Introduction

Math 318 is one of the nine courses offered for first-year mathematics graduate students at the University of Chicago. It is the second of three courses in the year-long geometry/topology sequence.

These notes are being live-TeXed, though I edit for typos and add diagrams requiring the TikZ package separately. I am using the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to chonoles@math.uchicago.edu.
Lecture 1 (2013-02-15)

Today we’ll talk about a method for reducing nth order differential equations to first order differential equations. Thus, for example, we can express acceleration as a function of position and velocity.

Let $\Omega \subset \mathbb{R}^n$ be open, and let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function (representing acceleration). Then given $(x_0, y_0) \in \Omega \times \mathbb{R}^n$, there is a unique $\gamma : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma(0) = x_0$, $\gamma'(0) = y_0$, and $\gamma''(t) = a(\gamma(t), \gamma'(t))$ for all $t \in (-\epsilon, \epsilon)$.

The trick was to introduce the space $\Omega \times \mathbb{R}^n$. Put $M = \Omega \times \mathbb{R}^n$, and let $\delta(t) = (\gamma(t), \gamma'(t))$. Any vector field on $M$ can be thought of as a function $M \to \mathbb{R}^n \times \mathbb{R}^n$; let $\vec{v} : M \to \mathbb{R}^n \times \mathbb{R}^n$ be the vector field defined by

$$\vec{v}(x, y) = (y, a(x, y)).$$

If $\gamma$ is as above, then we have $\delta(0) = (x_0, y_0) \in M$, and $\delta'(t) = (\gamma'(t), \gamma''(t)) = \vec{v}(\delta(t))$. Thus, curves $\gamma$ satisfying our requirements are precisely integral curves of the vector field $\vec{v}$ on $M$.

**Theorem.** Let $M$ be a $C^\infty$ manifold and let $v$ be $C^\infty$ vector field on $M$. Let $p \in M$. Then there is a neighborhood $U(p)$ of $p$ in $M$, some $a > 0$, and a $C^\infty$ function $\gamma : U(p) \times (-a, a) \to M$ such that for all $x \in U(p)$, $\gamma_x(t)$ is the (unique) integral curve of $v$ such that $\gamma_x(0) = x$, where $\gamma_x(t) := \gamma(x, t)$.

A reference for this theorem is Hurewicz’s *Lectures on Ordinary Differential Equations*.

**Remark 1.** Let $\gamma$ be an integral curve of a $C^\infty$ vector field. Then it is clear that $\gamma$ is $C^\infty$, because we have that

$$\gamma'(t) = v(\gamma(t)),$$

so that if $\gamma$ is $C^k$, then $v \circ \gamma$ is $C^k$, hence $\gamma'$ is $C^k$, so that $\gamma$ is $C^{k+1}$.

**Remark 2.** Assume that $v$ is a $C^1$ vector field. The contraction principle shows that $(x, t) \mapsto \gamma_x(t)$ is continuous in $(x, t)$ and defined on some neighborhood $U(p) \times (-a, a)$, as follows:

WLOG, let $p = 0$, and let $M = \Omega$, an open subset of $\mathbb{R}^n$. Then the space of continuous functions

$$\{ x \in \mathbb{R}^n \mid \|x\| \leq \alpha \} \times [-c, c] \to \{ x \in \mathbb{R}^n \mid \|x\| \leq R \}$$

with the $\| \cdot \|_\infty$ norm is a complete metric space.

**Remark 3.** Assume that $v$ is a $C^2$ vector field on $\Omega$, an open subset of $\mathbb{R}^n$. We want to show that the partial derivatives

$$\frac{\partial}{\partial x_i}(\gamma_x(t))$$

exist. We can simply check that the differential equation defining them is satisfied. Let $v : \Omega \to \mathbb{R}^n$ be a vector field on $\Omega$, and so that for any $x \in \Omega$ we have $v'(x) : \mathbb{R}^n \to \mathbb{R}^n$.

We introduce the (standard) notation $\phi_t(x) = \gamma_x(t)$. Thus, $\phi_t$ is defined at all $t \in (-a, a)$, and $\phi_t$ is a function $\phi_t : U(p) \to \Omega$, so that we have $\phi'_t(x) : \mathbb{R}^n \to \mathbb{R}^n$ for any $x \in U(p)$, where $\phi'_t(x)$ is the derivative of $x \mapsto v(\phi_t(x))$.

Let $\vec{v} : M \to \mathbb{R}^n \times M(h)(\mathbb{R})$ be the vector field on $M = \Omega \times \mathbb{R}^n$ defined by

$$\vec{v}(x, S) = (v(x), v'(x)S).$$

We claim that, for any fixed $t$, the map $x \mapsto (\phi_t(x), \phi'_t(x))$ is an integral curve for $\vec{v}$.
We have that 
\[ \frac{d}{dt} \phi_t(x) = v(\phi_t(x)), \]
so that 
\[ \tilde{v}(\phi_t(x), \phi'_t(x)) = \frac{d}{dt}(\phi_t(x), \phi'_t(x)) = (v(\phi_t(x)), v'(\phi_t(x))\phi'_t(x)). \]

This “reduces” the question to integral curves of $\tilde{v}$ on $\Omega \times M_n(\mathbb{R})$ (\(?\)).

**Corollary.** Let $M$ be a $C^\infty$ manifold, and let $v$ be a $C^\infty$ vector field on $M$. For all $x \in M$, we’ve shown that there is a maximal integral curve $\gamma_x : I_x \to M$ such that $\gamma_x(0) = x$. Let 
\[ D = \{(x,t) \in M \times \mathbb{R} \mid t \in I_x\}, \]
and write $\gamma_x(t) = \gamma(x,t)$ for all $(x,t) \in D$.

1. $D$ is an open subset of $M \times \mathbb{R}$.
2. If $\phi_t(x)$ is defined, then $\phi_s(x)$ is defined for all $0 \leq s \leq t$ (or all $t \leq s \leq 0$).
3. If $\phi_t(x)$ is defined and if $\phi_s(\phi_t(x))$ is defined, then $\phi_{t+s}(x)$ is defined and 
\[ \phi_{t+s}(x) = \phi_s(\phi_t(x)). \]
4. Assume $M$ is compact. Then the theorem implies that there is an open interval $(-a,a)$ such that $(-a,a) \subset I_x$ for all $x \in M$, so that $D \supseteq M \times (-a,a)$; statement 3 then implies $D = M \times \mathbb{R}$. Furthermore, for a fixed $t$, the map $x \mapsto \phi_t(x)$ is $C^\infty$ and induces an isomorphism on tangent spaces.

