(1) Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \Omega \to \mathbb{R}^n$ be C^∞. Assume that the linear transformation $f'(x)$ is an isomorphism for all $x \in \Omega$. Consider the vector fields v_i on Ω given by $v_i(x) = f'(x)^{-1}e_i$ (with $e_i, 1 \leq i \leq n$ denoting the basis vectors of $\mathbb{R}^n$). Let ϕ_t^i denote the flows given by the vector fields v_i. Prove that

$$f(\phi_{t_1}^1 \phi_{t_2}^2 ... \phi_{t_n}^n(x)) = f(x) + (t_1, t_2, ..., t_n)$$

In other words, the existence of solutions of ODE with parameters implies the most important assertion (open-ness) of the Inverse Function Theorem (but with the stronger differentiability hypthesis C^2).

(2) A one-form θ on a manifold M such that $\theta(p)$ is non-zero at every point $p \in M$ defines a sub-bundle S of the tangent-bundle TM given by $S(p) = \ker(\theta(p)) : T_pM \to \mathbb{R}$ for all $p \in M$.
   (a) Prove that $d\theta(X, Y) = -\theta[X, Y]$ for all sections X, Y of the bundle S.
   (b) Let $M = \mathbb{R}^{2n+1}$ and

$$\theta = x_1 dx_2 + x_3 dx_4 + ... + x_{2n-1} dx_{2n} + dx_{2n+1}$$

   Show that if $N \subset M$ is a locally closed submanifold such that $T_xN \subset S(x)$ for every point $x \in N$ then $\dim(N) \leq n$.

(3) Suppose $S \subset TM$ is a rank two sub-bundle spanned by linearly independent vector fields A and B. Define the vector fields B_k inductively, by setting $B_0 = B$ and $[A, B_k] = B_{k+1}$.
   (a) Assume that the vector fields $A, B, B_1, ..., B_{n-2}$, when evaluated at some point $p \in M$, give a basis of T_pM. Prove that if N is a locally closed connected submanifold such that (i) $p \in N$ and (ii) $S(q) \subset T_qN$ for all $q \in N$, then N is open in M.
   (b) For every $n \geq 2$, give an example where $A, B, ...B_{n-2}$ form a basis of the tangent bundle of M.

(4) It has been shown that if $f : X \to Y$ is a submersion then there is a sub-bundle H of the tangent-bundle TX such that $f'(x)|H(x) : H(x) \to T_{f(x)}Y$ is an isomorphism for every $x \in X$. If in addition H is an involutive sub-bundle, it will be referred to as 'special'.
   (a) Show that each point $x \in X$ has a nbhd $U(x)$ such that a special H exists when $f : X \to Y$ is replaced by $f|_{U(x)} : U(x) \to Y$.
   (b) Show that if
   (i)a special H exists,
   (ii)Y is connected and simply connected,
   (iii) $f : X \to Y$ is proper,
   then there is a diffeomorphism $g : X \to Y \times F$ such that $f = p_1 \circ g$ and $H(x) = \ker(p_2 \circ g)'(x)$ for all $x \in X$.
   (c) Deduce that when f is the Hopf fibration $S^3 \to S^2$ there is no special H.

(5) All objects and maps are C^∞. Let A and B be closed submanifolds of a manifold X that intersect transversally. Let $f : X \to Y$ be a proper map. Assume that $f, f|_A, f|_B, f|_{A\cap B}$ are submersions. Assume that Y is connected. Let $y, z \in Y$. Show there is a diffeomorphism $g : f^{-1}(y) \to f^{-1}(z)$ that restricts to diffeomorphisms

$$A \cap f^{-1}(y) \to A \cap f^{-1}(z) \text{ and } B \cap f^{-1}(y) \to B \cap f^{-1}(z)$$