Math 318 - Geometry/Topology 2

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Introduction

Math 318 is one of the nine courses offered for first-year mathematics graduate students at the University of Chicago. It is the second of three courses in the year-long geometry/topology sequence.

These notes are being live-TeXed, though I edit for typos and add diagrams requiring the Ti*k*Z package separately. I am using the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to `chonoles@math.uchicago.edu`.

Lecture 1 (2013-02-15)

Today we'll talk about a method for reducing nth order differential equations to first order differential equations. Thus, for example, we can express acceleration as a function of position and velocity.

Let $\Omega \subset \mathbb{R}^n$ be open, and let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function (representing acceleration). Then given $(x_0, y_0) \in \Omega \times \mathbb{R}^n$, there is a unique $\gamma : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma(0) = x_0$, $\gamma'(0) = y_0$, and $\gamma''(t) = a(\gamma(t), \gamma'(t))$ for all $t \in (-\epsilon, \epsilon)$.

The trick was to introduce the space $\Omega \times \mathbb{R}^n$. Put $M = \Omega \times \mathbb{R}^n$, and let $\delta(t) = (\gamma(t), \gamma'(t))$. Any vector field on M can be thought of as a function $M \to \mathbb{R}^n \times \mathbb{R}^n$; let $\widetilde{v} : M \to \mathbb{R}^n \times \mathbb{R}^n$ be the vector field defined by

$$\widetilde{v}(x, y) = (y, a(x, y)).$$

If γ is as above, then we have $\delta(0) = (x_0, y_0) \in M$, and $\delta'(t) = (\gamma'(t), \gamma''(t)) = \widetilde{v}(\delta(t))$. Thus, curves γ satisfying our requirements are precisely integral curves of the vector field $\widetilde{v}$ on M.

Theorem. *Let M be a C^∞ manifold and let v be C^∞ vector field on M. Let $p \in M$. Then there is a neighborhood $U(p)$ of p in M, some $a > 0$, and a C^∞ function $\gamma : U(p) \times (-a, a) \to M$ such that for all $x \in U(p)$, $\gamma_x(t)$ is the (unique) integral curve of v such that $\gamma_x(0) = x$, where $\gamma_x(t) := \gamma(x, t)$.*

A reference for this theorem is Hurewicz's *Lectures on Ordinary Differential Equations*.

Remark 1. Let γ be an integral curve of a C^∞ vector field. Then it is clear that γ is C^∞, because we have that

$$\gamma'(t) = v(\gamma(t)),$$

so that if γ is C^k, then $v \circ \gamma$ is C^k, hence γ' is C^k, so that γ is C^{k+1}.

Remark 2. Assume that v is a C^1 vector field. The contraction principle shows that $(x, t) \mapsto \gamma_x(t)$ is continuous in (x, t) and defined on some neighborhood $U(p) \times (-a, a)$, as follows:

WLOG, let $p = 0$, and let $M = \Omega$, an open subset of $\mathbb{R}^n$. Then the space of continuous functions

$$\{x \in \mathbb{R}^n \mid \|x\| \leq \alpha\} \times [-c, c] \quad \longrightarrow \quad \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$$

with the $\|\cdot\|_\infty$ norm is a complete metric space.

Remark 3. Assume that v is a C^2 vector field on Ω, an open subset of $\mathbb{R}^n$. We want to show that the partial derivatives

$$\frac{\partial}{\partial x_i}(\gamma_x(t))$$

exist. We can simply check that the differential equation defining them is satisfied. Let $v : \Omega \to \mathbb{R}^n$ be a vector field on Ω, and so that for any $x \in \Omega$ we have $v'(x) : \mathbb{R}^n \to \mathbb{R}^n$.

We introduce the (standard) notation $\phi_t(x) = \gamma_x(t)$. Thus, ϕ_t is defined at all $t \in (-a, a)$, and ϕ_t is a function $\phi_t : U(p) \to \Omega$, so that we have $\phi_t'(x) : \mathbb{R}^n \to \mathbb{R}^n$ for any $x \in U(p)$, where $\phi_t'(x)$ is the derivative of $x \mapsto v(\phi_t(x))$.

Let $\widetilde{v} : M \to \mathbb{R}^n \times \mathrm{M}_n(\mathbb{R})$ be the vector field on $M = \Omega \times \mathbb{R}^n$ defined by

$$\widetilde{v}(x, S) = (v(x), v'(x)S).$$

We claim that, for any fixed t, the map $x \mapsto (\phi_t(x), \phi_t'(x))$ is an integral curve for $\widetilde{v}$.

