K–THEORY OF COHERENT RINGS

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Abstract. We show that some basic results on the K–theory of noetherian rings can be extended to coherent rings.

1. Introduction

The main object of this paper is to show that $K_i(R[t]) = K_i(R)$ for coherent rings R which are regular (every finitely presented module has finite projective dimension). This gives a partial answer to a question of O. Braeunling who asked when this result holds for non–noetherian rings R. His question, which was suggested by [7], was forwarded to me by T. Y. Lam. At the same time , C. Quitté sent me a copy of his book (with H. Lombardi) [8] which recommends coherent rings as a substitute for noetherian rings in constructive mathematics. This suggested the above result.

An old result of Gersten [4, Th. 3.1] shows that $K_i(R[t]) = K_i(R)$ if R is regular and $R[x, y]$ is coherent. Here we show that it is sufficient to assume that only R is coherent using results of Quillen [6] not available when Gersten's paper was written. Recent work relating coherence properties to the vanishing of neagtive K-theory can be found in [1].

Most of this paper is expository since the proofs are modifications of standard proofs in Algebraic K–Theory. To avoid endless repetition, I will only consider the case of left modules. The results, of course, are also true for right modules with the obvious changes. The symbol t in $R[t]$ and $R[t, t^{-1}]$ will always be an indeterminate.

2. Coherent Modules

For the readers convenience, we recall here the basic facts about coherent modules and rings. For a detailed and comprehensive account see [5] (for the commutative case).

Definition 2.1. Let R be an associative ring. A left R–module M is called pseudo–coherent if every map $R^n \to M$ with $n < \infty$ has a finitely generated kernel. In other words, every finitely generated submodule of M is finitely presented. A coherent module is a finitely generated pseudo–coherent module. The ring R is called coherent if it is coherent as a left R-module.

In [8] the terminology has been changed. The pseudo-coherent modules are called coherent and coherent modules are called finitely generated coherent modules. I will stick to the more familiar terminology here to avoid confusion with the usage in algebraic geometry.

I would like to thank Claude Quitté for sending me a copy of his book (with H. Lombardi)[8] and for other relevant references. I would also like to thank T. Y. Lam for sending me the question which inspired this paper and O. Braeunling for useful comments and references.

Lemma 2.2. *If L is a finitely generated module and M is a pseudo–coherent module, every map $L \to M$ has a finitely generated kernel.*

Proof. Let $F = R^n$ map onto L. The kernel of $F \to L \to M$ is finitely generated and maps onto the kernel of $L \to M$. □

Corollary 2.3. *If L is a coherent module and M is a pseudo–coherent module, every map $L \to M$ has a coherent kernel.*

The kernel is pseudocoherent as a submodule of L. It is finitely generated by Lemma 2.2

Let $\mathcal{M}(R)$ be the category of all left R–modules, and let $\mathcal{F}g(R)$, $\mathcal{F}p(R)$, and $\mathcal{C}oh(R)$ be the full subcategories of $\mathcal{M}(R)$ of finitely generated, finitely presented, and coherent modules. If R is noetherian, $\mathcal{F}g(R) = \mathcal{F}p(R) = \mathcal{C}oh(R)$.

Theorem 2.4. *For any R, the subcategory $\mathcal{C}oh(R)$ of $\mathcal{M}(R)$ is closed under kernels, cokernels, images, and extensions and therefore is an abelian category.*

Proof. Let $f : M \to N$ with M and N coherent. Then $\ker f$ is coherent by Corollary 2.3 while $\operatorname{im} f$ is pseudocoherent as a submodule of N and finitely generated as an image of M. Let $I = \operatorname{im} f$ and $Q = \operatorname{ckr} f$. We have an exact sequence $0 \to I \to N \to Q \to 0$. Let $g : F \to Q$ with F free and finitely generated. Lift g to a map $h : F \to N$. Let $k : E \twoheadrightarrow I$ with E free and finitely generated. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & E \oplus F & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle (k,h)}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & I & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

gives us the exact sequence $\ker(k, h) \to \ker g \to 0$ showing that $\ker g$ is finitely generated as required. Finally let $0 \to M' \to M \to M'' \to 0$ be exact with M' and M'' coherent. Let $f : F \to M$ be a map with F free and finitely generated and let $g : F \to M \to M''$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

gives us the exact sequence $0 \to \ker f \to \ker g \to M'$. Since M' is coherent and $\ker g$ is finitely generated, Lemma 2.2 shows that $\ker f$ is finitely generated. □

Corollary 2.5. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left R–modules. If two of the modules M', M, M'' are coherent, so is the third.*

Corollary 2.6. *If M is coherent and N is finitely generated then the cokernel of any map $f : N \to M$ is coherent.*

The image L of f is coherent since it is a finitely generated submodule of M and $\operatorname{ckr} f = M/L$.

Corollary 2.7. *If R is left coherent then $\mathcal{F}p(R) = \mathcal{C}oh(R)$.*

If M is finitely presented it is the cokernel of a map $R^m \to R^n$ with $m, n < \infty$.

Lemma 2.8. *Let $\mathcal{A}$ be a full subcategory of $\mathcal{M}(R)$ such that $\mathcal{A}$ is abelian and $R \in \operatorname{ob}\mathcal{A}$. Then any map $f : M \to N$ in $\mathcal{A}$ has the same kernel in $\mathcal{A}$ as in $\mathcal{M}(R)$.*

Proof. Let $K = \ker f$ in $\mathcal{M}(R)$ and let $L = \ker f$ in $\mathcal{A}$. Then $L \to M \to N$ is 0 so $L \to M$ factors through K. If $x \in L$ maps to 0 in M, let $R \to L$ by $r \mapsto rx$. Then $R \to L \to M$ is 0. Since $L \to M$ is a monomorphism in $\mathcal{A}$ we see that $R \to L$ is 0 showing that $x = 0$. Therefore $L \to M$ is injective. We can now regard K and L as submodules of M and clearly $L \subseteq K$. Let $x \in K$ and let $R \to K$ by $r \mapsto rx$. Then $R \to M \to N$ is 0. Since this is in $\mathcal{A}$, $R \to M$ factors through L showing that $x \in L$. Therefore $L = K$. □

