Notation 0.1. We'll use the phrase: a deformation of C^∞ map $f_0 : X \to Y$ is a C^∞ map $F : X \times S \to Y$ where S is a nbhd of 0 in $\mathbb{R}^m$ and $F(x, 0) = f_0(x)$ for all $x \in X$. We set $f_s(x) = F(x, s)$ for all $x \in X, s \in S$.

Notation 0.2. Given $x \in A \subset M$ where A is C^∞ submanifold of a C^∞ manifold M we often regard T_xA as a linear subspace of T_xM. In reality, if $i : A \to M$ denotes the given inclusion, then i is a C^∞ map and we obtain $i'(x) : T_xA \to T_xM$. The linear transformation $i'(x)$ is one-to-one. Given $v \in T_xA$ we abuse notation by referring to the vector $i'(x)v \in T_xM$ simply as $v \in T_xM$.

In particular, if M is a finite dimensional real vector space V then we regard T_xA as a linear subspace of V.

Definition 0.3. A C^∞ map $f : X \to Y$ is a *submersion* (resp. *immersion*) if $f'(x) : T_xX \to T_{f(x)}Y$ is onto (resp. one-to-one) for all $x \in X$.

A C^∞ map $f : X \to Y$ is an *embedding* if (a) it is an immersion and (b) $x \mapsto f(x)$ gives a homeomorphism of X with a closed subset of Y.

Definition 0.4. A continuous map $f : X \to Y$ is *proper* if

$$K \subset Y, K \text{ compact} \implies f^{-1}(K)\text{compact}.$$

If Y is locally compact and Hausdorff, and if f is proper, then

$$F \subset X \text{ closed} \implies f(F) \text{ closed}$$

The projection $M \times Y \to Y$ is evidently proper if M is compact.

Homework problems

(1) Let $f : X \to Y$ be a C^∞ map of C^∞ manifolds. Define $g : X \times X \to Y \times Y$ by $g(p, q) = (f(p), f(q))$ for all $(p, q) \in X \times X$. Assume that g is transverse to the diagonal ΔY. Prove that f is a submersion. (Remark: this exercise is direct from the definition).

(2) Show that if $f : X \to Y$ is an embedding, then $f(X)$ is a C^∞ submanifold of Y and that f yields a diffeomorphism of X with $f(X)$.

Show that a one-to-one immersion is an embedding when X is compact (and Y is Hausdorff).

(3) *The fiber product of $f : X \to Z$ and $g : Y \to Z$,*denoted $X \times_Z Y$, is given by

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

Assume that X, Y, Z, f, g are all C^∞.

 (a) Define $F : X \times Y \to Z \times Z$ by $F(x, y) = (f(x), g(y))$ for all $(x, y) \in X \times Y$. Prove that if F is transverse to $\Delta(Z)$ (the diagonal of Z in $Z \times Z$) then $X \times_Z Y$ is a C^∞ submanifold of $X \times Y$.

 (b) Prove that if $f : X \to Z$ is a submersion then

  (i) $X \times_Z Y$ is a C^∞ submanifold of $X \times Y$, and

  (ii) the projection $X \times_Z Y \to Y$ is a submersion.

 (c) Is the analogous statement for $f : X \to Z$ an immersion true?

(4) Let V, W be finite dimensional real vector spaces. Let r be an integer. Let X_r be the subset of $\mathrm{Hom}_{\mathbb{R}}(V, W)$ consisting of those linear transformations $T : V \to W$ for which $\mathrm{rank}(T) \le r$.

 (a) Show that X_r is a closed subset of $\mathrm{Hom}_{\mathbb{R}}(V, W)$.

(b) Show that $Y_r = X_r \backslash X_{r-1}$ is a locally closed C^∞ submanifold of $\mathrm{Hom}_{\mathbb{R}}(V, W)$.

(c) Let $T \in Y_r$. Let $i : \ker(T) \to V$ denote the inclusion and let $p : W \to \mathrm{coker}T$ denote the projection, where $\mathrm{coker}T = W/T(V)$.
Show that the tangent-space of Y_r at T is the kernel of the linear transformation $\mathrm{Hom}_{\mathbb{R}}(V, W) \to \mathrm{Hom}_{\mathbb{R}}(\ker(T), \mathrm{coker}(T))$ given by $S \mapsto p \circ S \circ i$.

(5) Let $f : M \to N$ be C^∞. The critical locus of f, denoted by $\mathrm{Crit}(f)$ is $\{x \in M : f'(x) \text{ is not onto}\}$. Show that this a closed subset of M. Deduce that the set R of regular values of f is an open subset of N under the additional assumption that f is *proper*.

(6) Assume that X is compact C^∞. Let $F : X \times S \to Y$ be C^∞ and define $f_s : X \to Y$ by $f_s(x) = F(x, s)$ for all $x \in X, s \in S$. Assume that X is compact. Show that the subset of S consisting of the $\{s \in S : f_s \text{ has property P}\}$ is open in the four cases
P is (a) submersion, (b) immersion, (c) embedding, (d) transverse to A, where A is a closed C^∞ submanifold of Y.
Show that if $f_s : X \to Y$ is transverse to A then

(7) Let (Y, d) be a metric space. Given $f, g : Z \to Y$ we write $f \overset{\delta}{\sim} g$ if $d(f(z), g(z)) < \delta$ for all $z \in Z$.

(a) Assume that Z is compact. Let $\delta > 0$. Prove that if $f, g : Z \to Y$ are homotopic to each other, then there are continuous maps $Z \to Y$ denoted by $f = f_0, f_1, ..., f_k = g$ such that $f_{i-1} \overset{\delta}{\sim} f_i$ for all $i = 1, 2, 3, ...k$.

(b) Let Y be a compact C^∞ submanifold of $\mathbb{R}^n$. The metric on Y is what it inherits from $\mathbb{R}^n$. Prove there is some $\delta(Y) > 0$ such that if $f, g : Z \to Y$ are continuous and $f \overset{\delta(Y)}{\sim} g$ then f is homotopic to g. (Hint: an application of the tubular nbhd thm)
Show furthermore that if the above Z, f, g are C^∞ then the homotopy between f and g is also C^∞.