We have that

$$\frac{d}{dt}\phi_t(x) = v(\phi_t(x)),$$

so that

$$\widetilde{v}(\phi_t(x), \phi_t'(x)) = \frac{d}{dt}(\phi_t(x), \phi_t'(x)) = (v(\phi_t(x)), v'(\phi_t(x))\phi_t'(x)).$$

This "reduces" the question to integral curves of $\widetilde{v}$ on $\Omega \times \mathrm{M}_n(\mathbb{R})$ (?).

Corollary. *Let M be a C^∞ manifold, and let v be a C^∞ vector field on M. For all $x \in M$, we've shown that there is a maximal integral curve $\gamma_x : I_x \to M$ such that $\gamma_x(0) = x$. Let*

$$D = \{(x,t) \in M \times \mathbb{R} \mid t \in I_x\},$$

and write $\gamma_x(t) = \gamma(x,t)$ for all $(x,t) \in D$.

1. *D is an open subset of $M \times \mathbb{R}$.*
2. *If $\phi_t(x)$ is defined, then $\phi_s(x)$ is defined for all $0 \leq s \leq t$ (or all $t \leq s \leq 0$).*
3. *If $\phi_t(x)$ is defined and if $\phi_s(\phi_t(x))$ is defined, then $\phi_{t+s}(x)$ is defined and*

$$\phi_{t+s}(x) = \phi_s(\phi_t(x)).$$

4. *Assume M is compact. Then the theorem implies that there is an open interval $(-a,a)$ such that $(-a,a) \subset I_x$ for all $x \in M$, so that $D \supseteq M \times (-a,a)$; statement 3 then implies $D = M \times \mathbb{R}$. Furthermore, for a fixed t, the map $x \mapsto \phi_t(x)$ is C^∞ and induces an isomorphism on tangent spaces.*
5. *Observe that given a vector field v on M, if a constant path $\gamma(t) = p$ is an integral curve of v, then $v(p) = 0$. Conversely, if $v(p) = 0$, then the constant curve to p is an integral curve.*

Proof of 3. We know γ_x is defined on $[0, t+\epsilon)$, and that $\gamma_{\phi_t(x)}$ is defined on $[0, s+\epsilon')$. Now define

$$\delta(z) = \begin{cases} \gamma_x(z) & \text{for all } z \in [0,t], \\ \gamma_{\phi_x(t)}(z-t) & \text{for all } z \in [t, s+\epsilon). \end{cases}$$

Note that δ is an integral curve of v.

□

Exercise. Let $\mathbb{R} \xrightarrow{h} \mathrm{Diffeo}(M)$ be a group homomorphism such that the map $M \times \mathbb{R} \to M$ defined by $(x,t) \mapsto h(t)(x)$ is C^∞. Prove that there is a C^∞ vector field v on M such that $h(t) = \phi_t$ for the vector field v. This does not require M to be compact, just that the support of v is compact.

Remark. If v is a vector field on a manifold M and $v(p) \neq 0$, then there is a neighborhood $U(p)$ of p in M and a diffeomorphism $f : U(p) \to \Omega$, where Ω is an open subset of $\mathbb{R}^n$, such that the vector field v (restricted to $U(p)$) is carried by f to the constant vector field $\frac{\partial}{\partial x_1}$ on Ω.

Proof of Remark. Let's say that Z is a slice of M when $T_pZ \oplus \mathbb{R}v(p) = T_pM$. The hypotheses of the inverse function theorem hold at $(p,0)$, so that we can find coordinates in which the map from $Z \times (-\epsilon, \epsilon) \to M$ defined by $\phi_t(z) = \gamma_z(t)$ sends the vector field v to a constant vector field, such as for example $\frac{\partial}{\partial x_1}$. □

Next time, we'll start with Lie brackets. It's motivated from two points of view; maybe 100 in fact.

Lecture 2 (2013-02-18)

Though it'll seem like we're leaving integral curves, we'll return to them in the middle of the lecture.

Recall that given a C^∞ manifold M, a point $p \in M$, and a tangent vector $v \in T_pM$, there is an $\mathbb{R}$-linear functional $v : C^\infty(M) \to \mathbb{R}$, sending a C^∞ function $f : M \to \mathbb{R}$ to $v(f) \in \mathbb{R}$. It satisfies the Leibniz rule,

$$v(fg) = f(p)v(g) + g(p)v(f).$$

This is a generalization of the notion of directional derivative in Euclidean space.

