebraic geometry.

I would like to thank Claude Quitté for sending me a copy of his book (with H. Lombardi) [8] and for other relevant references. I would also like to thank T. Y. Lam for sending me the question which inspired this paper and O. Braeunling for useful comments and references.
Lemma 2.2. If $L$ is a finitely generated module and $M$ is a pseudo–coherent module, every map $L \rightarrow M$ has a finitely generated kernel.

Proof. Let $F = R^n$ map onto $L$. The kernel of $F \rightarrow L \rightarrow M$ is finitely generated and maps onto the kernel of $L \rightarrow M$. \hfill \Box

Corollary 2.3. If $L$ is a coherent module and $M$ is a pseudo–coherent module, every map $L \rightarrow M$ has a coherent kernel.

The kernel is pseudocoherent as a submodule of $L$. It is finitely generated by Lemma 2.2

Let $\mathcal{M}(R)$ be the category of all left $R$–modules, and let $\mathcal{F}g(R)$, $\mathcal{F}p(R)$, and $\text{Coh}(R)$ be the full subcategories of $\mathcal{M}(R)$ of finitely generated, finitely presented, and coherent modules. If $R$ is noetherian, $\mathcal{F}g(R) = \mathcal{F}p(R) = \text{Coh}(R)$.

Theorem 2.4. For any $R$, the subcategory $\text{Coh}(R)$ of $\mathcal{M}(R)$ is closed under kernels, cokernels, images, and extensions and therefore is an abelian category.

Proof. Let $f : M \rightarrow N$ with $M$ and $N$ coherent. Then $\ker f$ is coherent by Corollary 2.3 while $\text{im} f$ is pseudocoherent as a submodule of $N$ and finitely generated as an image of $M$. Let $I = \text{im} f$ and $Q = \text{cok} f$. We have an exact sequence $0 \rightarrow I \rightarrow N \rightarrow Q \rightarrow 0$. Let $g : F \rightarrow Q$ with $F$ free and finitely generated. Lift $g$ to a map $h : F \rightarrow N$. Let $k : E \rightarrow I$ with $E$ free and finitely generated. Applying the snake lemma to the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & E \oplus F & \longrightarrow & F & \longrightarrow & 0 \\
& & k & \downarrow & (k,h) & \downarrow & & g & \\
0 & \longrightarrow & I & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0
\end{array}
\]

gives us the exact sequence $\ker(k,h) \rightarrow \ker g \rightarrow 0$ showing that $\ker g$ is finitely generated as required. Finally let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be exact with $M'$ and $M''$ coherent. Let $f : F \rightarrow M$ be a map with $F$ free and finitely generated and let $g : F \rightarrow M \rightarrow M''$. Applying the snake lemma to the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & 0 \\
& & f & \downarrow & & \downarrow & g & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
\]

gives us the exact sequence $0 \rightarrow \ker f \rightarrow \ker g \rightarrow 0$. Since $M'$ is coherent and $\ker g$ is finitely generated, Lemma 2.2 shows that $\ker f$ is finitely generated. \hfill \Box

Corollary 2.5. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of left $R$–modules. If two of the modules $M'$, $M$, $M''$ are coherent, so is the third.

Corollary 2.6. If $M$ is coherent and $N$ is finitely generated then the cokernel of any map $f : N \rightarrow M$ is coherent.

The image $L$ of $f$ is coherent since it is a finitely generated submodule of $M$ and $\text{cok} f = M/L$.

Corollary 2.7. If $R$ is left coherent then $\mathcal{F}p(R) = \text{Coh}(R)$.

If $M$ is finitely presented it is the cokernel of a map $R^n \rightarrow R^n$ with $m,n < \infty$. 

Lemma 2.8. Let \( A \) be a full subcategory of \( M(R) \) such that \( A \) is abelian and \( R \in \text{ob} \, A \). Then any map \( f : M \to N \) in \( A \) has the same kernel in \( A \) as in \( M(R) \).

Proof. Let \( K = \ker f \) in \( M(R) \) and let \( L = \ker f \) in \( A \). Then \( L \to M \to N \) is 0 so \( L \to M \) factors through \( K \). If \( x \in L \) maps to 0 in \( M \), let \( R \to L \) by \( r \mapsto rx \). Then \( R \to L \to M \) is 0. Since \( L \to M \) is a monomorphism in \( A \) we see that \( R \to L \) is 0 showing that \( x = 0 \). Therefore \( L \to M \) is injective. We can now regard \( K \) and \( L \) as submodules of \( M \) and clearly \( L \subseteq K \). Let \( x \in K \) and let \( R \to K \) by \( r \mapsto rx \). Then \( R \to M \to N \) is 0. Since this is in \( A \), \( R \to M \) factors through \( L \) showing that \( x \in L \). Therefore \( L = K \).

\[ \square \]

Corollary 2.9. [4, Prop. 1.1(c)] A ring \( R \) is left noetherian if and only if \( \mathcal{F}g(R) \) is an abelian category. It is left coherent if and only if \( \mathcal{F}p(R) \) is an abelian category.

Proof. The ‘only if’ part follows from Theorem 2.4 and Corollary 2.7. Suppose that \( \mathcal{F}g(R) \) is an abelian category. Let \( I \) be a left ideal of \( R \). By Lemma 2.8 the kernel \( I \) of the map \( R \to R/I \) lies in \( \mathcal{F}g(R) \) and so is finitely generated. Finally, if \( \mathcal{F}p(R) \) is an abelian category then the kernel of \( f : R^n \to R \) lies in \( \mathcal{F}p(R) \) and so is finitely generated.

\[ \square \]

3. Examples

Lemma 3.1. If \( R \) is a coherent ring, so is any localization \( R_S \) (where \( S \) is a central multiplicative set) and for any coherent \( R_S \)–module \( M \) there is a coherent \( R \)–module \( N \) with \( NS \approx M \).

Proof. Let \( f : R^n_S \to R_S \). By multiplying \( f \) by an element of \( S \) we can assume that \( f \) lifts to \( g : R^n \to R \). The kernel of \( g \) is finitely generated and localizes to the kernel of \( f \). By Corollary 2.7 it is sufficient to prove the second part for finitely presented modules. Given \( R^n_S \to R^n_S \to M \to 0 \), some multiple \( sf \) with \( s \in S \) lifts to \( g : R^n \to R^n \) and we take \( N = \text{coker} \, g \). 

\[ \square \]

Lemma 3.2.