Corollary 2.9. [4, Prop. 1.1(c)] *A ring R is left noetherian if and only if $\mathcal{F}g(R)$ is an abelian category. It is left coherent if and only if $\mathcal{F}p(R)$ is an abelian category.*

Proof. The 'only if' part follows from Theorem 2.4 and Corollary 2.7. Suppose that $\mathcal{F}g(R)$ is an abelian category. Let I be a left ideal of R. By Lemma 2.8 the kernel I of the map $R \to R/I$ lies in $\mathcal{F}g(R)$ and so is finitely generated. Finally, if $\mathcal{F}p(R)$ is an abelian category then the kernel of $f : R^n \to R$ lies in $\mathcal{F}p(R)$ and so is finitely generated. □

3. Examples

Lemma 3.1. *If R is a coherent ring, so is any localization R_S (where S is a central multiplicative set) and for any coherent R_S–module M there is a coherent R–module N with $N_S \approx M$.*

Proof. Let $f : R_S^n \to R_S$. By multiplying f by an element of S we can assume that f lifts to $g : R^n \to R$. The kernel of g is finitely generated and localizes to the kernel of f. By Corollary 2.7 it is sufficient to prove the second part for finitely presented modules. Given $R_S^m \xrightarrow{f} R_S^n \to M \to 0$, some multiple sf with s in S lifts to $g : R^m \to R^n$ and we take $N = \operatorname{ckr} g$. □

Lemma 3.2.

1. *An R–module which is the filtered union of pseudo-coherent R–modules is pseudocoherent over R.*
2. *If a ring R is the filtered union of coherent subrings R_α and if R is flat over each R_α then R is coherent.*

Proof. The first statement is clear. For the second let $x_1, \dots, x_n \in R$ and map $f : R^n \to R$ by $e_i \to x_i$. All x_i lie in some R_α so we also get $g : R_\alpha^n \to R_\alpha$ with finitely generated kernel K. By flatness $R \otimes_{R_\alpha} K$ is the kernel of f which is therefore finitely generated. □

Corollary 3.3. *A polynomial ring in infinitely many variables over a noetherian ring is coherent. So are the rings of algebraic integers i.e. the integral closure of $\mathbb{Z}$ in an algebraic field extension of $\mathbb{Q}$.*

Remark 3.4. An example in [10] shows that $R[t]$ need not be coherent even if R is. In contrast to the noetherian case, a quotient R/I of a coherent ring R may not be a coherent ring. For example, any commutative ring can be a quotient of a polynomial ring over $\mathbb{Z}$ in sufficiently many variables.

Lemma 3.5. *Let I be a 2–sided ideal of a ring R and let M be an R–module annihilated by I so that M is also an R/I–module. If M is coherent over R then M is also coherent over R/I.*

Proof. M is clearly finitely generated. Let $f : (R/I)^n \to M$. Let $g : R^n \to M$ be the composition $R^n \to (R/I)^n \to M$. Then $\ker g$ maps onto $\ker f$ showing that $\ker f$ is finitely generated. □

Corollary 3.6. *If R is a coherent ring and I is a 2–sided ideal which is finitely generated as a left ideal, then R/I is a coherent ring.*

This is immediate from the lemma and Corollary 2.6. In particular, if the polynomial ring $R[x]$ is coherent so is R.

Corollary 3.7. *Let R and I be as in the Lemma. If M is a coherent R–module then M/IM is a coherent R/I–module.*

By Corollary 2.7 it is sufficient to prove this for finitely presented modules which is obvious.

Lemma 3.8. *Let M be a coherent module over a coherent ring R. If s is a central regular element of R then M/sM and ${}_sM = \{x \in R \mid sx = 0\}$ are coherent over the coherent ring R/Rs.*

Proof. R/Rs is coherent by Corollary 3.6 and M/sM is coherent by Corollary 3.7 even without the regularity assumption. For ${}_sM$ let F be a finitely generated R–module mapping onto M with kernel N which is coherent. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle s}\big\downarrow & & {\scriptstyle s}\big\downarrow & & {\scriptstyle s}\big\downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

we get an exact sequence $0 \to {}_sM \to N/sN \to F/sF$ showing that ${}_sM$ is coherent since N/sN and F/sF are. □

Recall that a subring R of a ring B is called a retract of B if there is a ring homomorphism $\epsilon : B \to R$ such that $\epsilon | R = id$.

Lemma 3.9. *If R is a retract of a coherent ring A which is flat over R then R is coherent.*

Proof. If $0 \to K \to R^n \to R$ with $n < \infty$, tensoring with A gives $0 \to A \otimes_R K \to A^n \to A$ so $A \otimes_R K$ is finitely generated and therefore so is $K = R \otimes_A A \otimes_R K$. □

4. A useful exact sequence

Let $R[t]$ be a polynomial ring in one variable over R. Let L be an R–module and let $F = R[t] \otimes_R L = L[t]$. Filter F by letting $F_n = \sum_{q=0}^{n} Rt^q \otimes L = L + Lt + Lt^2 + \cdots + Lt^n$. Let $F_n = 0$ for $n < 0$.

Lemma 4.1. *Let $f_0, f_1, \ldots, f_r \in F_k$ satisfy $\sum_{i=0}^{r} t^i f_i = 0$ Then $f_r \in F_{k-1}$*

Proof. Let $f_i = \sum_{j=0}^{k} t^j a_{ij}$ where $a_{ij} \in L$. Then $\sum_{i=0}^{r} \sum_{j=0}^{k} t^i t^j a_{ij} = 0$. The leading term $t^{r+k} a_{rk}$ must be 0 and the result follows. □

Let M be a left $R[t]$–module. Recall the following result from [2].

Theorem 4.2 ([2])**.** *There is an exact sequence ("The characteristic sequence")*

$$0 \to R[t] \otimes_R M \xrightarrow{\alpha} R[t] \otimes_R M \xrightarrow{\beta} M \to 0$$

where $\alpha(t^n \otimes x) = t^{n+1} \otimes x - t^n \otimes tx$ *and* $\beta(t^n \otimes x) = t^n x$.