5. Observe that given a vector field $v$ on $M$, if a constant path $\gamma(t) = p$ is an integral curve of $v$, then $v(p) = 0$. Conversely, if $v(p) = 0$, then the constant curve to $p$ is an integral curve.

**Proof of 3.** We know $\gamma_x$ is defined on $[0,t+\epsilon)$, and that $\gamma_{\phi_t(x)}$ is defined on $[0,s+\epsilon')$. Now define 
\[ \delta(z) = \begin{cases} 
\gamma_x(z) & \text{for all } z \in [0,t], \\
\gamma_{\phi_t(x)}(z-t) & \text{for all } z \in [t,s+\epsilon). 
\end{cases} \]

Note that $\delta$ is an integral curve of $v$. \[ \square \]

**Exercise.** Let $\mathbb{R} \xrightarrow{h} \text{Diffeo}(M)$ be a group homomorphism such that the map $M \times \mathbb{R} \to M$ defined by $(x,t) \mapsto h(t)(x)$ is $C^\infty$. Prove that there is a $C^\infty$ vector field $v$ on $M$ such that $h(t) = \phi_t$ for the vector field $v$. This does not require $M$ to be compact, just that the support of $v$ is compact.

**Remark.** If $v$ is a vector field on a manifold $M$ and $v(p) \neq 0$, then there is a neighborhood $U(p)$ of $p$ in $M$ and a diffeomorphism $f : U(p) \to \Omega$, where $\Omega$ is an open subset of $\mathbb{R}^n$, such that the vector field $v$ (restricted to $U(p)$) is carried by $f$ to the constant vector field $\frac{\partial}{\partial x_1}$ on $\Omega$.

**Proof of Remark.** Let’s say that $Z$ is a slice of $M$ when $T_pZ \oplus \mathbb{R}v(p) = T_pM$. The hypotheses of the inverse function theorem hold at $(p,0)$, so that we can find coordinates in which the map from $Z \times (-\epsilon, \epsilon) \to M$ defined by $\phi_t(z) = \gamma(z)$ sends the vector field $v$ to a constant vector field, such as for example $\frac{\partial}{\partial x_1}$. \[ \square \]

Next time, we’ll start with Lie brackets. It’s motivated from two points of view; maybe 100 in fact.
Lecture 2 (2013-02-18)

Though it’ll seem like we’re leaving integral curves, we’ll return to them in the middle of the lecture.

Recall that given a $C^\infty$ manifold $M$, a point $p \in M$, and a tangent vector $v \in T_p M$, there is an $\mathbb{R}$-linear functional $v : C^\infty(M) \rightarrow \mathbb{R}$, sending a $C^\infty$ function $f : M \rightarrow \mathbb{R}$ to $v(f) \in \mathbb{R}$. It satisfies the Leibniz rule,

$$v(fg) = f(p)v(g) + g(p)v(f).$$

This is a generalization of the notion of directional derivative in Euclidean space.

Now let $v$ be a vector field on $M$. Let $R = C^\infty(M)$. Now we have an $\mathbb{R}$-linear map $v : R \rightarrow R$, defined by $v(f)(p) = v(p)(f)$ for all $p \in M$. For example, if $M = \mathbb{R}^n$ and $v = (a_1, \ldots, a_n) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we have that

$$v(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

For any ring $S$, a function $D : S \rightarrow S$ is a derivation when $D(fg) = D(f) \cdot g + f \cdot D(g)$. Very often, we are given a subring $T \subset S$ contained in the center of $S$, that we require to satisfy $D(t) = 0$ for all $t \in T$. Note that the map $v : R \rightarrow R$ sending $f \mapsto v(f)$ is a derivation, and $v(\text{any constant function}) = 0$ (observe that we can $\mathbb{R}$ is a subring of $R$).

The following is an easy lemma.

**Lemma.** If $D_1, D_2 : R \rightarrow R$ are derivations, then $(D_1 \circ D_2) - (D_2 \circ D_1)$ is also a derivation.

In particular, if $v, w$ are $C^\infty$ vector fields on $M$, $U$ is an open subset of $M$, and $C^\infty(U)$ is the ring of $C^\infty$ functions on $U$, the map $f \mapsto v(w(f)) - w(v(f))$ is a derivation of $C^\infty(U)$. If we fix a point $p \in M$, we can consider neighborhoods $U$ of $p \in M$, and the map

$$f \mapsto (v(w(f)) - w(v(f))(p)$$

induces an $\mathbb{R}$-linear map on germs $C^\infty_{M,p} \rightarrow \mathbb{R}$. Being a derivation, this is equal to $h(p)(f)$ for a unique $h(p) \in T_p M$. It is true (though we won’t check) that $p \mapsto h(p)$ is a $C^\infty$ vector field on $M$, and we define the Lie bracket of $v$ and $w$ to be this $h$. We write $h = [v, w]$. Thus,

$$[v, w](f) = v(w(f)) - w(v(f))$$

for all $C^\infty$ maps $f : U \rightarrow \mathbb{R}$.

**Lemma.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, and let $v, w$ be $C^\infty$ vector fields on $\Omega$. Then

$$[v, w] = D_v w - D_w v,$$

where

$$(D_v w)(x) = \frac{d}{dt} w(x + tv) \bigg|_{t=0}.$$

**Proposition.** The $\mathbb{R}$-vector space of $C^\infty$ vector fields on $M$, together with the bracket, satisfies the axioms of a Lie algebra:

1. $[v, w] = -[w, v]$ for all $C^\infty$ vector fields $v$ and $w$.
2. $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$ for all $C^\infty$ vector fields $v_1, v_2, v_3$.
3. $[tv, w] = t[v, w]$ for all $t \in \mathbb{R}$.
**Definition.** Let $M$ and $N$ be $C^\infty$ manifolds, and let $\phi : M \to N$ be a $C^\infty$ map. Given vector fields $v$ on $M$ and $w$ on $N$, we say that $v$ and $w$ are $\phi$-related if for all $x \in M$,\[
abla \phi(x)v(x) = w(\phi(x)).\]

**Lemma 1.** Given vector fields $v$ on $M$ and $w$ on $N$, they are $\phi$-related if and only if $\phi(\gamma)$ is an integral curve of $w$ for any integral curve $\gamma$ of $v$.