1. An \( R \)–module which is the filtered union of pseudo-coherent \( R \)–modules is pseudocoherent over \( R \).

2. If a ring \( R \) is the filtered union of coherent subrings \( R_\alpha \) and if \( R \) is flat over each \( R_\alpha \) then \( R \) is coherent.

Proof. The first statement is clear. For the second let \( x_1, \ldots, x_n \in R \) and map \( f : R^n \to R \) by \( e_i \to x_i \). All \( x_i \) lie in some \( R_\alpha \) so we also get \( g : R^n_\alpha \to R_\alpha \) with finitely generated kernel \( K \). By flatness \( R \otimes_{R_\alpha} K \) is the kernel of \( f \) which is therefore finitely generated.

\[ \square \]

Corollary 3.3. A polynomial ring in infinitely many variables over a noetherian ring is coherent. So are the rings of algebraic integers i.e. the integral closure of \( \mathbb{Z} \) in an algebraic field extension of \( \mathbb{Q} \).

Remark 3.4. An example in [10] shows that \( R[t] \) need not be coherent even if \( R \) is. In contrast to the noetherian case, a quotient \( R/I \) of a coherent ring \( R \) may not be a coherent ring. For example, any commutative ring can be a quotient of a polynomial ring over \( \mathbb{Z} \) in sufficiently many variables.
Lemma 3.5. Let $I$ be a 2–sided ideal of a ring $R$ and let $M$ be an $R$–module annihilated by $I$ so that $M$ is also an $R/I$–module. If $M$ is coherent over $R$ then $M$ is also coherent over $R/I$.

Proof. $M$ is clearly finitely generated. Let $f : (R/I)^n \to M$. Let $g : R^n \to M$ be the composition $R^n \to (R/I)^n \to M$. Then $ker g$ maps onto $ker f$ showing that $ker f$ is finitely generated.

Corollary 3.6. If $R$ is a coherent ring and $I$ is a 2–sided ideal which is finitely generated as a left ideal, then $R/I$ is a coherent ring.

This is immediate from the lemma and Corollary 2.6. In particular, if the polynomial ring $R[x]$ is coherent so is $R$.

Corollary 3.7. Let $R$ and $I$ be as in the Lemma. If $M$ is a coherent $R$–module then $M/IM$ is a coherent $R/I$–module.

By Corollary 2.7 it is sufficient to prove this for finitely presented modules which is obvious.

Lemma 3.8. Let $M$ be a coherent module over a coherent ring $R$. If $s$ is a central regular element of $R$ then $M/sM = \{ x \in R | sx = 0 \}$ are coherent over the coherent ring $R/sR$.

Proof. $R/sR$ is coherent by Corollary 3.6 and $M/sM$ is coherent by Corollary 3.7 even without the regularity assumption. For $sM$ let $F$ be a finitely generated $R$–module mapping onto $M$ with kernel $N$ which is coherent. Applying the snake lemma to the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
& & s & \downarrow & s & \downarrow & s & \downarrow & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}
\]

we get an exact sequence $0 \to sM \to N/sN \to F/sF$ showing that $sM$ is coherent since $N/sN$ and $F/sF$ are.

Recall that a subring $R$ of a ring $B$ is called a retract of $B$ if there is a ring homomorphism $\epsilon : B \to R$ such that $\epsilon | R = id$.

Lemma 3.9. If $R$ is a retract of a coherent ring $A$ which is flat over $R$ then $R$ is coherent.

Proof. If $0 \to K \to R^n \to R$ with $n < \infty$, tensoring with $A$ gives $0 \to A \otimes_R K \to A^n \to A$ so $A \otimes_R K$ is finitely generated and therefore so is $K = R \otimes_A A \otimes_R K$.

4. A USEFUL EXACT SEQUENCE

Let $R[t]$ be a polynomial ring in one variable over $R$. Let $L$ be an $R$–module and let $F = R[t] \otimes_R L = L[t]$. Filter $F$ by letting $F_n = \sum_{i=0}^{n} R t^i \otimes L = L + Lt + Lt^2 + \cdots + Lt^n$. Let $F_n = 0$ for $n < 0$.

Lemma 4.1. Let $f_0, f_1, \ldots, f_r \in F_k$ satisfy $\sum_{i=0}^{r} t^i f_i = 0$ then $f_r \in F_{k-1}$.

Proof. Let $f_i = \sum_{j=0}^{k} t^{j} a_{ij}$ where $a_{ij} \in L$. Then $\sum_{i=0}^{r} t^i f_i = 0$. The leading term $t^{r+k} a_{rk}$ must be 0 and the result follows.
Let $M$ be a left $R[t]$-module. Recall the following result from [2].

**Theorem 4.2** ([2]). There is an exact sequence (“The characteristic sequence”)

$$0 \rightarrow R[t] \otimes_R M \xrightarrow{\alpha} R[t] \otimes_R M \xrightarrow{\beta} M \rightarrow 0$$

where $\alpha(t^n \otimes x) = t^{n+1} \otimes x - t^n \otimes tx$ and $\beta(t^n \otimes x) = t^n x$.

In [11] I gave a modified version with smaller terms as follows:

**Theorem 4.3.** Let $M$ be a finitely generated left $R[t]$-module which is contained in a free module $F$. Write $F = R[t] \otimes_R L$ where $L$ is free over $R$ and filter $F$ as above by $F_n = L + tL + \cdots + t^n L$. Let $M_n = M \cap F_n$. Then, for large $n$, there is an exact sequence

$$(1) \quad 0 \rightarrow R[t] \otimes_R M_{n-1} \xrightarrow{\alpha} R[t] \otimes_R M_n \xrightarrow{\beta} M \rightarrow 0$$

where $\alpha$ and $\beta$ define maps as indicated. Let $n$ be large enough that all chosen generators of $M$ lie in $M_n$. Then $\beta$ will be onto. That $\beta \alpha = 0$ is obvious. Suppose $\alpha(\sum_{i=0}^n t^i \otimes a_i) = 0$. Then $\sum_{i=0}^n t^i \otimes a_i - \sum_{i=0}^n t^i \otimes ta_i = 0$. The leading term, $t^{r+1} \otimes a_r$, is 0 so $a_r = 0$ and, by induction all $a_i = 0$ showing that $\alpha$ is injective.