Now let v be a vector field on M. Let $R = C^\infty(M)$. Now we have an $\mathbb{R}$-linear map $v : R \to R$, defined by $v(f)(p) = v(p)(f)$ for all $p \in M$. For example, if $M = \mathbb{R}^n$ and $v = (a_1, \dots, a_n) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we have that

$$v(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

For any ring S, a function $D : S \to S$ is a derivation when $D(fg) = D(f) \cdot g + f \cdot D(g)$. Very often, we are given a subring $T \subset S$ contained in the center of S, that we require to satisfy $D(t) = 0$ for all $t \in T$. Note that the map $v : R \to R$ sending $f \mapsto v(f)$ is a derivation, and $v(\text{any constant function}) = 0$ (observe that we can $\mathbb{R}$ is a subring of R).

The following is an easy lemma.

Lemma. *If $D_1, D_2 : R \to R$ are derivations, then $(D_1 \circ D_2) - (D_2 \circ D_1)$ is also a derivation.*

In particular, if v, w are C^∞ vector fields on M, U is an open subset of M, and $C^\infty(U)$ is the ring of C^∞ functions on U, the map $f \mapsto v(w(f)) - w(v(f))$ is a derivation of $C^\infty(U)$. If we fix a point $p \in M$, we can consider neighborhoods U of $p \in M$, and the map

$$f \mapsto (v(w(f)) - w(v(f)))(p)$$

induces an $\mathbb{R}$-linear map on germs $C^\infty_{M,p} \to \mathbb{R}$. Being a derivation, this is equal to $h(p)(f)$ for a unique $h(p) \in T_pM$. It is true (though we won't check) that $p \mapsto h(p)$ is a C^∞ vector field on M, and we define the Lie bracket of v and w to be this h. We write $h = [v, w]$. Thus,

$$[v, w](f) = v(w(f)) - w(v(f))$$

for all C^∞ maps $f : U \to \mathbb{R}$.

Lemma. *Let Ω be an open subset of $\mathbb{R}^n$, and let v, w be C^∞ vector fields on Ω. Then*

$$[v, w] = D_v w - D_w v,$$

where

$$(D_v w)(x) = \frac{d}{dt} w(x + tv)\bigg|_{t=0}.$$

Proposition. *The $\mathbb{R}$-vector space of C^∞ vector fields on M, together with the bracket, satisfies the axioms of a Lie algebra:*

1. *$[v, w] = -[w, v]$ for all C^∞ vector fields v and w.*
2. *$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$ for all C^∞ vector fields v_1, v_2, v_3.*
3. *$[tv, w] = t[v, w]$ for all $t \in \mathbb{R}$.*

Definition. Let M and N be C^∞ manifolds, and let $\phi : M \to N$ be a C^∞ map. Given vector fields v on M and w on N, we say that v and w are ϕ-related if for all $x \in M$,

$$\phi'(x)v(x) = w(\phi(x)).$$

Lemma 1. *Given vector fields v on M and w on N, they are ϕ-related if and only if $\phi(\gamma)$ is an integral curve of w for any integral curve γ of v.*

Proof. Assume that v and w are ϕ-relted. Let $\gamma : (a, b) \to M$ be an integral curve for v, so that for all $t \in (a, b)$, we have

$$\gamma'(t) = v(\gamma(t)).$$

Let $\delta = \phi \circ \gamma$. Then

$$\delta'(t) = \phi'(\gamma(t))\gamma'(t) = w(\delta(t)).$$

Everything is reversible, so we are done. □

Last time, I mentioned that if a vector field is non-zero at a point, then in some neighborhood it looks like $\frac{\partial}{\partial x_1}$. There is a proof of this in Warner's book on page 40.

Example. Let w be a vector field on N and suppose that $w(p) \neq 0$. Then there is a chart centered at p such that w is transformed to $\frac{\partial}{\partial x_n}$.

Proof. Let Zbe a codimension 1 closed submanifold of N containing p, and suppose that it is transverse, i.e. that $T_pZ \oplus \mathbb{R}w(p) = T_pN$. Let $\delta_y(t)$ be an integral curve of w with initial value y, i.e. $\delta_y(0) = y$. Let $M = Z \times (-c, c)$, and let $\phi : M \to N$ be the map defined by

$$\phi(z, t) = \delta_z(t).$$

This is a diffeomorphism in a neighborhood of $Z \times \{0\}$ by the inverse function theorem, and the curves $t \mapsto (z, t)$ on M are sent by ϕ to the curves $\delta_z(t)$ on N, which are integral curves of w. Thus, $t \mapsto (z, t)$ is an integral curve for $\frac{\partial}{\partial x_n}$. □

Lemma 2. *Let M and N be C^∞ manifolds, and let $\phi : M \to N$ be C^∞.*

(a) If v on M and w on N are ϕ-related, then $v(\phi^* f) = \phi^* w(f))$ for any C^∞ map $f : N \to \mathbb{R}$; this is just a restatement of the definition.