In [11] I gave a modified version with smaller terms as follows:

Theorem 4.3. *Let M be a finitely generated left $R[t]$–module which is contained in a free module F. Write $F = R[t] \otimes_R L$ where L is free over R and filter F as above by $F_n = L + tL + \cdots + t^n L$. Let $M_n = M \cap F_n$. Then, for large n, there is an exact sequence*

$$0 \to R[t] \otimes_R M_{n-1} \xrightarrow{\alpha} R[t] \otimes_R M_n \xrightarrow{\beta} M \to 0 \tag{1}$$

where α and β are as in Theorem 4.2.

Proof. It is easy to see that α and β define maps as indicated. Let n be large enough that all chosen generators of M lie in M_n. Then β will be onto. That $\beta\alpha = 0$ is obvious. Suppose $\alpha(\sum_{i=0}^r t^i \otimes a_i) = 0$. Then $\sum_{i=0}^r t^{i+1} \otimes a_i - \sum_{i=0}^r t^i \otimes ta_i = 0$ The leading term, $t^{r+1} \otimes a_r$, is 0 so $a_r = 0$ and, by induction all $a_i = 0$ showing that α is injective,

Suppose $\beta(\sum_{i=0}^r t^i \otimes a_i) = 0$ where all a_i are in M_n. Then $\sum_{i=0}^r t^i a_i = 0$. Since $a_i \in F_n$, Lemma 4.1 shows that $a_r \in F_{n-1}$. Therefore $\alpha(t^{r-1} \otimes a_r)$ is defined. It is $t^r \otimes a_r - t^{r-1} \otimes ta_r$ so by subtracting it from $\sum_{i=0}^r t^i \otimes a_i$ we can reduce the degree. It follows by induction that $\ker \beta = \operatorname{im} \alpha$. □

5. K_0

In this section and the next we examine the case of projective modules.

Lemma 5.1. *If R is a coherent ring any finitely generated projective R-module is coherent and, if M is a coherent R-module, there is a resolution*

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

with all P_i finitely generated projective. If M also has finite projective dimension there is such a resolution with $P_n = 0$ for all large n.

Proof. The first statement follows from Corollary 2.7. The resolution is constructed in the standard way. Let P_0 be projective, finitely generated, and map onto M with kernel N. Similarly let P_1 map onto N etc. If M has finite projective dimension then $\ker(P_{n+1} \to P_n)$ will be projective for large n and we can stop there. □

Theorem 5.2. *Let R be a left coherent ring such that each finitely presented R–module has finite projective dimension. Then each finitely generated projective $R[t]$–module P has a finite resolution by extended projective modules*

$$0 \to R[t] \otimes_R Q_n \to R[t] \otimes_R Q_{n-1} \to \cdots \to R[t] \otimes_R Q_0 \to P \to 0.$$

where each Q_i is finitely generated projective over R.

Proof. Let $P \oplus S = F$ be free and finitely generated. Filter F as in Theorem 4.3 and let $P_n = P \cap F_n$. Since P_n is the kernel of $F_n \to S$ it is coherent by Corollary 2.3. By Theorem 4.3 we get an exact sequence

$$0 \to R[t] \otimes_R P_{n-1} \xrightarrow{\alpha} R[t] \otimes_R P_n \to P \to 0.$$

Choose finite projective resolutions $A'_\bullet$ for P_{n-1} and $B'_\bullet$ for P_n and extend these to get resolutions $A_\bullet = R[t] \otimes_R A'_\bullet$ for $R[t] \otimes_R P_{n-1}$ and $B_\bullet = R[t] \otimes_R B'_\bullet$ for $R[t] \otimes_R P_n$. Cover α by a map $f : A_\bullet \to B_\bullet$, and let $C_\bullet$ be the mapping cone of f i.e. $C_m = A_{m-1} \oplus B_m$ with $\partial(a, b) = (-\partial a, \partial b + f(a))$. Note that $C_m = R[t] \otimes_R A'_{m-1} \oplus R[t] \otimes_R B'_m$ is extended from R. The exact sequence

$$\cdots \to H_m(A_\bullet) \to H_m(B_\bullet) \to H_m(C_\bullet) \to H_{m-1}(A_\bullet) \to \dots$$

shows that $H_m(C_\bullet) = 0$ for $m \neq 0$ and is P for $m = 0$, so $C_\bullet$ is the required resolution. □

Corollary 5.3. *If R is a left coherent ring such that each finitely presented R–module has finite projective dimension, then $[M] \mapsto [R[t] \otimes_R M]$ induces an isomorphism $K_0(R) \approx K_0(R[t])$.*

The map is onto by the theorem and is split injective by the map $[N] \mapsto [N/tN]$.

Remark 5.4. Since $R[t]$ need not be coherent even if R is, it is not clear whether this result can be extended to $R[t_1, \dots, t_n]$ for $n > 1$.

6. K_i

Theorem 6.1. *If R is a left coherent ring such that each finitely presented R–module has finite projective dimension, then $[M] \mapsto [R[t] \otimes_R M]$ induces isomorphisms $K_i(R) = K_i(R[t])$ and $K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R)$ for all $i > 0$.*

Proof. By the Fundamental Theorem [6] we have $K_i(R[t]) = K_i(R) \oplus NK_i(R)$ and $K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R) \oplus NK_i(R) \oplus NK_i(R)$ and $NK_i(R) = Nil_{i-1}(R)$. (When $i = 1$ there is a more elementary proof of these results in [2]). Therefore it will suffice to show that, under the hypothesis, $Nil_i(R) = 0$ for $i \geq 0$.

Recall that for any exact category $\mathcal{C}$, $\mathcal{N}il(\mathcal{C})$ is the category with objects (A, α) where $A \in \mathcal{C}$, and α is a nilpotent endomorphism of A. A morphism $(A, \alpha) \to (B, \beta)$ in $\mathcal{N}il(\mathcal{C})$ is a morphism $f : A \to B$ such that $f\alpha = \beta f$. Taking $\mathcal{C}$ to be the category of projective modules $\mathcal{P}(R)$ we get a category $\mathcal{N} = \mathcal{N}il(\mathcal{P}(\mathcal{R}))$. There are exact functors $\mathcal{N} \to \mathcal{P}(R)$ by $(A, \alpha) \mapsto A$ and $\mathcal{P}(R) \to \mathcal{N}$ by $P \mapsto (P, 0)$. These show that $K_i(R)$ is a direct summand of $K_i(\mathcal{N})$. Define $Nil_i(R)$ to be the cokernel of $K_i(R) \to K_i(\mathcal{N})$. Then $K_i(\mathcal{N}) = K_i(R) \oplus Nil_i(R)$