**Proof.** Assume that $v$ and $w$ are $\phi$-related. Let $\gamma : (a, b) \to M$ be an integral curve for $v$, so that for all $t \in (a, b)$, we have\[
abla \gamma(t) = v(\gamma(t)).\]
Let $\delta = \phi \circ \gamma$. Then\[
abla \delta(t) = \nabla \phi(\gamma(t))\nabla \gamma(t) = w(\phi(t)).\]
Everything is reversible, so we are done. \hfill \Box

Last time, I mentioned that if a vector field is non-zero at a point, then in some neighborhood it looks like $\frac{\partial}{\partial x^i}$. There is a proof of this in Warner’s book on page 40.

**Example.** Let $w$ be a vector field on $N$ and suppose that $w(p) \neq 0$. Then there is a chart centered at $p$ such that $w$ is transformed to $\frac{\partial}{\partial x^i}$.

**Proof.** Let $Z$ be a codimension 1 closed submanifold of $N$ containing $p$, and suppose that it is transverse, i.e. that $T_p Z \oplus \mathbb{R}^w(p) = T_p N$. Let $\delta_y(t)$ be an integral curve of $w$ with initial value $y$, i.e. $\delta_y(0) = y$. Let $M = Z \times (-c, c)$, and let $\phi : M \to N$ be the map defined by\[
\phi(z, t) = \delta_z(t).
\]
This is a diffeomorphism in a neighborhood of $Z \times \{0\}$ by the inverse function theorem, and the curves $t \mapsto (z, t)$ on $M$ are sent by $\phi$ to the curves $\delta_z(t)$ on $N$, which are integral curves of $w$. Thus, $t \mapsto (z, t)$ is an integral curve for $\frac{\partial}{\partial x^i}$.

**Lemma 2.** Let $M$ and $N$ be $C^\infty$ manifolds, and let $\phi : M \to N$ be $C^\infty$.

(a) If $v$ on $M$ and $w$ on $N$ are $\phi$-related, then $v(\phi^* f) = \phi^* w(f)$ for any $C^\infty$ map $f : N \to \mathbb{R}$; this is just a restatement of the definition.

(b) If $v_1$ is $\phi$-related to $w_1$ and $v_2$ is $\phi$-related to $w_2$, then $[v_1, v_2]$ and $[w_1, w_2]$ are $\phi$-related.

**Proof of (b).** We have\[
\phi(\phi^*(f)) = \phi^*(\phi^*(w_2(f))) = \phi^*(w_2(f))).\]
Now interchange and subtract. \hfill \Box

**Remark.** This has an important consequence. If $M$ is a locally closed submanifold of $N$, $\phi : M \to N$ is the inclusion, and $w$ is a vector field on $N$, then to say that there is some $v$ on $M$ that is $\phi$-related to $w$ is equivalent to saying that $w(x) \in T_x M$ for all $x \in M$ (because $w(x) = v(x)$). Thus, Lemma 2 is saying something about vector fields that are tangent to submanifolds; if $w_1$ and $w_2$ are vector fields on $N$ such that $w_1(x), w_2(x)$ belong to $T_x M$ for all $x \in M$, then $[w_1, w_2]$ has the same property.

**Definition.** Let $M$ be a $C^\infty$ manifold, and let $W$ be a $C^\infty$ subbundle of $TM$ of rank $r$. A locally closed submanifold $A$ of $M$ is a leaf if for all $x \in A$, $T_x A = W(x)$. 
Suppose that there is a leaf of $W$ through every point of $M$. If $w_1, w_2$ are $C^\infty$ sections of $W$, then $[w_1, w_2]$ is necessarily also a section of $W$; we can see this easily as follows. Let $p \in M$ and let $Z$ be a leaf through $p$. Because $Z$ is a leaf, $w_1$ and $w_2$ are tangential to $Z$, so $[w_1, w_2]$ is tangential to $Z$, i.e. $[w_1, w_2](p) \in T_p Z = W(p)$ for all $p \in M$.

**Definition.** A $C^\infty$ subbundle $W$ of $TM$ is said to be involutive (alternatively, integrable) if for all $C^\infty$ sections $w_1, w_2$ of $W$, $[w_1, w_2]$ is also a section of $W$.

We have already proven one piece of the following theorem:

**Theorem (Frobenius).** Let $W$ be a subbundle of $TM$. The following are equivalent:

1. $W$ is involutive.
2. There is a leaf of $W$ through every point.
3. For all $p \in M$, there is a diffeomorphism $h$ from a neighborhood of $p$ to $U_1 \times U_2$, where $U_i$ is an open subset of $\mathbb{R}^{n_i}$ for $i = 1, 2$, such that $h(W)$ is the constant $\mathbb{R}^{n_1} \times \{0\}$ bundle on $U_1 \times U_2$.

**Proof.** It is clear that $3 \implies 2$, and we have already proven that $2 \implies 1$, so it remains to prove that $1 \implies 3$. This proof is taken from Narasimhan (the proof is originally due to Volterra).

**Step 1.** Let $W$ be an involutive subbundle of rank $r$. Then in a neighborhood of any $p \in M$, we can find vector fields $w_1, \ldots, w_r$ which are a frame for $W$, i.e. $w_1(x), \ldots, w_r(x)$ are a basis for $W(x)$ for all $x$ in the neighborhood, and such that $[w_i, w_j] = 0$ for all $i, j$.

Let me make a linear algebra observation: given a vector space $V = V_1 \oplus V_2$, subspaces $W \subset V$ such that the projection to $V_1$ is an isomorphism, i.e.

$$
\begin{array}{ccc}
V & \xrightarrow{p_1} & V_1 \\
\cong & & \\
W & \xrightarrow{} & V_1
\end{array}
$$

can be identified with graphs of linear transformations $S : V_1 \to V_2$.

