CORRECTION TO: VECTOR BUNDLES AND PROJECTIVE MODULES

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Abstract. This corrects an error in the statement of Theorem 6 of my paper Vector Bundles and Projective Modules

Example 4 of [5] concerns the following question: If a projective module becomes free after extending the ground field (say from \( \mathbb{R} \) to \( \mathbb{C} \)) is it stably free? The example considered was the coordinate ring of the \( n \)-sphere \( A_n = \mathbb{R}[x_0, \ldots, x_n]/(\sum_0^n x_i^2 - 1) \) and Theorem 6 claimed that there was an example over this ring for \( n = 4 \). This is incorrect. The error is in the assertion that \( \mathbb{C} \otimes A \approx P \oplus P' \) in line -5 on page 276. I had originally worked out this example for the case of the 2-sphere mentioned in lines 5 and 6 on page 277. In this case, the argument is correct. If \( P \) is a \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \)-module, then \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \approx P \oplus P' \) where \( P' \) is the conjugate module defined by letting \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \) act on \( P \) via the automorphism \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \rightarrow \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \) sending \( z \otimes a \) to \( \bar{z} \otimes a \) so the assertion \( \mathbb{C} \otimes A \approx P \oplus P' \) is correct in this case. When writing this up, it struck me that the argument could be extended to the case of the 4-sphere by replacing \( \mathbb{C} \) by the quaternions \( \mathbb{H} \), but I failed to check the details with sufficient care and the argument runs afoot of the non-commutativity of the quaternions. Here \( P \) is an \( \mathbb{H} \otimes_{\text{mathbb{R}}} \Lambda \)-module. To find \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \) we can regard \( P \) as a \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \)-module via the inclusion \( \mathbb{C} \otimes_{\text{mathbb{R}}} \Lambda \subset \mathbb{H} \otimes_{\text{mathbb{R}}} \Lambda \). The automorphism sending \( z \otimes a \) to \( \bar{z} \otimes a \) extends to an automorphism of \( \mathbb{H} \otimes_{\text{mathbb{R}}} \Lambda \) sending \( q \otimes a \) to \( \bar{q} \otimes a \) but this sends \( j \) to \( j \) not \( -j \) and so does not agree with the definition of \( P' \) in [5]. If, as in [5], we try the map sending \( q \otimes a \) to \( \bar{q} \otimes a \), we see that this is an anti-automorphism making the resulting module \( P' \) a right module so again this does not produce the module \( P' \) used in [5] and the statement that \( P' \) is the conjugate module of \( P \) on page 276, line -9 is incorrect.

The correct version of the theorem can be easily deduced from the results of [6]. The question can be reformulated in K-theoretic terms as follows: Let \( A \) be an \( \mathbb{R} \)-algebra. The base change map \( \text{bch} : K_0(A) \rightarrow K_0(\mathbb{C} \otimes_{\mathbb{R}} A) \) is defined by sending \([M]\) to \( \mathbb{C} \otimes_{\mathbb{R}} M \). The question then is whether this map is injective. Here is the correct answer for \( A = A_n \).

Theorem 0.1. The base change map \( K_0(A_n) \rightarrow K_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) \) is not injective if and only if \( n \equiv 1, 2 \pmod{8} \).

Proof. The restriction map \( \text{res} : K_0(\mathbb{C} \otimes_{\mathbb{R}} A) \rightarrow K_0(A) \) is defined by sending \([M]\) to \([M]\) i.e. we forget the complex structure. The composition \( \text{bch} \circ \text{res} \) is multiplication by 2 since \( \mathbb{C} \otimes_{\mathbb{R}} M \) is isomorphic to \( M \oplus M \) as an \( A \)-module. Therefore, the kernel of \( \text{bch} \) is annihilated by 2. It is well-known that \( K_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) \) is 0 for \( n \) odd and \( \mathbb{Z} \) for \( n \) even [2, 3, 4] or [7, Theorem 10.2]. Therefore the kernel of \( \text{bch} \) for \( A = A_n \) is equal to the torsion submodule of \( K_0(A_n) \). Since \( K_0(A_n) = \mathbb{Z} \oplus \tilde{K}_0(A_n) \) this
is also the torsion submodule of $\tilde{K}_0(A_n)$. By [6], $\tilde{K}_0(A_n)$ is the same as $\tilde{K}^0(S^n)$ which is periodic with period 8, the first 8 values, beginning with $n = 1$ being $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ [1, Table 2]. Therefore $K_0(A_n)$ has non–trivial torsion if and only if $n \equiv 1, 2 \pmod{8}$.