(b) If v_1 is ϕ-related to w_1 and v_2 is ϕ-related to w_2, then $[v_1, v_2]$ and $[w_1, w_2]$ are ϕ-related.

Proof of (b). We have

$$v_1(v_2(\phi^*(f))) = v_1(\phi^*(w_2(f))) = \phi^*(w_1(w_2(f))).$$

Now interchange and subtract. □

Remark. This has an important consequence. If M is a locally closed submanifold of N, $\phi : M \to N$ is the inclusion, and w is a vector field on N, then to say that there is some v on M that is ϕ-related to w is equivalent to saying that $w(x) \in T_xM$ for all $x \in M$ (because $w(x) = v(x)$). Thus, Lemma 2 is saying something about vector fields that are tangent to submanifolds; if w_1 and w_2 are vector fields on N such that $w_1(x), w_2(x)$ belong to T_xM for all $x \in M$, then $[w_1, w_2]$ has the same property.

Definition. Let M be a C^∞ manifold, and let W be a C^∞ subbundle of TM of rank r. A locally closed submanifold A of M is a leaf if for all $x \in A$, $T_xA = W(x)$.

(The notion of leaf can defined in more generality than what is given here.)

Suppose that there is a leaf of W through every point of M. If w_1, w_2 are C^∞ sections of W, then $[w_1, w_2]$ is necessarily also a section of W; we can see this easily as follows. Let $p \in M$ and let Z be a leaf through p. Because Z is a leaf, w_1 and w_2 are tangential to Z, so $[w_1, w_2]$ is tangential to Z, i.e. $[w_1, w_2](p) \in T_pZ = W(p)$ for all $p \in M$.

Definition. A C^∞ subbundle W of TM is said to be involutive (alternatively, integrable) if for all C^∞ sections w_1, w_2 of W, $[w_1, w_2]$ is also a section of W.

We have already proven one piece of the following theorem:

Theorem (Frobenius)**.** *Let W be a subbundle of TM. The following are equivalent:*

1. *W is involutive.*
2. *There is a leaf of W through every point.*
3. *For all $p \in M$, there is a diffeomorphism h from a neighborhood of p to $U_1 \times U_2$, where U_i is an open subset of $\mathbb{R}^{n_i}$ for $i = 1, 2$, such that $h(W)$ is the constant $\mathbb{R}^{n_1} \times \{0\}$ bundle on $U_1 \times U_2$.*

Proof. It is clear that $3 \implies 2$, and we have already proven that $2 \implies 1$, so it remains to prove that $1 \implies 3$. This proof is taken from Narasimhan (the proof is originally due to Volterra).

Step 1. Let W be an involutive subbundle of rank r. Then in a neighborhood of any $p \in M$, we can find vector fields $w_1, \ldots, w_r$ which are a frame for W, i.e. $w_1(x), \ldots, w_r(x)$ are a basis for $W(x)$ for all x in the neighborhood, and such that $[w_i, w_j] = 0$ for all i, j.

Let me make a linear algebra observation: given a vector space $V = V_1 \oplus V_2$, subspaces $W \subset V$ such that the projection to V_1 is an isomorphism, i.e.

$$W \hookrightarrow V \xrightarrow{p_1} V_1, \qquad W \xrightarrow{\cong} V_1$$

can be identified with graphs of linear transformations $S : V_1 \to V_2$.

Now write $\mathbb{R}^N = V_1 \times V_2$, where $N = \dim(M)$, where V_1 and V_2 have been chosen such that $p|_{W(p)} : W(p) \to V_1$ is an isomorphism (p is the projection $\mathbb{R}^N \to V_1$), so that $W(x) \cong V_1$ for all x in some neighborhood of p. Thus, for each x, we get $S(x) : V_1 \to V_2$, and

$$W(x) = \{(v_1, S(x)v_1) \mid v_1 \in V_1\}.$$

Let $\Omega \subset V_1 \times V_2 = \mathbb{R}^N$ be open. WLOG we have $V_1 = \mathbb{R}^r$, where $e_1, \ldots, r_r$ are the standard basis of $\mathbb{R}^r$. We have $S(x)e_i = u_i(x)$, where $u_i : \Omega \to V_2$ is some C^∞ function. Thus $W(x)$ is the linear space of the $e_i + u_i$. For any i, j, we have that $[e_i + u_i, e_j + u_j]$ is a section of W, and using the formula

$$[\alpha, \beta] = D_\alpha \beta - D_\beta \alpha$$

on Euclidean space, we have that $[e_i + u_i + e_j + u_j]$ is a section of V_2 (i.e. a function $\Omega \to V_2$); but it also has to be a section of W, so it has to be 0 since $V_2 \cap W(x) = 0$ for all $x \in \Omega$. ☐

We'll finish the proof of this with Step 2 next time.