Suppose $\beta(\sum_{i=0}^r t^i \otimes a_i) = 0$ where all $a_i$ are in $M_n$. Then $\sum_{i=0}^r t^i a_i = 0$. Since $a_i \in F_n$, Lemma 4.1 shows that $a_r \in F_{n-1}$. Therefore $\alpha(t^{r-1} \otimes a_r)$ is defined. It is $t^r a_r - t^{r-1} \otimes a_r$ so by subtracting it from $\sum_{i=0}^r t^i \otimes a_i$ we can reduce the degree. It follows by induction that $\ker \beta = \text{im} \alpha$. \qed

5. $K_0$

In this section and the next we examine the case of projective modules.

**Lemma 5.1.** If $R$ is a coherent ring any finitely generated projective R-module is coherent and, if $M$ is a coherent $R$-module, there is a resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

with all $P_i$ finitely generated projective. If $M$ also has finite projective dimension there is such a resolution with $P_n = 0$ for all large $n$.

**Proof.** The first statement follows from Corollary 2.7. The resolution is constructed in the standard way. Let $P_0$ be projective, finitely generated, and map onto $M$ with kernel $N$. Similarly let $P_1$ map onto $N$ etc. If $M$ has finite projective dimension then $\ker(P_{n+1} \rightarrow P_n)$ will be projective for large $n$ and we can stop there. \qed

**Theorem 5.2.** Let $R$ be a left coherent ring such that each finitely presented $R$-module has finite projective dimension. Then each finitely generated projective $R[t]$-module $P$ has a finite resolution by extended projective modules

$$0 \rightarrow R[t] \otimes_R Q_n \rightarrow R[t] \otimes_R Q_{n-1} \rightarrow \cdots \rightarrow R[t] \otimes_R Q_0 \rightarrow P \rightarrow 0,$$

where each $Q_i$ is finitely generated projective over $R$.

**Proof.** Let $P \otimes S = F$ be free and finitely generated. Filter $F$ as in Theorem 4.3 and let $P_n = P \cap F_n$. Since $P_n$ is the kernel of $F_n \rightarrow S$ it is coherent by Corollary 2.3. By Theorem 4.3 we get an exact sequence

$$0 \rightarrow R[t] \otimes_R P_{n-1} \xrightarrow{\alpha} R[t] \otimes_R P_n \rightarrow P \rightarrow 0.$$
Choose finite projective resolutions $A'_\bullet$ for $P_{n-1}$ and $B'_\bullet$ for $P_n$ and extend these to get resolutions $A_\bullet = R[t] \otimes_R A'_\bullet$ for $R[t] \otimes_R P_{n-1}$ and $B_\bullet = R[t] \otimes_R B'_\bullet$ for $R[t] \otimes_R P_n$. Cover $\alpha$ by a map $f : A_\bullet \to B_\bullet$, and let $C_\bullet$ be the mapping cone of $f$ i.e. $C_m = A_{m-1} \oplus B_m$ with $\partial(a,b) = (-\partial a, \partial b + f(a))$. Note that $C_m = R[t] \otimes_R A_{m-1}' \oplus R[t] \otimes_R B_m'$ is extended from $R$. The exact sequence
\[
\cdots \to H_m(A_\bullet) \to H_m(B_\bullet) \to H_m(C_\bullet) \to H_{m-1}(A_\bullet) \to \cdots
\]
shows that $H_m(C_\bullet) = 0$ for $m \neq 0$ and is $P$ for $m = 0$, so $C_\bullet$ is the required resolution.

Corollary 5.3. If $R$ is a left coherent ring such that each finitely presented $R$–module has finite projective dimension, then $[M] \mapsto [R[t] \otimes_R M]$ induces an isomorphism $K_i(R) \approx K_0(R[t])$.

The map is onto by the theorem and is split injective by the map $[N] \mapsto [N/tN]$.

Remark 5.4. Since $R[t]$ need not be coherent even if $R$ is, it is not clear whether this result can be extended to $R[t_1, \ldots, t_n]$ for $n > 1$.

6. $K_i$

Theorem 6.1. If $R$ is a left coherent ring such that each finitely presented $R$–module has finite projective dimension, then $[M] \mapsto [R[t] \otimes_R M]$ induces isomorphisms $K_i(R) = K_i(R[t])$ and $K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R)$ for all $i > 0$.

Proof. By the Fundamental Theorem [6] we have $K_i(R[t]) = K_i(R) \oplus NK_i(R)$ and $K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R) \oplus NK_i(R) \oplus NK_i(R)$ and $NK_i(R) = Nil_{i-1}(R)$. (When $i = 1$ there is a more elementary proof of these results in [2]). Therefore it will suffice to show that, under the hypothesis, $Nil_i(R) = 0$ for $i \geq 0$.

Recall that for any exact category $C$, $Nil(C)$ is the category with objects $(A, \alpha)$ where $A \in C$, and $\alpha$ is a nilpotent endomorphism of $A$. A morphism $(A, \alpha) \to (B, \beta)$ in $Nil(C)$ is a morphism $f : A \to B$ such that $f \alpha = \beta f$. Taking $C$ to be the category of projective modules $\mathcal{P}(R)$ we get a category $\mathcal{N} = Nil(\mathcal{P}(R))$. There are exact functors $\mathcal{N} \to \mathcal{P}(R)$ by $(A, \alpha) \mapsto A$ and $\mathcal{P}(R) \to \mathcal{N}$ by $P \mapsto (P, 0)$. These show that $K_i(R)$ is a direct summand of $K_i(\mathcal{N})$. Define $Nil_i(R)$ to be the cokernel of $K_i(\mathcal{R}) \to K_i(\mathcal{N})$. Then $K_i(\mathcal{N}) = K_i(R) \oplus Nil_i(R)$

Let $Nil'_i(R)$ be defined like $Nil_i(R)$ with the category of projective modules replaced by the category $\text{Coh}(R)$ of coherent modules and $\mathcal{N}$ replaced by the category $\mathcal{N}' = Nil(\text{Coh}(R))$. The previous remarks also apply to this case showing that $K_i(\mathcal{N}) = K_i(\text{Coh}(R)) \oplus Nil'_i(R)$.