Now write $\mathbb{R}^N = V_1 \times V_2$, where $N = \dim(M)$, where $V_1$ and $V_2$ have been chosen such that $p_1|_{W(p)} : W(p) \to V_1$ is an isomorphism ($p$ is the projection $\mathbb{R}^N \to V_1$), so that $W(x) \cong V_1$ for all $x$ in some neighborhood of $p$. Thus, for each $x$, we get $S(x) : V_1 \to V_2$, and

$$
W(x) = \{(v_1, S(x)v_1) \mid v_1 \in V_1\}.
$$

Let $\Omega \subset V_1 \times V_2 = \mathbb{R}^N$ be open. WLOG we have $V_1 = \mathbb{R}^r$, where $e_1, \ldots, e_r$ are the standard basis of $\mathbb{R}^r$. We have $S(x)e_i = u_i(x)$, where $u_i : \Omega \to V_2$ is some $C^\infty$ function. Thus $W(x)$ is the linear space of the $e_i + u_i$. For any $i, j$, we have that $[e_i + u_i, e_j + u_j]$ is a section of $W$, and using the formula

$$
[\alpha, \beta] = D_\alpha \beta - D_\beta \alpha
$$

on Euclidean space, we have that $[e_i + u_i + e_j + u_j]$ is a section of $V_2$ (i.e. a function $\Omega \to V_2$); but it also has to be a section of $W$, so it has to be 0 since $V_2 \cap W(x) = 0$ for all $x \in \Omega$.

We’ll finish the proof of this with Step 2 next time.
Lecture 3 (2013-02-20)

Everything we’re talking about today will be $C^\infty$.

To finish the proof of the Frobenius theorem from last time, it remains to show the following result:

**Lemma 1.** If $w_1, \ldots, w_r$ are linearly independent, commuting vector fields (commuting in the sense that their pairwise Lie brackets are 0), then there is a chart centered at any given point where the $w_i$ are transformed to the coordinate vector fields $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, r$.

**Remark.** Let $v$ and $w$ be vector fields on $M$. Let $\phi_t$ and $\psi_s$ denote the one-parameter groups for $v$ and $w$ respectively (i.e. the flows). Then for all $p \in M$, there is some neighborhood $U(p)$ of $p$ and $(-\epsilon, \epsilon)$ such that $\phi_t(\psi_s(x))$ and $\psi_s(\phi_t(x))$ are defined for all $x \in U(p)$ and $t, s \in (-\epsilon, \epsilon)$.

**Lemma 2.** With notation as above, if $[v, w] = 0$, then $\phi_t(\psi_s(x)) = \psi_s(\phi_t(x))$ for any $x \in U(p)$ and $s, t \in (-\epsilon, \epsilon)$.

**Proof that Lemma 2 $\implies$ Lemma 1.** Let’s assume the result of Lemma 2 in the case that $v(p) \neq 0$. Let $\phi_i$ denote the one-parameter groups with respect to $w_i$ for each $i = 1, \ldots, r$. Let $p \in M$, and select a locally closed $C^\infty$ submanifold $Z \subset M$ with $p \in Z$ such that $T_p Z \oplus \mathbb{R} w_1(p) \oplus \cdots \oplus \mathbb{R} w_r(p) = T_p M$. Note that by assuming this is true at $p$, we can assume this is true in a neighborhood of $p$.

Let $h : (-\epsilon, \epsilon)^r \times (Z \cap U(p)) \to M$ be defined by

$$h(x_1, \ldots, x_r, z) = \phi_{x_1}^1 \phi_{x_2}^2 \cdots \phi_{x_r}^r(z).$$

We see that $h$ induces an isomorphism from the tangent space at $(0, \ldots, 0, z)$ to $T_z M$ for all $z \in Z \cap U(p)$. Note that $h(t, x_2, \ldots, x_r, z)$ is an integral curve for $w_1$, so that $h'(z) \frac{\partial}{\partial x_1} = w(h(z))$ for all $z$ in the domain of $h$ (this is not a $z$ in the sense of “I didn’t get down what was on the board”, but rather “?” itself what was written on the board). This is

$$\phi_{x_2}^2 \phi_{x_1}^1 \cdots,$$

and thus we see that $h'(z) \frac{\partial}{\partial x_1} = w_2(h(z))$, etc. (not sure I understand this part). \qed

**Proof of Lemma 2.** We have that $w_1(p) \neq 0$, so (as we have shown earlier) we can assume WLOG that $w = \frac{\partial}{\partial x_1}$. For any vector $v = \sum a_i \frac{\partial}{\partial x_i}$, we have that

$$[w, v] = \sum \frac{\partial a_i}{\partial x_1} \cdot \frac{\partial}{\partial x_i}.$$

By assumption, this is zero, so the $a_i$’s are (in some neighborhood) functions of $(x_2, \ldots, x_n)$. Because the statement is local, we can assume that we are working on $(-\epsilon, \epsilon) \times \Omega$ for an open subset $\Omega \subset \mathbb{R}^{n-1}$. Let $c \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Let $h_c : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \to (-\epsilon, \epsilon) \times \Omega$ be defined by

$$h_c(x_1, x_2, \ldots) = (x_1 + c, x_2, \ldots).$$

Then $v$ and $v|_{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}$ are $h_c$-related.

Therefore, if $\delta$ is an integral curve of $v$, then $h_c \circ \delta$ is also an integral curve. Let $\phi_t$ denote the one-parameter group associated to $v$. Then we have that

$$h_c \circ \phi_t = \phi_t \circ h_c.$$

But $h_c = \psi_c$ where $\psi_c$ is the one-parameter group associated to $w$. \qed
**Theorem** (Thom’s ambient isotopy lemma). Let \( I = [0, 1] \), let \( A \) and \( B \) be \( C^\infty \) manifolds where \( A \) is compact, and let \( F : A \times I \to B \) be a \( C^\infty \) map. Let \( f_t : A \to B \) be defined by \( f_t(a) = F(a, t) \) for all \( a \in A \) and \( t \in [0, 1] \). If \( f_t \) is an embedding for all \( t \in I \), then there is a \( C^\infty \) map \( G : B \times I \to B \) such that \( g_t \) is a diffeomorphism for all \( t \in I \), and \( f_t = g_t \circ f_0 \) for all \( t \in I \), where \( g_t(b) = G(b, t) \).