Let $Nil'_i(R)$ be defined like $Nil_i(R)$ with the category of projective modules replaced by the category $\mathcal{C}oh(R)$ of coherent modules and $\mathcal{N}$ replaced by the category $\mathcal{N}' = \mathcal{N}il(\mathcal{C}oh(R))$. The previous remarks also apply to this case showing that $K_i(\mathcal{N}) = K_i(\mathcal{C}oh(R)) \oplus Nil'_i(R)$. □

Proposition 6.2. *If R is a left coherent ring such that each finitely presented R–module has finite projective dimension, then $Nil_i(R) \xrightarrow{\approx} Nil'_i(R)$.*

Proof. We can regard $\mathcal{N}'$ as the full subcategory of the category of $R[t]$–modules consisting of modules A which are coherent over R and are such that $t|A$ is nilpotent. It is closed under subobjects, quotients, and extensions and so is an abelian category. The category $\mathcal{N}$ is a full subcategory of $\mathcal{N}'$ which is closed under kernels of epimorphisms and extensions. The hypothesis and the next lemma show that each module in $\mathcal{N}'$ has a finite resolution by modules in $\mathcal{N}$ so it follows from [9, Th. 3,Cor 1] that $K_i(\mathcal{N}) = K_i(\mathcal{N}')$. It also follows from [9, Th. 3,Cor 1] that

$K_i(\mathcal{P}(R)) = K_i(\mathcal{C}oh(R))$, so Proposition 6.2 follows from the 5–Lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Nil_i(R) & \longrightarrow & K_i(\mathcal{N}) & \longrightarrow & K_i(\mathcal{P}(R)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\approx & & \downarrow\approx & & \\ 0 & \longrightarrow & Nil'_i(R) & \longrightarrow & K_i(\mathcal{N}') & \longrightarrow & K_i(\mathcal{C}oh(R)) & \longrightarrow & 0. \end{array}$$

□

Lemma 6.3. *A finitely presented module M with a nilpotent endomorphism α can be covered by a finitely generated projective module P with a nilpotent endomorphism β so that $(P,\beta) \twoheadrightarrow (M,\alpha)$.*

Proof. Suppose $\alpha^{n+1} = 0$. Let Q be projective with $f : Q \twoheadrightarrow M$, let $P = Q^{n+1}$, and let $\beta(x_0, \ldots, x_n) = (0, x_1, x_2, \ldots, x_{n-1})$. Map P to M by sending $(x_0, \ldots, x_n)$ to $fx_0 + \alpha f x_1 + \alpha^2 f x_2 + \ldots$. □

Theorem 6.1 now follows from the next lemma.

Lemma 6.4. *If R is a coherent ring then $Nil'_i(R) = 0$.*

Proof. If $[M,\alpha] \in \mathcal{N}'$, filter M by $M_i = \alpha^i M$. The M_i and their quotients are all coherent and α induces 0 on each M_i/M_{i+1}. Theorem 4 of [9] now applies to the categories $\mathcal{C}oh(R) \subseteq \mathcal{N}'$. It follows that $K_i(\mathcal{C}oh(R)) = K_i(\mathcal{N}')$ showing that $Nil'_i(R) = 0$. Although $\mathcal{C}oh(R)$ is not closed under extensions in $\mathcal{N}'$, it is closed under subobjects, quotient objects, and finite products, which is all that is needed for [9, Theorem 4]. □

7. G_0

For a noetherian ring R, $G_0(R)$ is the Grothendieck group of the category of finitely generated modules. For general rings I will define $G_0(R)$ as the Grothendieck group of the category of coherent modules. This seems, at the moment, to be a good choice since $\mathcal{C}oh(R)$ is an abelian category and, in the noetherian case, it agrees with the standard definition.

Throughout this section $R[t]$ will denote a polynomial ring in one variable t.

Theorem 7.1. *If R is a left coherent ring such that $R[t]$ is also left coherent then $G_0(R) \approx G_0(R[t])$ by the map sending $[M]$ to $[R[t] \otimes_R M]$ and $G_0(R) \approx G_0(R[t,t^{-1}])$ sending $[M]$ to $[R[t,t^{-1}] \otimes_R M]$.*

Proof. Let R be as in Theorem 7.1 so that $R[t]$ is also coherent, and so is $R[t,t^{-1}]$ by Lemma 3.1. Therefore $R[t] \otimes_R M$ and $R[t,t^{-1}] \otimes_R M]$ will be coherent if M is since coherent is the same as finitely presented over these rings by Corollary 2.7. If M is a coherent $R[t]$–module then M/tM and ${}_tM = \{x | x \in M, tx = 0\}$ are coherent R-modules by Lemma 3.8 We can therefore define a map $G_0(R[t]) \to G_0(R)$ by sending $[M]$ to $[M/tM] - [{}_tM]$. That this preserves the relations follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t & & \downarrow t & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Since $G_0(R) \to G_0(R[t]) \to G_0(R)$ is easily seen to be the identity, all that remains is to show that $G_0(R) \to G_0(R[t])$ is onto. Let M be a coherent $R[t]$–module and let $0 \to N \to F \to F' \to M \to 0$ be exact with F and F' free and finitely generated. Filter F as in Theorem 4.3 and let $N_n = N \cap F_n$. For large n we get an exact sequence

$$0 \to R[t] \otimes_R N_{n-1} \to R[t] \otimes_R N_n \to N \to 0.$$

Now for any k, N_k is the kernel of $F_k \to F'$ and so is coherent over R by Corollary 2.3. It follows that $[N]$ lies in the image of $G_0(R) \to G_0(R[t])$ and therefore so does $[M] = [N] - [F] + [F']$.

For the last statement it suffices to show that $G_0(R[t]) \to G_0(R[t, t^{-1}])$ is an isomorphism. We use the following standard fact (adapted to the coherent case).

Lemma 7.2. *Let A be a coherent ring with a central multiplicative set S. Let $\mathcal{N}$ be the full subcategory of $N \in \mathcal{C}oh(A)$ such that $N_S = 0$. Then*

$$K_0(\mathcal{N}) \xrightarrow{i} G_0(A) \xrightarrow{j} G_0(A_S) \to 0$$

is exact.