Proposition 6.2. If $R$ is a left coherent ring such that each finitely presented $R$–module has finite projective dimension, then $Nil_i(R) \approx Nil'_i(R)$.

Proof. We can regard $\mathcal{N}'$ as the full subcategory of the category of $R[t]$–modules consisting of modules $A$ which are coherent over $R$ and are such that $t \cdot A$ is nilpotent. It is closed under subobjects, quotients, and extensions and so is an abelian category. The category $\mathcal{N}$ is a full subcategory of $\mathcal{N}'$ which is closed under kernels of epimorphisms and extensions. The hypothesis and the next lemma show that each module in $\mathcal{N}'$ has a finite resolution by modules in $\mathcal{N}$ so it follows from [9, Th. 3, Cor 1] that $K_i(\mathcal{N}) = K_i(\mathcal{N}')$. It also follows from [9, Th. 3, Cor 1] that
Lemma 6.3. A finitely presented module $M$ with a nilpotent endomorphism $\alpha$ can be covered by a finitely generated projective module $P$ with a nilpotent endomorphism $\beta$ so that $(P, \beta) \to (M, \alpha)$.

Proof. Suppose $\alpha^{n+1} = 0$. Let $Q$ be projective with $f : Q \to M$, let $P = Q^{n+1}$, and let $\beta(x_0, \ldots, x_n) = (0, x_1, x_2, \ldots, x_{n-1})$. Map $P$ to $M$ by sending $(x_0, \ldots, x_n)$ to $fx_0 + \alpha fx_1 + \alpha^2 fx_2 + \ldots$.

Theorem 6.1 now follows from the next lemma.

Lemma 6.4. If $R$ is a coherent ring then $\text{Nil}_i'(R) = 0$.

Proof. If $[M, \alpha] \in \mathcal{N}'$, filter $M$ by $M_i = \alpha^i M$. The $M_i$ and their quotients are all coherent and $\alpha$ induces 0 on each $M_i/M_{i+1}$. Theorem 4 of [9] now applies to the categories $\text{Coh}(R) \subseteq \mathcal{N}'$. It follows that $K_i(\text{Coh}(R)) = K_i(\mathcal{N}')$ showing that $\text{Nil}_i'(R) = 0$. Although $\text{Coh}(R)$ is not closed under extensions in $\mathcal{N}'$, it is closed under subobjects, quotient objects, and finite products, which is all that is needed for [9, Theorem 4].

7. $G_0$

For a noetherian ring $R$, $G_0(R)$ is the Grothendieck group of the category of finitely generated modules. For general rings I will define $G_0(R)$ as the Grothendieck group of the category of coherent modules. This seems, at the moment, to be a good choice since $\text{Coh}(R)$ is an abelian category and, in the noetherian case, it agrees with the standard definition.

Throughout this section $R[t]$ will denote a polynomial ring in one variable $t$.

Theorem 7.1. If $R$ is a left coherent ring such that $R[t]$ is also left coherent then $G_0(R[t]) \approx G_0(R[t])$ by the map sending $[M]$ to $[R[t] \otimes_R M]$ and $G_0(R) \approx G_0(R[t, t^{-1}])$ sending $[M]$ to $[R[t, t^{-1}] \otimes_R M]$.

Proof. Let $R$ be as in Theorem 7.1 so that $R[t]$ is also coherent, and so is $R[t, t^{-1}]$ by Lemma 3.1. Therefore $R[t] \otimes_R M$ and $R[t, t^{-1}] \otimes_R M$ will be coherent if $M$ is since coherent is the same as finitely presented over these rings by Corollary 2.7. If $M$ is a coherent $R[t]$-module then $M/tM$ and $M = \{x|x \in M, tx = 0\}$ are coherent $R$-modules by Lemma 3.8. We can therefore define a map $G_0(R[t]) \to G_0(R)$ by sending $[M]$ to $[M/tM] - [\hat{i}M]$. That this preserves the relations follows from the snake lemma applied to the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow t & & \downarrow t & & \downarrow t & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
$$
Since $G_0(R) \rightarrow G_0(R[t]) \rightarrow G_0(R)$ is easily seen to be the identity, all that remains is to show that $G_0(R) \rightarrow G_0(R[t])$ is onto. Let $M$ be a coherent $R[t]$–module and let $0 \rightarrow N \rightarrow F \rightarrow F' \rightarrow M \rightarrow 0$ be exact with $F$ and $F'$ free and finitely generated. Filter $F$ as in Theorem 4.3 and let $N_n = N \cap F_n$. For large $n$ we get an exact sequence

$$0 \rightarrow R[t] \otimes_R N_{n-1} \rightarrow R[t] \otimes_R N_n \rightarrow N \rightarrow 0.$$  

Now for any $k$, $N_k$ is the kernel of $F_k \rightarrow F'$ and so is coherent over $R$ by Corollary 2.3. It follows that $[N]$ lies in the image of $G_0(R) \rightarrow G_0(R[t])$ and therefore so does $[M] = [N] - [F] + [F']$.

For the last statement it suffices to show that $G_0(R[t]) \rightarrow G_0(R[t, t^{-1}])$ is an isomorphism. We use the following standard fact (adapted to the coherent case).

**Lemma 7.2.** Let $A$ be a coherent ring with a central multiplicative set $S$. Let $N$ be the full subcategory of $N \in \text{Coh}(A)$ such that $N_S = 0$. Then

$$K_0(N) \xrightarrow{i} G_0(A) \xrightarrow{j} G_0(A_S) \rightarrow 0$$

is exact.