Recall that if \( A \) is an arbitrary subset of a \( C^\infty \) manifold \( M \), then given a map \( f : A \to \mathbb{R} \), we say that it is \( C^\infty \) map when there exist open sets \( U_\lambda \subset M \) for all \( \lambda \in \Lambda \) such that \( f|_{A \cap U_\lambda} = f_\lambda|_{A \cap U_\lambda} \) and \( W := \bigcup U_\lambda \) contains \( A \). Then \( \{U_\lambda\}_{\lambda \in \Lambda} \) is an open cover of \( W \), so there is a partition of unity subordinate to this cover. Let \( \varphi : W \to \mathbb{R} \) be subordinate to \( U_\lambda \).

Consider \( \varphi_{\lambda|_{U_\lambda}} \circ f_\lambda : U_\lambda \to \mathbb{R} \), which has support contained in \( U_\lambda \), and extends by zero to a \( C^\infty \) function on \( W \) denoted by \( \varphi f_\lambda \). If we then define \( \tilde{f} = \sum_{\lambda \in \Lambda} \varphi f_\lambda \), then \( \tilde{f} \) is a \( C^\infty \) function defined on \( W \) that extends \( f \). More generally, if we have a \( C^\infty \) bundle

\[
\begin{array}{ccc}
V & \longrightarrow & W \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
\]

where \( A \) is arbitrary, then what we’ve shown is that it extends to a \( C^\infty \) section on an open \( W \supset A \).

A variant of this result is that if \( A \) is a closed set, then note that \( \{U_\lambda \mid \lambda \in \Lambda\} \cup \{M - A\} \) is also an open cover, so we can create a partition of unity \( \{\varphi_\lambda \mid \lambda \in \Lambda\} \cup \{\varphi_0\} \). If we define \( f_0 : (M - A) \to \mathbb{R} \) to be zero, then let

\[
\tilde{f} = \sum_{\lambda} \varphi f_\lambda + \varphi_0 f_0.
\]

Once again, \( \tilde{f} : M \to \mathbb{R} \) and \( \tilde{f}|_A = f \). Finally, if \( A \) is compact, then we see that \( \tilde{f} \) can be chosen to have compact support.

**Proof of Thom’s lemma.** WLOG, we can assume that \( B \subset \mathbb{R}^N \), so that \( F : A \times I \to B \) can be extended to a \( C^\infty \) map \( F : A \times (-\epsilon, 1+\epsilon) \to B \). This is because we can extend to a map \( A \times \mathbb{R} \to \mathbb{R}^N \), and letting \( U \) be a tubular neighborhood around \( B \) in \( \mathbb{R}^N \), we can find an open neighborhood \( V \) around \( A \times I \) in \( A \times \mathbb{R} \) that maps into \( U \), and because \( A \) is compact we can take \( V \) to be of the form \( A \times (-\epsilon, 1+\epsilon) \), and then we can use the retraction from \( U \) to \( B \) to map everything into \( B \).

\[
\begin{array}{ccc}
A \times I & \subset & V \\
\downarrow & & \downarrow \\
B & \subset & U \\
\downarrow & & \downarrow \\
\subset & & \subset \\
A \times \mathbb{R} & \subset & \mathbb{R}^N
\end{array}
\]

Because \( A \) is compact, we can assume that \( f_t \) is an embedding for all \( t \in (-\epsilon, 1+\epsilon) \). Define \( \tilde{F} : A \times (-\epsilon, 1+\epsilon) \to B \times (-\epsilon, 1+\epsilon) \) to be the map sending \( (a, t) \mapsto (F(a, t), t) \). Then \( \tilde{F} \) sends \( (0, \frac{d}{dt}) \) to a vector field \( (w, \frac{d}{dt}) \). Let \( C = \tilde{F}(A \times (-\epsilon, 1+\epsilon)) \).

We have that \( C \hookrightarrow B \times (-\epsilon, 1+\epsilon) \) is closed and a section \( w \) of \( p_1^*TB|C \), where \( p_1 : B \times (-\epsilon, 1+\epsilon) \to B \). There exists a global \( C^\infty \) section \( \tilde{w} \) that extends \( w \). Consider \( v = (\tilde{w}, \frac{d}{dt}) \), which is a vector field on \( B \times (-\epsilon, 1+\epsilon) \). Let \( \phi_t \) be the flow associated to \( v \).

**Fact 1:** We know that for all \( a \in A \), the map \( t \mapsto (f_t(a), t) \) is an integral curve.

**Fact 2:** We may assume that \( \text{supp}(\tilde{w}) \overset{p_2}{\to} (-\epsilon, 1+\epsilon) \) is proper. This implies that for all \( z \in (-\epsilon, 1+\epsilon) \), the flow \( \phi_t(B \times z) \) is defined for all \( t \) with \( |t| < \delta \), say. In particular, \( \phi_t(B \times z) \) is defined
for all $z \in I$ and for all $t$ with $|t| < \delta$.

**Fact 3:** We have that $\phi_t(B \times z) \subset B \times \{z + t\}$, from which it follows that for all $0 \leq z \leq 1$, $\phi_t$ is defined on $B \times z$ for all $-z \leq t \leq 1 - z$.

From these facts, we have that $\phi_t|_{B \times 0} \sim B \times t$ is a diffeomorphism for all $0 \leq t \leq 1$. Now define $g_t = \phi_t$ and we are done. \qed
Lecture 6 (2013-02-27)

Let $M$ be a $C^\infty$ manifold and $v$ a $C^\infty$ vector field on $M$. Let $\phi_t(x) = \gamma_x(t)$ be the integral curve for $v$ with $\gamma_x(0) = x$. Let $\omega$ be any object attached to the manifold, such as for example a section of $TM^{\otimes m} \otimes T^*M^{\otimes n}$. Then the Lie derivative of $\omega$ with respect to $v$ makes sense:

$$L_v \omega = \frac{d}{dt} \phi^*_t \omega \bigg|_{t=0}$$

In particular, $L_v w$ is defined when $w$ is a vector field.

