Notation and assumptions.

1. $\gamma_x(t)$ denotes the maximal integral curve of a vector field $v$ with initial position $\gamma_x(0) = x$. Furthermore $\phi_t(x)$ is defined to be $\gamma_x(t)$.

2. All vector fields below are assumed to be $C^\infty$.

3. Let $v$ be a vector field on a manifold $N$ and let $f : M \to N$ be $C^\infty$. The vector field $f^*v$ on $M$ is defined under the assumption $f'(x) : T_xM \to T_{f(x)}N$ is an isomorphism for all $x \in M$ by the formula $(f^*v)_x = f'(x)^{-1}v(f(x))$ for all $x \in M$.

Homework

1. Let $v$ be a vector field on $\mathbb{R}$. Assume that $v(a) = v(b) = 0$ and $a < b$. Let $a < x < b$.

   (a) Prove that the integral curve $\gamma_x(t)$ is defined for all $t \in \mathbb{R}$.

   (b) Show that $\gamma_x(\mathbb{R})$ is contained in the open interval $(a, b)$.

2. With assumptions and notation as above, prove that $\gamma_x(\mathbb{R}) = (a, b)$ if and only if $\{y \in (a, b) : v(y) = 0\}$ is empty.

3. Let $v(x) = P(x) \frac{dx}{dt}$ where $P$ is a polynomial in one variable.

   (a) Find an example of a $P$ and $x \in \mathbb{R}$ such that the integral curve $\gamma_x(t)$ is not defined for all $t \in \mathbb{R}$.

   (b) Show that if $\deg(P) \leq 1$ the integral curves of $v$ are defined for all $(x, t) \in \mathbb{R} \times \mathbb{R}$.

4. Consider the vector field $v(x) = f(x) \frac{dx}{dt}$ defined for $x \in V$ a nbhd of 0. Assume that $f(0) = 0$ and $f'(0) = c \neq 0$. Show there is a nbhd $U$ of 0 in $\mathbb{R}$ and a diffeomorphism $h$ of $U$ to an open subset of $V$ with $h(0) = 0$ such that $h^*v$ equals $cx \frac{dx}{dt}$.

5. Find the condition a function $f$ defined on an open subset $\Omega \subset \mathbb{R}^3$ must satisfy in order that the vector sub-bundle of the tangent bundle spanned by vector fields $\partial_1$ and $\partial_2 + f \partial_3$ is involutive.

6. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Consider the vector field $v$ on $\mathbb{R}^n$ given by $v(x) = Sx$ for all $x \in \mathbb{R}^n$. Show that $\text{Re}(a) < 0$ (resp. $> 0$) for every eigenvalue of $S$ if and only if $\|\gamma_x(t)\| \to 0$ (resp. $\to \infty$) as $t \to \infty$ for every non-zero $x \in \mathbb{R}^n$.

7. Let $v$ be a vector field on $M$ that vanishes at a given point $p \in M$. There is a nbhd $U(p)$ of $p$ in $M$ and an interval $(-c, c)$ such that $\phi_t : U(p) \to M$ is defined for all $t \in (-c, c)$. Recall that $\phi_t(p) = p$ and thus we obtain the linear transformation $\phi'_t(p) \in \text{End}(T_pM)$. The map $t \mapsto \phi'_t(p)$ is $C^\infty$. Denote its derivative $\frac{d}{dt}\phi'_t(p)|_{t=0}$ by $\rho(v) \in \text{End}(T_pM)$.

   (a) Show that if vector fields $v$ and $w$ both vanish at $p$, then $[v, w]$ also vanishes at $p$.

   (b) For $v, w$ as above, show that $\rho[v, w] = [\rho(v), \rho(w)]$. (One way is to do the next two parts first.)

   (c) When $M$ be an open subset of $\mathbb{R}^n$, the vector field is given by $v : M \to \mathbb{R}^n$. Its derivative at $p \in M$ is a linear transformation $v'(p) : \mathbb{R}^n \to \mathbb{R}^n$. Under the assumption that $v, w$ are vector fields on $M$ that vanish at $p$, show that $[v, w]'(p) = [v'(p), w'(p)]$.

   (d) Notation as in part (c) with $v(p) = 0$. Show that $\rho(v) = v'(p)$.