Proof. We define a mapping $G_0(A_S) \to \operatorname{ckr} i$ as follows. If N is a coherent A_S-module then by Lemma 3.1 there is a coherent A-module M with $M_S \approx N$. If L is another such module Lemma 9.1 shows we can multiply the isomorphism $L_S \approx M_S$ by an element of S so that it lifts to a map $g : L \to M$. The kernel and cokernel of g are in $\mathcal{N}$ so $[L] = [M]$ in $\operatorname{ckr} i = G_0(A)/\operatorname{im} K_0(\mathcal{N})$. Define $f : G_0(A_S) \to \operatorname{ckr} i$ by $f([N]) = [M]$. If $0 \to N' \xrightarrow{p} N \to N'' \to 0$ multiply p by an element of S and lift it to a map $q : M' \to M$. We get $0 \to K \to M' \xrightarrow{q} M \to M'' \to 0$. Localizing shows $M''_S \approx N''$ and K lies in $\mathcal{N}$ so $[M'] + [M''] \equiv [M] \mod K_0(\mathcal{N})$ showing that our map is well defined. The two maps between $G_0(A)/\operatorname{im} K_0(\mathcal{N})$ and $G_0(A_S)$ are easily seen to be inverses, proving the lemma. □

Apply this to $A = R[t]$ and $A_S = R[t, t^{-1}]$ with $S = \{t^n | n \geq 0\}$. We have to show that $N_S = 0$ implies $[N] = 0$ in $G_0(R[t])$. Since N is finitely generated, $t^n N = 0$ for some n. Filtering N by coherent submodules $N \supseteq tN \supseteq t^2 N \supseteq \cdots \supseteq t^n N = 0$ shows that it will suffice to consider the case where $tN = 0$. By Lemma 3.5 N is coherent as an R-module so the characteristic sequence Theorem 4.2

$$0 \to R[t] \otimes_R N \to R[t] \otimes_R N \to N \to 0$$

shows that $[N] = 0$ in $G_0(R[t])$ so $G_0(R[t]) \to G_0(R[t, t^{-1}])$ is an isomorphism. □

8. Graded Rings

The next three sections are devoted to the proof of the analogues of the above results for the functors G_i. We follow Quillen [9] closely (except for writing $G_i(R)$ instead of $K'_i(R)$) and begin by looking at the case of graded polynomial rings in preparation for the treatment of G_i in the final section.

Let $B = \bigoplus_{n=0}^{\infty} B_n$ be a graded ring and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a graded B–module. I will use $\overline{M}$ to denote M as an ungraded module. For the present purposes I will call M coherent if $\overline{M}$ is a coherent module and write $\mathcal{C}ohgr(R)$ for the category of coherent graded R–modules and degree preserving morphisms.

If $M = \bigoplus_{n=0}^{\infty} M_n$ we let $M(-p)$ be M with a new grading $M(-p)_n = M_{n-p}$. We have $\operatorname{Hom}_B(B(-p), M) = M_p$. By a free graded module I will mean a direct sum $\bigoplus_i B(-p_i)$ where $p_i \geq 0$.

Lemma 8.1. *Let M be a graded module over a graded ring.*

1. *M is finitely generated if and only if $\overline{M}$ is finitely generated.*
2. *M is finitely presented if and only if $\overline{M}$ is finitely presented.*

Proof. The "only if" statements are clear. If $a_1, \dots, a_n$ generate $\overline{M}$ let $a_i = \sum_j a_{ij}$ where $a_{ij} \in M_j$. The a_{ij} are then homogeneous generators of M. If M is finitely presented it is finitely generated so we can map a finitely generated free module F onto M getting an exact sequence $0 \to N \to F \to M \to 0$. After dropping the grading we see that $\overline{N}$ is finitely generated and therefore so is N. □

Let R be a coherent ring and let $B = R[x_1, \dots, x_r]$ with $r < \infty$. Grade B by $\deg x_i = 1$ and assume that B is coherent and therefore so is R by Corollary 3.6 or Lemma 3.9. Our aim is to compute $G_i(B) = K_i(\mathcal{C}ohgr(B))$. Define exact functors $b_p : \mathcal{C}oh(R) \to \mathcal{C}ohgr(B)$ by $b_p(M) = B(-p) \otimes_R M$. They are exact since B is free over R. They are graded using the grading of $B(-p)$ so that $b_p(M)_n = B(-p)_n \otimes_R M$. Since $M \in \mathcal{C}oh(R)$ is finitely presented by Corollary 2.7 so is $b_p(M)$ which is therefore coherent so we get a map $\beta_p : G_i(R) \to G_i(B)$.

Theorem 8.2. *Let $B = R[x_1, \dots, x_r]$, with $r < \infty$ be a polynomial ring over a ring R graded by $\deg x_i = 1$ for all i. If B is coherent then $\beta = \bigoplus_{p \geq 0} \beta_p : \bigoplus_{p \geq 0} G_i(R) \to G_i(B)$ is an isomorphism.*

Proof. We define an inverse mapping $\gamma = \bigoplus_{p \geq 0} \gamma_p : G_i(B) \to \bigoplus_{p \geq 0} G_i(R)$ as follows. If $N \in \mathcal{C}ohgr(B)$ let $Q(N) = R \otimes_B N = N/B^+N$ where $B^+ = \bigoplus_{n>0} B_n$ is the kernel of the retraction $B \to R$. We have $Q(N) = \bigoplus Q_n(N)$ with $Q_n(N) = N_n/D_n(N)$ where $D_n(N) = (B^+N)_n = \bigoplus_{i=1}^n B_i N_{n-i}$, the decomposable elements of N_n. Since N is coherent it is finitely presented and therefore so is $Q(N)$ which, as well as its summands $Q_n(N)$, is therefore coherent. Q is right exact but not exact in general. The Tor sequence for $0 \to N' \to N \to N'' \to 0$ is $\cdots \to \mathrm{Tor}_1^B(R, N'') \to Q(N') \to Q(N) \to Q(N'') \to 0$. Therefore Q will be exact on the full subcategory $\mathcal{N}$ of $\mathcal{C}ohgr(B)$ of objects N such that $\mathrm{Tor}_i^B(R, N) = 0$ for all $i > 0$. Since R has Tor-dimension $r < \infty$ over B the resolution theorem [9, §4,Cor. 3] shows that $K_i(\mathcal{N}) = K_i(\mathcal{C}ohgr(B))$. Define $\gamma_p : K_i(\mathcal{N}) \to K_i(\mathcal{C}oh(R))$ to be the map induced by the functor $Q_p : \mathcal{N} \to \mathcal{C}oh(R)$.