**Proposition.** For all vector fields $v, w$ on $M$, we have $L_v w = [v, w]$.

**Lemma** (Leibniz rule for sections of bundles). Let $v$ be a vector field.

1. $L_v (\omega \wedge \eta) = (L_v \omega) \wedge \eta + \omega \wedge L_v (\eta)$, where $\omega$ is a $k$-form and $\eta$ is an $\ell$-form
2. $L_v i_w \theta = i_{L_v (w)} \theta + i_w L_v \theta$ where $w$ is a vector field, and $\theta$ is a $k$-form
3. $v(\theta(w)) = \theta(L_v(w) + (L_v \theta)(w))$, where $\theta$ is a 1-form (this is just a special case of 2)

**Proof.** Let $V_1, V_2, V_3$ be vector bundles on $M$, and let $B$ be a bilinear map

$$
\begin{array}{cc}
V_1 \times_M V_2 & \rightarrow & V_3 \\
B & \downarrow & \\
M & \leftarrow & \\
\end{array}
$$

i.e. $B(x) : V_1(x) \times V_2(x) \rightarrow V_3(x)$ is bilinear for all $x \in M$. Let $s^1_t, s^2_t$ be families of $C^\infty$ sections of $V_1$ and $V_2$ respectively, indexed by $t \in (-\epsilon, \epsilon)$. Let $p_1$ be the projection $p_1 : M \times (-\epsilon, \epsilon) \rightarrow M$, so that each $s_t$ is a section of $p_1^* V_i$. Then

$$
\frac{d}{dt} B(s^1_t, s^2_t) = B \left( \frac{d}{dt} s^1_t, s^2_t \right) + B \left( s^1_t, \frac{d}{dt} s^2_t \right).
$$

How will we apply this - we want to choose $s_t = \phi^*_t(?)$.

Let $V_1 = TM, V_2 = \Lambda^k T^* M, V_3 = \Lambda^{k-1} T^* M$, and let $B(x) : T_x M \times \Lambda^k T^*_x M \rightarrow \Lambda^{k-1} T^*_x M$ be defined by $B(x)(\omega, \theta) = i_{v_x}(\theta)$.

Part 2 is then an application of the Leibniz rule

$$
i_{v_x^*}(\omega \wedge \eta) = i_{v_x^*}(\omega) \wedge \eta + (-1)^{\deg(\omega) \deg(\eta)} \omega \wedge i_{v_x^*}(\eta)
$$

where $v_x^* \in V^*$ and $\omega \in \Lambda^k V$, and 3 is just 2 for $k = 1$.

Given $\theta = df$, where $f : M \rightarrow \mathbb{R}$ is a $C^\infty$ map, then

$$
\theta(w) = (df)(w) = w(f)
$$

That $v(\theta(w)) = v(w(f))$ is just the left side of 3. But

$$
\theta(L_v w) = (L_v w)(f),
$$

hence

$$
L_v(\theta) = L_v(df) = dL_v f = d(v(f)),
$$
hence
\[(L_v \theta)(w) = w(v(f)).\]

Now 3 reads as
\[v(w(f)) = w(v(f)) + (L_v w)(f),\]
i.e.
\[(L_v w)(f) = v(w(f)) - w(v(f)) = [v, w](f)\]
for all \(C^\infty\) maps \(f : M \to \mathbb{R}.\)

**Corollary** (Special case of Cartan’s formula). Let \(\omega\) be a 1-form, and let \(v_1\) and \(v_2\) be vector fields. Then
\[d\omega(v_1, v_2) = i_{v_1} d\omega(v_2) - i_{v_2} d\omega(v_1) + (L_{v_1} w)(v_2) - (L_{v_2} w)(v_1),\]

\[\text{for } v_1, v_2 \in T_x M.\]

**Proof.** We have that \(L_v = i_v d + di_v.\) Thus,
\[i_{v_1} d\omega = L_{v_1} \omega - d(i_{v_1} \omega),\]
so that
\[d\omega(v_1, v_2) = i_{v_1} d\omega(v_2) = (L_{v_1} \omega)v_2 - v_2(\omega(v_1))\]
\[= L_{v_1} (\omega(v_2)) - \omega(L_{v_1} v_2) - v_2(\omega(v_1))\]
\[= v_1(\omega(v_2)) - \omega([v_1, v_2]) - v_2(\omega(v_1)).\]

**Remark.** Note that we can identify \(\Lambda^k T^*_x M\) with \((\Lambda^k T^*_x M)^*\) as follows: given \(\omega \in \Lambda^k T^*_x M\), we define
\[\omega(v_1, v_2, \ldots, v_k) = i_{v_k} i_{v_{k-1}} \cdots i_{v_1} \omega \in \Lambda^k T^*_x M \in \mathbb{R}\]
for \(v_1, \ldots, v_k \in T_x M.\)

**Remark.** We defined
\[L_v \omega = \left. \frac{d}{dt} \phi_t^* \omega \right|_{t=0}.\]

It is more generally true that
\[\left. \frac{d}{dt} \phi_t^* (\omega) \right|_{t=t_0} = \phi_{t_0}^* (L_v \omega).\]

Note that we haven’t said what kind of thing \(\omega\) is; it only makes sense for certain natural bundles. But this works in particular when \(\omega\) is some vector field \(w\). Then \(\phi_t\) is the flow associated to \(v\); also, let \(\psi_s\) be the flow associated to \(w\). Then
\[[v, w] = 0 \iff L_v(w) = 0 \iff \frac{d}{dt}(\phi_t^* w) = 0 \text{ for all } t \iff \phi_t^* w = w \text{ for all } t.\]

Assume that \([v, w] = 0\), so that \(\phi_t^* w = w\), and thus \(\phi_t\) (integral curve of \(w\)) is an integral curve of \(w\). This is equivalent to saying that \(\phi_t \circ \psi_s = \psi_s \circ \phi_t\). Thus, we have established the following:

**Corollary.** \([v, w] = 0 \iff \phi_t \circ \psi_s = \psi_s \circ \phi_t\)

**Theorem** (Ehresmann’s theorem). Let \(f : X \to Y\) be a proper submersion. Then \(f\) is a \(C^\infty\) fiber bundle.