**Proof.** We define a mapping $G_0(A_S) \rightarrow \text{ckr} i$ as follows. If $N$ is a coherent $A_S$-module then by Lemma 3.1 there is a coherent $A$-module $M$ with $M_S \approx N$. If $L$ is another such module Lemma 9.1 shows we can multiply the isomorphism $L_S \approx M_S$ by an element of $S$ so that it lifts to a map $g : L \rightarrow M$. The kernel and cokernel of $g$ are in $\mathcal{N}$ so $[L] = [M]$ in $\text{ckr} i = G_0(A)/\text{im} K_0(N)$. Define $f : G_0(A_S) \rightarrow \text{ckr} i$ by $f([N]) = [M]$. If $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ multiply $p$ by an element of $S$ and lift it to a map $q : M' \rightarrow M$. We get $0 \rightarrow K \rightarrow M' \rightarrow M'' \rightarrow 0$. Localizing shows $M''_S \approx N''$ and $K$ lies in $\mathcal{N}$ so $[M'] + [M''] \equiv [M]$ mod $K_0(N)$ showing that our map is well defined. The two maps between $G_0(A)/\text{im} K_0(N)$ and $G_0(A_S)$ are easily seen to be inverses, proving the lemma. 

Apply this to $A = R[t]$ and $A_S = R[t, t^{-1}]$ with $S = \{t^n | n \geq 0\}$. We have to show that $N_S = 0$ implies $[N] = 0$ in $G_0(R[t])$. Since $N$ is finitely generated, $t^n N = 0$ for some $n$. Filtering $N$ by coherent submodules $N \supseteq tN \supseteq t^2N \supseteq \cdots \supseteq t^n N = 0$ shows that it will suffice to consider the case where $tN = 0$. By Lemma 3.5 $N$ is coherent as an $R$-module so the characteristic sequence Theorem 4.2

$$0 \rightarrow R[t] \otimes_R N \rightarrow R[t] \otimes_R N \rightarrow N \rightarrow 0$$

shows that $[N] = 0$ in $G_0(R[t])$ so $G_0(R[t]) \rightarrow G_0(R[t, t^{-1}])$ is an isomorphism. 

8. Graded Rings

The next three sections are devoted to the proof of the analogues of the above results for the functors $G_i$. We follow Quillen [9] closely (except for writing $G_i(R)$ instead of $K_i^*(R)$) and begin by looking at the case of graded polynomial rings in preparation for the treatment of $G_i$ in the final section.

Let $B = \bigoplus_{n=0}^{\infty} B_n$ be a graded ring and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a graded $B$–module. I will use $M$ to denote $M$ as an ungraded module. For the present purposes I will call $M$ coherent if $\overline{M}$ is a coherent module and write $\text{Cohgr}(R)$ for the category of coherent graded $R$–modules and degree preserving morphisms.

If $M = \bigoplus_{n=0}^{\infty} M_n$ we let $M(-p)$ be $M$ with a new grading $M(-p)_n = M_{n+p}$. We have $\text{Hom}_B(B(-p), M) = M_p$. By a free graded module I will mean a direct sum $\bigoplus_i B(-p_i)$ where $p_i \geq 0$. 
Lemma 8.1. Let $M$ be a graded module over a graded ring.

1. $M$ is finitely generated if and only if $\overline{M}$ is finitely generated.
2. $M$ is finitely presented if and only if $\overline{M}$ is finitely presented.

Proof. The "only if" statements are clear. If $a_1, \ldots, a_n$ generate $\overline{M}$ let $a_i = \sum_j a_{ij}$ where $a_{ij} \in M_j$. The $a_{ij}$ are then homogeneous generators of $M$. If $M$ is finitely presented it is finitely generated so we can map a finitely generated free module $F$ onto $M$ getting an exact sequence $0 \to N \to F \to M \to 0$. After dropping the grading we see that $N$ is finitely generated and therefore so is $N$. \qed

Let $R$ be a coherent ring and let $B = R[x_1, \ldots, x_r]$ with $r < \infty$. Grade $B$ by $\deg x_i = 1$ and assume that $B$ is coherent and therefore so is $R$ by Corollary 3.6 or Lemma 3.9. Our aim is to compute $G_i(B) = K_i(\text{Cohgr}(B))$. Define exact functors $b_p : \text{Coh}(R) \to \text{Cohgr}(B)$ by $b_p(M) = B(-p) \otimes_R M$. They are exact since $B$ is free over $R$. They are graded using the grading of $B(-p)$ so that $b_p(M)_n = B(-p)_n \otimes_R M$. Since $M \in \text{Coh}(R)$ is finitely presented by Corollary 2.7 so is $b_p(M)$ which is therefore coherent so we get a map $\beta_p : G_i(R) \to G_i(B)$.

Theorem 8.2. Let $B = R[x_1, \ldots, x_r]$, with $r < \infty$ be a polynomial ring over a ring $R$ graded by $\deg x_i = 1$ for all $i$. If $B$ is coherent then $\beta = \bigoplus_{p \geq 0} \beta_p : \bigoplus_{p \geq 0} G_i(R) \to G_i(B)$ is an isomorphism.

Proof. We define an inverse mapping $\gamma = \bigoplus_{p \geq 0} \gamma_p : G_i(B) \to \bigoplus_{p \geq 0} G_i(R)$ as follows. If $N \in \text{Cohgr}(B)$ let $Q(N) = R \otimes_B N = N/B^+ N$ where $B^+ = \bigoplus_{n > 0} B_n$ is the kernel of the retraction $B \to R$. We have $Q(N) = \bigoplus Q_n(N)$ with $Q_n(N) = N_n/D_n(N)$ where $D_n(N) = (B^+ N)_n = \bigoplus_{i=1}^r B_i N_{n-i}$, the decomposable elements of $N_n$. Since $N$ is coherent it is finitely presented and therefore so is $Q(N)$ which, as well as its summands $Q_n(N)$, is therefore coherent. $Q$ is right exact but not exact in general. The Tor sequence for $0 \to N' \to N \to N'' \to 0$ is $\cdots \to \text{Tor}_i^B(R,N'') \to \text{Tor}_i^B(R,N') \to Q(N') \to Q(N) \to Q(N'') \to 0$. Therefore $Q$ will be exact on the full subcategory $N$ of $\text{Cohgr}(B)$ of objects $N$ such that $\text{Tor}_i^B(R,N) = 0$ for all $i > 0$. Since $R$ has Tor-dimension $r < \infty$ over $B$ the resolution theorem [9, §4, Cor. 3] shows that $K_i(N) = K_i(\text{Cohgr}(B))$. Define $\gamma_p : K_i(N) \to K_i(\text{Coh}(R))$ to be the map induced by the functor $Q_p : N \to \text{Coh}(R)$.