For $N \in \mathcal{N}$ let F_pN be the submodule of N generated by all N_i with $i \leq p$ and let $F_p\mathcal{N}$ be the full subcategory of the F_pN for all $N \in \mathcal{N}$.

Lemma 8.3. *$F_p\mathcal{N}$ is closed under extensions and so is an exact category. Moreover $\mathcal{N}$ is the union of these subcategories.*

Proof. If $0 \to N' \to N \to N'' \to 0$ is exact and N' and N'' are in $F_p\mathcal{N}$ then N' and N are generated by homogeneous elements $\{x_i'\}$ and $\{y_j''\}$ of degree at most p. Lift the y_j'' to elements y_j of the same degree in N. Then $\{x_i'\}$ and $\{y_i\}$ generate N so $F_pN = N$. Any N in $\mathcal{N}$ is finitely generated and so lies in $F_p\mathcal{N}$ when p is greater than the degrees of its generators. □

Since Q_n is 0 on $F_p\mathcal{N}$ for $n > p$, the same is true of $\gamma_n : K_i(F_p\mathcal{N}) \to G_i(R)$ so we can define a map $\gamma = \bigoplus_0^\infty \gamma_n : K_i(F_p\mathcal{N}) \to \bigoplus_0^\infty G_i(R)$. Taking the limit as $p \to \infty$ we get the required map $\gamma = \bigoplus_0^\infty \gamma_n : K_i(\mathcal{N}) \to \bigoplus_0^\infty G_i(R)$.

We can also replace $\mathcal{C}ohgr(B)$ by $\mathcal{N}$ in discussing the maps β_p since $b_p(M)$ lies in $\mathcal{N}$ by the following lemma.

Lemma 8.4. [9, §7, Proof of Lemma 1] *If R is a subring of B, B is flat over R, and X is any R–module then* $\operatorname{Tor}_i^B(R, B\otimes_R X) = 0$ *for all* $i > 0$

Proof. Let $T_i(X) = \operatorname{Tor}_i^B(R, B\otimes_R X)$. This is an exact ∂–functor because B is flat over R. It is effaceable since $B\otimes_R X$ is projective over B if X is projective over R. Thefore the T_i are the derived functors of T_0 but $T_0(X) = R\otimes_B B\otimes_R X = X$ which is exact so its higher derived functors are 0. □

We now have the required maps.

$$\bigoplus_0^\infty G_i(R) \xrightarrow{\beta} K_i(\mathcal{N}) \xrightarrow{\gamma} \bigoplus_0^\infty G_i(R)$$

The composition is induced by the functor taking M in $\mathcal{C}oh(R)$ to $Q_n\circ b_p(M) = Q_n(B(-p)\otimes_R M)$ which is the degree n part of $R\otimes_B B(-p)\otimes_R M = M$ graded by assigning the degree p to all elements. This is M if $n = p$ and otherwise 0. This shows that $\gamma\circ\beta$ is the identity.

The composition $\beta\circ\gamma$ is the sum of the compositions $\beta_p\circ\gamma_p$. where $\beta_p\circ\gamma_p$ is induced by $b_p\circ Q_p$. The next lemma shows that the functor $\beta_p\circ\gamma_p$ is isomorphic to the functor $N\mapsto F_pN/F_{p-1}N$. It follows that the functors F_p are exact on $\mathcal{N}$. If we replace $\mathcal{N}$ with $F_p\mathcal{N}$ we have a finite filtration and can apply the theorem on characteristic filtrations [9, §3 Cor.3] to conclude that the endomorphisms of K_i induced by the functors $N\mapsto F_pN/F_{p-1}N$ sum to the identity showing that $\beta\circ\gamma$ is the identity. Taking the limit as $p\to\infty$ then shows that this is true for $\mathcal{N}$, proving Theorem 8.2. □

The following lemma makes no use of coherence. The definitions of F_pN, $Q(N) = \bigoplus_n Q_n(N)$, and $Q_p(N) = N_p/D_p(N)$ are the same as above.

Lemma 8.5. [9, §7, Lemma 1] *Let $B = \bigoplus_{n=0}^\infty B_n$ be a graded ring with $B_0 = R$ and let N be a finitely generated graded B-module. If* $\operatorname{Tor}_1^B(R, N) = 0$ *then there is an isomorphism* $\theta_p : B(-p)\otimes_R Q_p(N) \xrightarrow{\approx} F_pN/F_{p-1}N$.

Proof. Since $N_p\subseteq F_pN$ and $D_p(N)\subseteq F_{p-1}N$ there is a map $Q_p(N) = N_p/D_p(N)\to F_pN/F_{p-1}N$. The right hand side is a B-module so this extends to give us our map θ_p. Since F_pN is generated by $F_{p-1}N$ and N_p, θ_p is onto.

Remark 8.6. To be consistent with the previous notation we regard $Q_p(N)$ as an ungraded module and write $B(-p)\otimes_R Q_p(N)$ instead of $B\otimes_R Q_p(N)$.

We use the following facts.

(1) $Q(N) = 0$ implies $N = 0$.
(2) θ_p is onto.
(3) $Q(F_{p-1}N)\rightarrowtail Q(F_pN)$ is injective.
(4) $Q(\theta_p)$ is an isomorphism.

For (1), if $N_n = 0$ for all $n\le m$ then $D_m(N) = 0$ so $N_m = Q_m(N) = 0$. (2) was proved above. For (3) and (4) we observe that $Q_n(F_pN) = 0$ for $n > p$ while $Q_n(F_pN) = Q_n(N)$ for $n\le p$. This implies (3). For (4) observe that if M is a graded module generated by M_p then $Q(M) = M_p$ because $B^+M = \sum_{n>p} M_n$. This condition is satisfied by $B(-p)\otimes_R Q_p(N)$ which has $(B(-p)\otimes_R Q_p(N))_p = Q_p(N)$ and by $F_pN/F_{p-1}N$ which has $(F_pN/F_{p-1}N)_p = N_p/D_p(N) = Q_p(N)$. $Q(\theta)$ corresponds to the identity map $Q_p(N)\to Q_p(N)$, the map used to define θ.