We will give a second proof of this using flows.
**Proof.** Let \( X \to \mathbb{R}^m \) be an embedding of \( X \) in Euclidean space. Thus, for any \( x \in X \), \( T_x X \) gets an inner product. We have a short exact sequence

\[
0 \to T_x f^{-1}(f(x)) \to T_x X \xrightarrow{f'(x)} T_{f(x)}Y \to 0
\]

where we have used that \( f \) is a submersion. Let \( W(x) = T_x f^{-1}(f(x))^\perp \), so that we get a subbundle \( W \) of \( TX \) such that \( f'(x)(W(x)) = T_{f(x)}Y \).

Assume that \( Y = (-1, 1)^n \subset \mathbb{R}^n \). Then \( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \) are vector fields on \( Y \), i.e. sections of \( TY \), and so we get corresponding sections \( w_1, \ldots, w_n \) of \( W \) such that \( f'(x)(w_i) = \frac{\partial}{\partial y_i} \) for all \( i \). Note that even though the \( \frac{\partial}{\partial y_i} \) all commute with each other, we need not have that the \( w_i \) all commute with each other.

Let \( \phi_t^i \) denote the flow associated to \( w_i \). One sees that for any compact \( K \subseteq X \), there is an \( \epsilon > 0 \) such that \( \phi_{t_1}^1 \cdots \phi_{t_n}^n(x) \) are defined for all \( x \in K \) and \( t_i \in (-\epsilon, \epsilon) \).

Let \( K = f^{-1}(0) \), which is compact because \( f \) is proper. Then we have a commutative diagram

\[
\begin{array}{ccc}
K \times (-\epsilon, \epsilon)^n & \xrightarrow{h} & X \\
\downarrow{p_2} & & \downarrow{f} \\
(-\epsilon, \epsilon)^n & \longrightarrow & (-1, 1)^n
\end{array}
\]

and \( h \) induces isomorphisms on tangent spaces at \( K \times 0 \), so it must do so in a neighborhood of \( K \times 0 \). Because \( p_2 \) is proper, it follows that by shrinking \( \epsilon \) if necessary, we may assume that \( h \) induces isomorphisms on tangent spaces everywhere, and that \( h \) is one-to-one. Then \( h \) is then a diffeomorphism onto its image \( U \), which is open in \( X \). We want to show that \( U = X \); thus, let \( F = X \setminus U \). Then \( F \) is closed in \( X \), and because \( f \) is proper, we have that \( f(F) \) is a closed set (we’re using Hausdorffness here). Then \( F \cap f^{-1}(0) = \emptyset \), because \( 0 \notin f(F) \), and now replace \( (-1, 1)^n \) by the complement of \( f(F) \). 

We can now state a refinement of Ehresmann’s theorem.

**Theorem.** Let \( f : X \to Y \) be a proper submersion, and let \( A \subseteq X \) be a closed \( C^\infty \) submanifold. Assume also that \( f|_A : A \to Y \) is a submersion. Then \( f : (X, A) \to Y \) is a fiber-bundle pair.
Lecture 7 (2013-03-01)

Last time, we were discussing the Ehresmann theorem for fiber bundles of pairs. There was just one thing left to prove.

In the notation of the last lecture, we had $C^\infty$ manifolds $X$ and $Y$, a closed $C^\infty$ submanifold $A \subseteq X$, and a $C^\infty$ map $f : X \to Y$ such that both $f$ and $f|_A$ are submersions. (Note that for the Ehresmann theorem, we would assume properness, but for now we just want to extract the subbundle $W$ which did not need that hypothesis.)

**Proposition.** There exists a subbundle $W \subset TX$ such that

1. For all $x \in X$, the derivative $f'(x)|_{W(x)} : W(x) \to T_{f(x)}Y$ is an isomorphism.
2. For all $x \in A$, we have $W(x) \subset TA$ (both interpreted as subspaces of $T_xX$).

This proposition implies the Ehresmann theorem for pairs.

The secret code phrase here is that

$$H^1(\text{any sheaf of modules over the sheaf of } C^\infty \text{ functions}) = 0$$

**Proof.** For the first step, note that the problem makes sense on any open $U \subset X$, so it will suffice to show that $W$ exists locally, i.e. that for all $x \in X$, there is a neighborhood $U(x)$ where the theorem holds.

If $x \notin A$, then we’re done, so suppose that $x \in A$. WLOG, we can take $X = \mathbb{R}^n$, $A = \{ x \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0 \}$, and $f : X \to Y$ the map $f(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$ where $r \leq m$. In this case, we can just take $W$ to be the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}$.

Now to Step 2; we want to providing an algebraic description of $W$. This is essential. We can’t add subbundles, but we can add / do other linear things to sections of bundles.

For each $x \in X$, let $s(x)$ be the inverse of the isomorphism described in (i); in other words, we want to demonstrate the existence of a map of bundles $s : f^*TY \to TX$ such that

1. $f'(x) \circ s(x) : T_{f(x)}Y \to T_{f(x)}Y$ is the identity for all $x \in X$
2. For all $x \in A$, we have $s(x)(T_{f(x)}Y) \subseteq T_xA$.

Step 3: Suppose that $s_1$ and $s_2$, both maps $f^*TY \to TX$, satisfy conditions (i’) and (ii’). Then $h = s_2 - s_1 : f^*TY \to TX$ satisfies

1. $f'(x) \circ h(x) = 0$ for all $x \in X$
2. $h(x)(T_{f(x)}Y) \subseteq T_xA$ for all $x \in A$.

so that

$$Z = \{ h : f^*TY \to TX \mid (i’’) \text{ and } (ii’’)) \text{ hold} \}$$

is a module over the ring of $C^\infty$ functions on $X$. Note that this is a characterization; in other words, if $s_1$ satisfies (i’) and (ii’), then $s_1 + h$ satisfies them if and only if $h \in Z$.