For $N \in N$ let $F_p N$ be the submodule of $N$ generated by all $N_i$ with $i \leq p$ and let $F_p N$ be the full subcategory of the $F_p N$ for all $N \in N$.

Lemma 8.3. $F_p N$ is closed under extensions and so is an exact category. Moreover $N$ is the union of these subcategories.

Proof. If $0 \to N' \to N \to N'' \to 0$ is exact and $N'$ and $N''$ are in $F_p N$ then $N'$ and $N''$ are generated by homogeneous elements $\{x^p_i\}$ and $\{y^p_i\}$ of degree at most $p$. Lift the $y^p_i$ to elements $y_j$ of the same degree in $N$. Then $\{x^p_i\}$ and $\{y_i\}$ generate $N$ so $F_p N = N$. Any $N$ in $N$ is finitely generated and so lies in $F_p N$ when $p$ is the degrees of its generators. \qed

Since $Q_n$ is 0 on $F_p N$ for $n > p$, the same is true of $\gamma_n : K_i(F_p N) \to G_i(R)$ so we can define a map $\gamma = \bigoplus_{n=0}^\infty \gamma_n : K_i(F_p N) \to \bigoplus_{n=0}^\infty G_i(R)$. Taking the limit as $p \to \infty$ we get the required map $\gamma = \bigoplus_{n=0}^\infty \gamma_n : K_i(N) \to \bigoplus_{n=0}^\infty G_i(R)$.

We can also replace $\text{Cohgr}(B)$ by $N$ in discussing the maps $\beta_p$ since $b_p(M)$ lies in $N$ by the following lemma.
Lemma 8.4. [9, §7, Proof of Lemma 1] If $R$ is a subring of $B$, $B$ is flat over $R$, and $X$ is any $R$–module then $\text{Tor}_i^B(R, B \otimes_R X) = 0$ for all $i > 0$.

Proof. Let $T_i(X) = \text{Tor}_i^B(R, B \otimes_R X)$. This is an exact $\partial$–functor because $B$ is flat over $R$. It is effaceable since $B \otimes_R X$ is projective over $B$ if $X$ is projective over $R$. Therefore the $T_i$ are the derived functors of $T_0$ but $T_0(X) = R \otimes_B B \otimes_R X = X$ which is exact so its higher derived functors are 0.

We now have the required maps.

$$
\bigoplus_0 \infty G_i(R) \xrightarrow{\beta} K_i(N) \xrightarrow{\gamma} \bigoplus_0 \infty G_i(R)
$$

The composition is induced by the functor taking $M$ in $\text{Coh}(R)$ to $Q_n \circ b_p(M) = Q_n(B(-p) \otimes_R M)$ which is the degree $n$ part of $R \otimes_B B(-p) \otimes_R M = M$ graded by assigning the degree $p$ to all elements. This is $M$ if $n = p$ and otherwise 0. This shows that $\gamma \circ \beta$ is the identity.

The composition $\beta \circ \gamma$ is the sum of the compositions $\beta_p \circ \gamma_p$, where $\beta_p \circ \gamma_p$ is induced by $b_p \circ Q_p$. The next lemma shows that the functor $\beta_p \circ \gamma_p$ is isomorphic to the functor $N \mapsto F_pN/F_{p-1}N$. It follows that the functors $F_p$ are exact on $\mathcal{N}$. If we replace $\mathcal{N}$ with $F_p\mathcal{N}$ we have a finite filtration and can apply the theorem on characteristic filtrations [9, §3 Cor.3] to conclude that the endomorphisms of $K_i$ induced by the functors $N \mapsto F_pN/F_{p-1}N$ sum to the identity showing that $\beta \circ \gamma$ is the identity. Taking the limit as $p \to \infty$ then shows that this is true for $\mathcal{N}$, proving Theorem 8.2.

The following lemma makes no use of coherence. The definitions of $F_p\mathcal{N}, Q(N) = \bigoplus_n Q_n(N)$, and $Q_p(N) = N_p/D_p(N)$ are the same as above.

Lemma 8.5. [9, §7, Lemma 1] Let $B = \bigoplus_{n=0}^{\infty} B_n$ be a graded ring with $B_0 = R$ and let $N$ be a finitely generated graded $B$–module. If $\text{Tor}_i^B(R, N) = 0$ then there is an isomorphism $\theta_p : B(-p) \otimes_R Q_p(N) \xrightarrow{\sim} F_pN/F_{p-1}N$.

Proof. Since $N_p \subseteq F_p\mathcal{N}$ and $D_p(N) \subseteq F_{p-1}N$ there is a map $Q_p(N) = N_p/D_p(N) \to F_p\mathcal{N}/F_{p-1}\mathcal{N}$. The right hand side is a $B$–module so this extends to give us our map $\theta_p$. Since $F_p\mathcal{N}$ is generated by $F_{p-1}N$ and $N_p$, $\theta_p$ is onto.

Remark 8.6. To be consistent with the previous notation we regard $Q_p(N)$ as an ungraded module and write $B(-p) \otimes_R Q_p(N)$ instead of $B \otimes_R Q_p(N)$.

We use the following facts.

1. $Q(N) = 0$ implies $N = 0$.
2. $\theta_p$ is onto.
3. $Q(F_{p-1}N) \to Q(F_pN)$ is injective.
4. $Q(\theta_p)$ is an isomorphism.

For (1), if $N_n = 0$ for all $n \leq m$ then $D_m(N) = 0$ so $N_m = Q_m(N) = 0$. (2) was proved above. For (3) and (4) we observe that $Q_n(F_pN) = 0$ for $n > p$ while $Q_n(F_pN) = Q_n(N)$ for $n \leq p$. This implies (3). For (4) observe that if $M$ is a graded module generated by $M_p$ then $Q(M) = M_p$ because $B^+M = \sum_{n \geq p} M_n$. This condition is satisfied by $B(-p) \otimes_R Q_p(N)$ which has $(B(-p) \otimes_R Q_p(N))_p = Q_p(N)$ and by $F_p\mathcal{N}/F_{p-1}\mathcal{N}$ which has $(F_p\mathcal{N}/F_{p-1}\mathcal{N})_p = N_p/D_p(N) = Q_p(N)$. $Q(\theta)$ corresponds to the identity map $Q_p(N) \to Q_p(N)$, the map used to define $\theta$. 