Define $T_i(N) = \operatorname{Tor}_i^B(R, N)$, an exact ∂-functor with $T_0 = Q$. We have $T_1(N) = 0$ by the hypothesis. For large p $F_pN = N$ because N is finitely generated. Assuming $T_1(F_pN) = 0$, we will show that $T_1(F_{p-1}N) = 0$, proving that this is true for all p. At the same time we show that $K^p = \ker\theta_p = 0$ so θ_p is an isomorphism. The exact sequence $0 \to K^p \to B(-p) \otimes_R Q_p(N) \to F_pN/F_{p-1}N \to 0$ gives us an exact sequence

$$0 \to T_1(F_pN/F_{p-1}N) \to Q(K^p) \to Q(B(-p) \otimes_R Q_p(N)) \xrightarrow[\approx]{Q(\theta)} Q(F_pN/F_{p-1}N)$$

Note that $T_1(B(-p) \otimes_R Q_p(N)) = 0$ by Lemma 8.4. The map on the right is an isomorphism by (4). This shows that $T_1(F_pN/F_{p-1}N) = Q(K^p)$.

Assume now that $T_1(F_pN) = 0$. The exact T_1-sequence for $0 \to F_{p-1}N \to F_pN \to F_pN/F_{p-1}N \to 0$ is

$$0 = T_1(F_pN) \to T_1(F_pN/F_{p-1}N) \to Q(F_{p-1}N) \rightarrowtail Q(F_pN)$$

The map on the right is injective by (3). This shows that $T_1(F_pN/F_{p-1}N) = Q(K^p)$ is 0 so $K^p = 0$ by (1). From the same sequence we get $T_2(F_pN/F_{p-1}N) \to T_1(F_{p-1}N) \to T_1(F_pN) = 0$. Since $K^p = 0$, θ_p is an isomorphism so $T_2(F_pN/F_{p-1}N) = T_2(B(-p) \otimes_R Q_p(N)) = 0$ by Lemma! 8.4. It follows that $T_1(F_{p-1}N) = 0$ completing the induction. □

9. Localization

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring and let S be a central multiplicative subset of A consisting of homogeneous elements. Then A_S is a graded ring with $\deg x/s = \deg x - \deg s$. We allow elements of negative degree here.

Lemma 9.1. *Let S be a central multiplicative subset of a ring A and let M and N be A–modules with M finitely presented. If $\gamma : M_S \to N_S$ is an A_S–homomorphism there is an A–homomorphism $g : M \to N$ and an element s of S with $g_S = s\gamma$.*

Proof. Suppose $M = F$ is free and finitely generated by $e_1, \dots, e_n$. Let $\gamma(e_k) = x_k/s$ with s in S. Then $g(e_k) = x_k$ is the required map. Let $F' \xrightarrow{i} F \xrightarrow{j} M \to 0$ be a finite presentation of M. By the previous remark we can find $h : F \to N$ such that $h_S = s\gamma \circ j_S$. Now $h \circ i$ localizes to 0 since $j_S \circ i_S = 0$ so the image of $h \circ i$, being finitely generated, is annihilated by some $t \in S$. Therefore $th \circ i = 0$ so th factors through the cokernel M of i giving us the required map g. We have $g \circ j = th$ so $g_S \circ j_S = th_S = ts\gamma \circ j_S$ and therefore $g_S = th_S = ts\gamma$ since j_S is an epimorphism. □

If $F : \mathcal{A} \to \mathcal{B}$ is an exact covariant functor of abelian categories then the full subcategory $\mathcal{S}$ of objects A of $\mathcal{A}$ with $F(A) = 0$ is a Serre subcategory and we have an exact functor $\mathcal{A}/\mathcal{S} \to \mathcal{B}$. If this is an equivalence of categories we get an exact localization sequence

$$\cdots \to K_i(\mathcal{S}) \to K_i(\mathcal{A}) \to K_i(\mathcal{B}) \to K_{i-1}(\mathcal{S}) \to \dots$$

by the Localization Theorem [9, §5 Th. 5].

Theorem 9.2 ([11],Theorem 5.11)**.** *In this situation if the following two conditions are satisfied then $\mathcal{A}/\mathcal{S} \to \mathcal{B}$ is an equivalence of categories..*

(1) *For each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ with $F(A) \approx B$.*

(2) *If $A, A' \in \mathcal{A}$ and $f : F(A) \to F(A')$ in $\mathcal{B}$ there is an object A'' and maps $A \overset{h}{\leftarrow} A'' \overset{g}{\to} A'$ such that $F(h)$ is an isomorphism and $f = F(g)F(h)^{-1}$*

Corollary 9.3. *Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring and let S be a central multiplicative subset of A consisting of homogeneous elements. Let $\mathcal{S}$ be the full subcategory of $\mathcal{C}ohgr(A)$ of all modules M with $M_S = 0$. Then $\mathcal{C}ohgr(A)/\mathcal{S} \approx \mathcal{C}ohgr(A_S)$.*

Proof. We have to check the two conditions of Theorem 9.2 with $\mathcal{A} = \mathcal{C}ohgr(A)$ and $\mathcal{B} = \mathcal{C}ohgr(A_S)$.

(1) Any $N \in \mathcal{B}$ is finitely presented so we can write $F'_S \overset{\gamma}{\to} F_S \to N \to 0$ with F_S and F'_S free and finitely generated over A_S. By Lemma 9.1 we can write $s\gamma = g_S$ and $N = \operatorname{ckr}\gamma = \operatorname{ckr} g_S$ since s is a unit in A_S. Therefore $N = M_S$ where $M = \operatorname{ckr} g$.

(2) Suppose $\gamma : M_S \to N_S$ in $\mathcal{C}ohgr(A_S)$. By Lemma 9.1 write $s\gamma = g_S$ where $g : M \to N$. Then we have $M \overset{s}{\leftarrow} M(-d) \overset{g}{\to} N$ and s localizes to an isomorphism as required. Note that $M(-d) = M$ as a module but we have changed the grading by $d = \deg s = \deg g$ to make the two maps have degree 0.