We can also determine the maps $\text{bch}$ and $\text{res}$ explicitly for $A_n$. The summands of $K_0(A_n) = \mathbb{Z} \oplus \tilde{K}_0(A_n)$ are clearly stable under these maps and on the summand $\mathbb{Z}$ $\text{bch} = 1$ and $\text{res} = 2$. On the other summand the maps are as follows.

**Theorem 0.2.** The maps $\tilde{K}_0(A_n) \xrightarrow{\text{bch}} K_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) \xrightarrow{\text{res}} \tilde{K}_0(A_n)$ are 0 if $n$ is odd or $n \equiv 6 \pmod{8}$. Otherwise they are as follows for appropriate choices of generators.

(1) $\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ if $n \equiv 2 \pmod{8}$.

(2) $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}$ if $n \equiv 4 \pmod{8}$.

(3) $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ if $n \equiv 0 \pmod{8}$.

**Proof.** If $n$ is odd, $\tilde{K}_0(\mathbb{C} \otimes_{\mathbb{R}} A_n) = 0$ while $\tilde{K}_0(A_n) = 0$ if $n \equiv 6 \pmod{8}$. Let $q_n = \sum x_i^2$ as a quadratic form over $\mathbb{R}$ and let $C(q_n)$ be the Clifford algebra of $q_n$. The group $\text{ABS}(q_n)$ is defined to be the cokernel of $C(q_n, 1, 1) \rightarrow C(q_n)$. This is the same as the group $A_n$ of [1] as noted in [7, page 457]. It was shown in [6] that the map $\text{ABS}(q_n) \rightarrow \tilde{K}_0(A_n)$ is an isomorphism. The same is true for the complex analogue $\text{ABS}_\mathbb{C}(q_n) \rightarrow \tilde{K}_0(\mathbb{C} \otimes_{\mathbb{R}} A_n)$ as was observed much earlier by Fossum [3]. These maps clearly commute with $\text{bch}$ and $\text{res}$ so it is enough to prove the theorem for the maps $\text{ABS}(q_n) \xrightarrow{\text{bch}} \text{ABS}_\mathbb{C}(q_n) \xrightarrow{\text{res}} \text{ABS}(q_n)$. By [7, 7] $\text{ABS}(q)$ is generated by any simple $C(q)$–module. From [1, Table 1] where $C(q_n)$ is denoted $C_n^{\ast+1}$ we see that for $n \equiv 2 \pmod{8}$ we have $C(q_n) = M_m(\mathbb{C})$ an $m \times m$–matrix algebra over $\mathbb{C}$ for an appropriate $m$ and $C(q_n) = M_m(\mathbb{C}) \times M_m(\mathbb{C})$. Therefore a simple $C(q_n)$–module restricts to a simple $C(q_n)$–module showing that the map $\text{res}$ is onto. If $n \equiv 4 \pmod{8}$, then $C(q_n) = M_m(\mathbb{H}) \times M_m(\mathbb{H})$ and $C(q_n) = M_{2m}(\mathbb{C}) \times M_{2m}(\mathbb{C})$. Here a simple $C(q_n)$–module again restricts to a simple $C(q_n)$–module as one sees by comparing dimensions so res = 1. If $n \equiv 0 \pmod{8}$, then $C(q_n) = M_m(\mathbb{R}) \times M_m(\mathbb{R})$ and $C(q_n) = M_m(\mathbb{C}) \times M_m(\mathbb{C})$. Here a simple $C(q_n)$–module extends to a simple $C(q_n)$–module so $\text{bch} = 1$.

**References**