Define $T_i(N) = \text{Tor}^p(R, N)$, an exact $\partial$-functor with $T_0 = Q$. We have $T_1(N) = 0$ by the hypothesis. For large $p F_p N = N$ because $N$ is finitely generated. Assuming $T_1(F_p N) = 0$, we will show that $T_1(F_{p-1} N) = 0$, proving that this is true for all $p$. At the same time we show that $K^p = \ker \theta_p = 0$ so $\theta_p$ is an isomorphism. The exact sequence $0 \to K^p \to B(-p) \otimes_R Q_p(N) \to F_p N/F_{p-1} N \to 0$ gives us an exact sequence

$$0 \to T_1(F_p N/F_{p-1} N) \to Q(K^p) \to Q(B(-p) \otimes_R Q_p(N)) \xrightarrow{Q(\theta_p)} Q(F_p N/F_{p-1} N)$$

Note that $T_1(B(-p) \otimes_R Q_p(N)) = 0$ by Lemma 8.4. The map on the right is an isomorphism by (4). This shows that $T_1(F_p N/F_{p-1} N) = Q(K^p)$.

Assume now that $T_1(F_p N) = 0$. The exact $T_1$-sequence for $0 \to F_p N \to F_p N/F_{p-1} N \to 0$ is

$$0 = T_1(F_p N) \to T_1(F_p N/F_{p-1} N) \to Q(F_{p-1} N) \to Q(F_p N)$$

The map on the right is injective by (3). This shows that $T_1(F_p N/F_{p-1} N) = Q(K^p)$ is $0$ so $K^p = 0$ by (1). From the same sequence we get $T_2(F_p N/F_{p-1} N) \to T_1(F_{p-1} N) \to T_1(F_p N) = 0$. Since $K^p = 0$, $\theta_p$ is an isomorphism so $T_2(F_p N/F_{p-1} N) = T_2(B(-p) \otimes_R Q_p(N)) = 0$ by Lemma 8.4. It follows that $T_1(F_{p-1} N) = 0$ completing the induction. □

9. Localization

Let $A = \bigoplus_{n \geq 0} A_n$ be a graded ring and let $S$ be a central multiplicative subset of $A$ consisting of homogeneous elements. Then $A_S$ is a graded ring with $\deg x/s = \deg x - \deg s$. We allow elements of negative degree here.

Lemma 9.1. Let $S$ be a central multiplicative subset of a ring $A$ and let $M$ and $N$ be $A$–modules with $M$ finitely presented. If $\gamma : M_S \to N_S$ is an $A_S$–homomorphism there is an $A$–homomorphism $g : M \to N$ and an element $s$ of $S$ with $g_S = s \gamma$.

Proof. Suppose $M = F$ is free and finitely generated by $e_1, \ldots, e_n$. Let $\gamma(e_k) = x_k/s$ with $s$ in $S$. Then $g(e_k) = x_k$ is the required map. Let $F' \xrightarrow{i} F \xrightarrow{j} M \to 0$ be a finite presentation of $M$. By the previous remark we can find $h : F \to N$ such that $h_S = s \gamma \circ j_S$. Now $h \circ i$ localizes to $0$ since $j_S \circ i_S = 0$ so the image of $h \circ i$, being finitely generated, is annihilated by some $t \in S$. Therefore $th \circ i = 0$ so $th$ factors through the cokernel of $i$ giving us the required map $g$. We have $g \circ j = th$ so $g_S \circ j_S = th_S = ts \gamma \circ j_S$ and therefore $g_S = th_S = ts \gamma$ since $j_S$ is an epimorphism. □

If $F : A \to B$ is an exact covariant functor of abelian categories then the full subcategory $\mathcal{S}$ of objects $A$ of $A$ with $F(A) = 0$ is a Serre subcategory and we have an exact functor $A/\mathcal{S} \to B$. If this is an equivalence of categories we get an exact localization sequence

$$\cdots \to K_i(\mathcal{S}) \to K_i(A) \to K_i(B) \to K_{i-1}(\mathcal{S}) \to \cdots$$

by the Localization Theorem [9, §5 Th. 5].

Theorem 9.2 ([11], Theorem 5.11). In this situation if the following two conditions are satisfied then $A/\mathcal{S} \to B$ is an equivalence of categories.

1. For each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ with $F(A) \approx B$. 
(2) If \( A, A' \in \mathcal{A} \) and \( f : F(A) \to F(A') \) in \( \mathcal{B} \) there is an object \( A'' \) and maps
\[A \leftarrow A'' \twoheadrightarrow A'\]
such that \( F(h) \) is an isomorphism and \( f = F(g)F(h)^{-1} \).

**Corollary 9.3.** Let \( A = \bigoplus_{n \geq 0} A_n \) be a graded ring and let \( S \) be a central multiplicative subset of \( A \) consisting of homogeneous elements. Let \( \mathcal{S} \) be the full subcategory of \( \text{Cohgr}(A) \) of all modules \( M \) with \( M_S = 0 \). Then \( \text{Cohgr}(A)/\mathcal{S} \approx \text{Cohgr}(A_S) \).

**Proof.** We have to check the two conditions of Theorem 9.2 with \( \mathcal{A} = \text{Cohgr}(A) \) and \( \mathcal{B} = \text{Cohgr}(A_S) \).

1. Any \( N \in \mathcal{B} \) is finitely presented so we can write \( F_S' \xrightarrow{\gamma} F_S \to N \to 0 \) with \( F_S \) free and finitely generated over \( A_S \). By Lemma 9.1 we can write \( s\gamma = gs \) and \( N = \text{coker} \gamma = \text{coker} g_S \) since \( s \) is a unit in \( A_S \). Therefore \( N = M_S \) where \( M = \text{coker} g \).

2. Suppose \( \gamma : M_S \to N_S \) in \( \text{Cohgr}(A_S) \). By Lemma 9.1 write \( s\gamma = gs \) where \( g : M \to N \). Then we have \( M \xleftarrow{\alpha} M(-d) \xrightarrow{\beta} N \) and \( s \) localizes to an isomorphism as required. Note that \( M(-d) = M \) as a module but we have changed the grading by \( d = \text{deg} s = \text{deg} g \) to make the two maps have degree 0.

\( \square \)

In the next lemma we allow \( A \) and its modules to have elements of negative degree.