As a corollary of Step 3, we see that if $s_1, \ldots, s_m$ are as in Step 2, and $\varphi_1, \ldots, \varphi_m : W \to \mathbb{R}$ are a $C^\infty$ partition of unity (so that $\sum \varphi_i = 1$), then $\sum \varphi_i s_i$ also satisfies the conditions of step 2, because

$$\sum_{i \in Z} \varphi_i s_i = \sum_{i=1}^m \varphi_i (s_i - s_1) + \left( \sum_{i=1}^m \varphi_i \right) s_1.$$

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Math 318 - Geometry/Topology 2

Page 14

Lecture 7
Now we come to the proof of the proposition itself. Let \( \{U_\lambda\}_{\lambda \in \Lambda} \) be an open cover equipped with \( s_\lambda : f^*TY|_{U_\lambda} \rightarrow TX|_{U_\lambda} \) all satisfying (i') and (ii'). There is a partition of unity \( \varphi_\lambda \) subordinate to \( U_\lambda \); then \( \varphi_\lambda s_\lambda \) (originally defined only on \( U_\lambda \)) can be extended by 0 to a \( C^\infty \) map \( \varphi_\lambda s_\lambda : f^*TY \rightarrow TX \).

Now let \( s = \sum \varphi_\lambda s_\lambda : f^*TY \rightarrow TX \); the corollary above implies that \( s \) satisfies (i') and (ii').

**Existence of inner products on vector bundles**

Given a \( C^\infty \) vector bundle \( f : V \rightarrow M \), we want to construct a map \( B : V \times_M V \rightarrow \mathbb{R} \) such that \( B : V(x) \times V(x) \rightarrow \mathbb{R} \) is a positive definite, symmetric, bilinear form.

If \( W \) is a vector space, and \( B : W \times W \rightarrow \mathbb{R} \) is symmetric and bilinear, we say that \( B \) is positive semi-definite if \( B(w,w) \geq 0 \) for all \( w \in W \), and positive definite if it is positive semi-definite and \( B(w,w) = 0 \) implies \( w = 0 \).

**Proof.** Step 1. Assume that \( V|_U \) is a trivial bundle, i.e. there exist sections \( s_1, \ldots, s_k \) of \( V|_U \) such that \( s_1(x), \ldots, s_k(x) \) form a basis for \( V(x) \) for all \( x \in U \).

Define \( B_U(s_i(x), s_j(x)) = \delta_{ij}(x) \). Given an open cover \( \mathcal{U} \), and a partition of unity \( \varphi_U \) subordinate to \( \mathcal{U} \), then \( \sum \varphi_U B_U \) is a symmetric bilinear positive semi-definite form. But for any \( x \in X \), if \( v \in V(x) \) is non-zero, then there is some \( U \) such that \( \varphi_U(x) > 0 \), so that \( x \in U \) and moreover \( B_U(v,v) > 0 \), hence \( B(v,v) \geq \varphi_U(x)B_U(v,v) > 0 \). Thus, this is in fact positive definite.

**Existence of connections on a vector bundle**

A good reference for this is Milnor’s *Morse Theory*.

Let \( p : V \rightarrow M \) be a \( C^\infty \) vector bundle. A connection is essentially a way of taking a derivative of a section \( s \) of a vector bundle \( v \) with respect to a vector field on \( M \).

Suppose that \( x \in U \) and that \( V|_U \) is trivial, and that \( s_1, \ldots, s_k \) are sections of \( V|_U \) that give a basis for \( V(x) \) for each \( x \in U \). For any \( v \in T_xM \), we define

\[
v \left( \sum f_i s_i \right) = \sum v(f_i) s_i.
\]

A connection, or a covariant derivative, \( \nabla \) on \( V \) is a map taking in a vector field \( v \) on \( M \), and a section \( s \) of \( V \), and outputting \( \nabla_v s \), another section of \( V \). We also require that a connection satisfy certain properties: for any \( C^\infty \) map \( f : M \rightarrow \mathbb{R} \),

1. \( \nabla_v (s_1 + s_2) = \nabla_v (s_1) + \nabla_v (s_2) \)
2. \( \nabla_v (fs) = v(f) s + f \nabla_v (s) \) (this is the Leibniz rule)
3. \( \nabla_{fv}(s) = f \nabla_v (s) \)

We could have stated this definition sheaf-theoretically, which is after all necessary to do it on analytic manifolds, but for \( C^\infty \) manifolds, they are equivalent.

We want to show that any \( C^\infty \) vector bundle \( V \rightarrow M \) has a connection.

The argument is the same as we’ve been doing. Step 1 is to show that they exist locally (this is just the trivial connection). Step 2 is to take two connections \( \nabla^1, \nabla^2 \) and define \( h \) via \( \nabla^2 = \nabla^1 + h \), i.e. \( \nabla^2_v(s) = \nabla^1_v(s) + h_v(s) \) for all sections \( s \), and note that \( h \) satisfies three properties: \( h \) is additive in \( s 

\[
h_v(fs) = fh_v(s) \]
for all $C^\infty$ functions $f$, and $h_{fv}(s) = fh_v(s)$.

Then, if $\nabla^1$ is a connection and $\nabla^2 = \nabla^1 + h$, then $\nabla^2$ is a connection if and only if $h$ satisfies the above three properties. The collection of all such $h$ can be thought of being comprised of precisely the sections of $\text{Hom}(TM, \text{End}(V))$, which is a module over $C^\infty$ functions $M \to \mathbb{R}$.

We then conclude by using a partition of unity and noting that $\sum \varphi_U \nabla_U$ gives a connection.

Let’s examine connections in a basic case; let $M$ be an open interval $(a, b)$. By the properties of a connection, all we have to look at is $\nabla_{\frac{d}{ds}}(s)$. In particular, what is

$$\{\text{sections } s : M \to V \mid \nabla_{\frac{d}{ds}}(s) = 0\} \ ?$$

We know that $V$ is trivial because we’re working on an interval; choose a specific trivialization, so that we will think of sections as maps $s : (a, b) \to \mathbb{R}^k$. Define vectors of $C^\infty$ functions $m_i$ by

$$\nabla_{\frac{d}{ds}}(e_i) = m_i,$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

the 1 being in the $i$th position. Then