□

In the next lemma we allow A and its modules to have elements of negative degree.

Lemma 9.4. *If A is a coherant graded ring having a central unit s in A_1 then A_0 is also coherent, $\mathcal{C}ohgr(A) \approx \mathcal{C}oh(A_0)$, and $G_i(A_0) \to G_i(A)$ induced by $M \mapsto A \otimes_{A_0} M$ is an isomorphism.*

Proof. Since s is a unit $s^n : A_0 \approx A_n$ so $A = \bigoplus A_0 s^n$ showing that A is a Laurent polynomial ring $A_0[s, s^{-1}]$ and therefore A_0 is coherent by Lemma 3.9. Similarly if N is a graded A-module $s^n : N_0 \approx N_n$ so $N = N_0[s, s^{-1}]$ Define $f : \mathcal{C}oh(A_0) \to \mathcal{C}ohgr(A)$ by $f(M) = A \otimes_{A_0} M$ and $g : \mathcal{C}ohgr(A) \to \mathcal{C}oh(A_0)$ by $g(N) = N_0$. These maps are inverse equivalences of categories. □

10. G_i

Theorem 10.1. *If R is a ring such that the polynomial ring $R[x, y]$ is coherent then $G_i(R) \to G_i(R[x])$, induced by the functor $M \mapsto R[x] \otimes_R M$, is an isomorphism.*

Proof. We can assume that $i > 0$ because of Theorem 7.1. Let $A = R[t, s]$ and $B = R[t]$ be polynomial rings graded by $\deg t = \deg s = 1$. The localization sequence [9, §5 Th. 5] for $A \to A_s$ is $\cdots \to G_i(\mathcal{N}) \to G_i(A) \to G_i(A_s) \to G_{I-1}(\mathcal{N}) \dots$ where $\mathcal{N}$ is the Serre subcategory of A–modules M such that $M_s = 0$. By the Devissage Theorem [9, §5 Th.4] $G_i(\mathcal{N}) \approx G_i(A/(s)) = G_i(B)$.

By Lemma 9.4 $G_i(A_s) = G_i((A_s)_0)$. Since $A_s = R[t, s, s^{-1}]$ we see that $(A_s)_0 = R[z]$ where $z = t/s$.Therefore the localization sequence takes the form

$$\cdots \to G_i(B) \xrightarrow{j_*} G_i(A) \to G_i(R[z]) \to G_{i-1}(B) \to \dots$$

Here A and B are graded rings and the map $j_* : G_i(B) \to G_i(A)$ is induced by the inclusion of $\mathcal{C}ohgr(B)$ in $\mathcal{C}ohgr(A)$ while the map $G_i(A) \to G_i(R[z])$ is induced by $N \mapsto (N_s)_0$.

By Theorem 8.2 we have an isomorphism $\alpha = \bigoplus_{p\geq 0} \alpha_p : \bigoplus_{p\geq 0} G_i(R) \to G_i(A)$ induced by $a_p : M \mapsto A(-p)\otimes_R M$. and similarly $\beta = \bigoplus_{p\geq 0} \beta_p : \bigoplus_{p\geq 0} G_i(R) \to G_i(B)$ given by $b_p(M) \mapsto B(-p)\otimes_R M$. We have a diagram

$$\begin{array}{ccc} \bigoplus_{p\geq 0} G_i(R) & \xrightarrow{f} & \bigoplus_{p\geq 0} G_i(R) \\ \approx\big\downarrow\beta & & \approx\big\downarrow\alpha \\ G_i(B) & \xrightarrow{j_*} & G_i(A) \end{array}$$

The exact sequence $0 \to A(-p-1) \xrightarrow{s} A(-p) \to B(-p) \to 0$ tensored with M gives us an exact sequence of functors $0 \to a_{p+1} \to a_p \to jb_p \to 0$. By the characteristic filtration theorem [9, §3 Cor.3] this implies that $j_*\beta_p = \alpha_p - \alpha_{p+1}$. If $x = (x_p) \in \bigoplus_{p\geq 0} G_i(R)$ then $j_*\beta(x) = \sum_{p\geq 0} j_*\beta_p(x_p) = \sum_{p\geq 0} \alpha_p(x_p) - \sum_{p\geq 0} \alpha_{p+1}(x_p)$ so $j_*\beta(x) = \sum_{p\geq 0} \alpha_p(x_p - x_{p-1}) = \alpha(y)$ with $y_p = x_p - x_{p-1}$ where we set $x_p = 0$ when $p < 0$.

The map $G_i(B) \to G_i(A)$ is therefore isomorphic to the map $f : \bigoplus_{p\geq 0} G_i(R) \to \bigoplus_{p\geq 0} G_i(R)$ given by $(x_p) \mapsto (y_p)$ where $y_p = x_p - x_{p-1}$. We can recover the x_p from the y_p by $x_p = y_0 + y_1 + \cdots + y_p$ so the map is injective. The image is the set of (y_p) for which $\sum_p y_p = 0$ so the cokernel is $G_i(R)$ via the map $\bigoplus_p G_i(R) \to G_i(R)$ sending (y_p) to $\sum_p y_p$. The map $G_i(R) \xrightarrow{\alpha_0} G_i(A) \to \operatorname{ckr} j_* \approx G_i(R[z])$ is therefore an isomorphism. It is induced by the functor sending an R-module M to $(A\otimes_R M)_s)_0 = (A_s)_0 \otimes_R M = R[z]\otimes_R M$ as required. □

Corollary 10.2. *If R is a ring such that the polynomial ring $R[x, y]$ is coherent then $G_i(R[x, x^{-1}]) = G_i(R) \oplus G_{i-1}(R)$ for $i > 0$.*

Proof. Observe that in the localization sequence

$$\cdots \to G_i(R) \to G_i(R[x]) \to G_i(R[x, x^{-1}]) \to G_{i-1}(R) \to G_{i-1}(R[x]) \to \dots$$

the map $G_i(R) \to G_i(R[x])$ is 0 because of the exact characteristic sequence $0 \to M[x] \to M[x] \to M \to 0$ (Theorem 4.2) and the theorem on characteristic filtrations [9, §3 Cor.3]. □

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