**Lemma 9.4.** If \( A \) is a coherent graded ring having a central unit \( s \) in \( A_1 \) then \( A_0 \) is also coherent, \( \text{Cohgr}(A) \approx \text{Coh}(A_0) \), and \( G_i(A_0) \to G_i(A) \) induced by \( M \to A \otimes A_0 \) and \( M \) is an isomorphism.

**Proof.** Since \( s \) is a unit \( s^n : A_0 \approx A_n \) so \( A = \bigoplus A_0 s^n \) showing that \( A \) is a Laurent polynomial ring \( A_0[s, s^{-1}] \) and therefore \( A_0 \) is coherent by Lemma 3.9. Similarly if \( N \) is a graded \( A \)-module \( s^n : N_0 \approx N_n \) so \( N = N_0[s, s^{-1}] \) Define \( f : \text{Coh}(A_0) \to \text{Cohgr}(A) \) by \( f(M) = A \otimes A_0 M \) and \( g : \text{Cohgr}(A) \to \text{Coh}(A_0) \) by \( g(N) = N_0 \). These maps are inverse equivalences of categories.

\( \square \)

10. \( G_i \)

**Theorem 10.1.** If \( R \) is a ring such that the polynomial ring \( R[x, y] \) is coherent then \( G_i(R) \to G_i(R[x]) \), induced by the functor \( M \mapsto R[x] \otimes_R M \), is an isomorphism.

**Proof.** We can assume that \( i > 0 \) because of Theorem 7.1. Let \( A = R[t, s] \) and \( B = R[t] \) be polynomial rings graded by \( \deg t = \deg s = 1 \). The localization sequence \([9, \S 5 \text{Th. 5}]\) for \( A \to A_s \) is \( \cdots \to G_i(N) \to G_i(A) \to G_i(A_s) \to G_{i-1}(N) \to \cdots \) where \( N \) is the Serre subcategory of \( A \)-modules \( M \) such that \( M_s = 0 \). By the Devissage Theorem \([9, \S 5 \text{Th.4}]\) \( G_i(N) \approx G_i(A/(s)) = G_i(B) \).

By Lemma 9.4 \( G_i(A_s) = G_i((A_0)_0) \). Since \( A_s = R[t, s, s^{-1}] \) we see that \( (A_s)_0 = R[z] \) where \( z = t/s \). Therefore the localization sequence takes the form
\[\cdots \to G_i(B) \xrightarrow{j_s} G_i(A) \to G_i(R[z]) \to G_{i-1}(B) \to \cdots\]

Here \( A \) and \( B \) are graded rings and the map \( j_s : G_i(B) \to G_i(A) \) is induced by the inclusion of \( \text{Cohgr}(B) \) in \( \text{Cohgr}(A) \) while the map \( G_i(A) \to G_i(R[z]) \) is induced by \( N \mapsto (N_s)_0 \).
By Theorem 8.2 we have an isomorphism $\alpha = \bigoplus_{p \geq 0} \alpha_p : \bigoplus_{p \geq 0} G_i(R) \to G_i(A)$ induced by $a_p : M \mapsto A(-p) \otimes_R M$, and similarly $\beta = \bigoplus_{p \geq 0} \beta_p : \bigoplus_{p \geq 0} G_i(R) \to G_i(B)$ given by $b_p(M) \mapsto B(-p) \otimes_R M$. We have a diagram

$$
\begin{array}{cccc}
\bigoplus_{p \geq 0} G_i(R) & \xrightarrow{f} & \bigoplus_{p \geq 0} G_i(R) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
G_i(B) & \xrightarrow{j_*} & G_i(A)
\end{array}
$$

The exact sequence $0 \to A(-p - 1) \to A(-p) \to B(-p) \to 0$ tensored with $M$ gives us an exact sequence of functors $0 \to M_{p+1} \to M_p \to M_{p-1} \to 0$. By the characteristic filtration theorem [9, §3 Cor.3] this implies that $j_*\beta_p = \alpha_p - \alpha_{p+1}$. If $x = (x_p) \in \bigoplus_{p \geq 0} G_i(R)$ then $j_*\beta(x) = \sum_{p \geq 0} j_*\beta_p(x_p) = \sum_{p \geq 0} \alpha_p(x_p) - \sum_{p \geq 0} \alpha_{p+1}(x_p)$ so $j_*\beta(x) = \sum_{p \geq 0} \alpha_p(x_p - x_{p-1}) = \alpha(y)$ with $y_p = x_p - x_{p-1}$ where we set $x_p = 0$ when $p < 0$.

The map $G_i(B) \to G_i(A)$ is therefore isomorphic to the map $f : \bigoplus_{p \geq 0} G_i(R) \to \bigoplus_{p \geq 0} G_i(R)$ given by $(x_p) \mapsto (y_p)$ where $y_p = x_p - x_{p-1}$. We can recover the $x_p$ from the $y_p$ by $x_p = y_0 + y_1 + \cdots + y_p$ so the map is injective. The image is the set of $(y_p)$ for which $\sum_p y_p = 0$ so the cokernel is $G_i(R)$ via the map $\bigoplus_p G_i(R) \to G_i(R)$ sending $(y_p)$ to $\sum_p y_p$. The map $G_i(R) \xrightarrow{\alpha} G_i(A) \to \mathrm{cok} j_* \approx G_i(R[z])$ is therefore an isomorphism. It is induced by the functor sending an $R$-module $M$ to $(A \otimes_R M)_{j_*}$ if $R[z] \otimes_R M$ as required.

**Corollary 10.2.** If $R$ is a ring such that the polynomial ring $R[x, y]$ is coherent then $G_i(R[z, x^{-1}]) = G_i(R) \oplus G_{i-1}(R)$ for $i > 0$.

**Proof.** Observe that in the localization sequence

$$
\cdots \to G_i(R) \to G_i(R[x]) \to G_i(R[x, x^{-1}]) \to G_{i-1}(R) \to G_{i-1}(R[x]) \to \cdots
$$

the map $G_i(R) \to G_i(R[x])$ is 0 because of the exact characteristic sequence $0 \to M[x] \to M[x] \to M \to 0$ (Theorem 4.2) and the theorem on characteristic filtrations [9, §3 Cor.3].

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