Lecture 3 (2013-02-20)

Everything we're talking about today will be C^∞.

To finish the proof of the Frobenius theorem from last time, it remains to show the following result:

Lemma 1. *If $w_1, \ldots, w_r$ are linearly independent, commuting vector fields (commuting in the sense that their pairwise Lie brackets are 0), then there is a chart centered at any given point where the w_i are transformed to the coordinate vector fields $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, r$.*

Remark. Let v and w be vector fields on M. Let ϕ_t and ψ_s denote the one-parameter groups for v and w respectively (i.e. the flows). Then for all $p \in M$, there is some neighborhood $U(p)$ of p and $(-\epsilon, \epsilon)$ such that $\phi_t(\psi_s(x))$ and $\psi_s(\phi_t(x))$ are defined for all $x \in U(p)$ and $t, s \in (-\epsilon, \epsilon)$.

Lemma 2. *With notation as above, if $[v, w] = 0$, then $\phi_t(\psi_s(x)) = \psi_s(\phi_t(x))$ for any $x \in U(p)$ and $s, t \in (-\epsilon, \epsilon)$.*

Proof that Lemma 2 $\implies$ Lemma 1. Let's assume the result of Lemma 2 in the case that $v(p) \neq 0$. Let ϕ_t^i denote the one-parameter groups with respect to w_i for each $i = 1, \ldots, r$. Let $p \in M$, and select a locally closed C^∞ submanifold $Z \subset M$ with $p \in Z$ such that $T_pZ \oplus \mathbb{R}w_1(p) \oplus \cdots \oplus \mathbb{R}w_r(p) = T_pM$. Note that by assuming this is true at p, we can assume this is true in a neighborhood of p.

Let $h : (-\epsilon, \epsilon)^r \times (Z \cap U(p)) \to M$ be defined by

$$h(x_1, \ldots, x_r, z) = \phi^1_{x_1} \phi^2_{x_2} \cdots \phi^r_{x_r}(z).$$

We see that h induces an isomorphism from the tangent space at $(0, \ldots, 0, z)$ to T_zM for all $z \in Z \cap U(p)$. Note that $h(t, x_2, \ldots, x_r, z)$ is an integral curve for w_1, so that $h'(?)\frac{\partial}{\partial x_1} = w(h(?))$ for all ? in the domain of h (this is not a ? in the sense of "I didn't get down what was on the board", but rather "?" itself what was written on the board). This is

$$\phi^2_{x_2} \phi^1_{x_1} \cdots ,$$

and thus we see that $h'(?)\frac{\partial}{\partial x_2} = w_2(h(?))$, etc. (not sure I understand this part). □

Proof of Lemma 2. We have that $w_1(p) \neq 0$, so (as we have shown earlier) we can assume WLOG that $w = \frac{\partial}{\partial x_1}$. For any vector $v = \sum a_i \frac{\partial}{\partial x_i}$, we have that

$$[w, v] = \sum \frac{\partial a_i}{\partial x_1} \cdot \frac{\partial}{\partial x_i}.$$

By assumption, this is zero, so the a_i's are (in some neighborhood) functions of $(x_2, \ldots, x_n)$. Because the statement is local, we can assume that we are working on $(-\epsilon, \epsilon) \times \Omega$ for an open subset $\Omega \subset \mathbb{R}^{n-1}$. Let $c \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Let $h_c : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \to (-\epsilon, \epsilon) \times \Omega$ be defined by

$$h_c(x_1, x_2, \ldots) = (x_1 + c, x_2, \ldots).$$

Then v and $v|_{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}$ are h_c-related.

Therefore, if δ is an integral curve of v, then $h_c \circ \delta$ is also an integral curve. Let ϕ_t denote the one-parameter group associated to v. Then we have that

$$h_c \circ \phi_t = \phi_t \circ h_c.$$

But $h_c = \psi_c$ where ψ_c is the one-parameter group associated to w. □

Theorem (Thom's ambient isotopy lemma). *Let $I = [0,1]$, let A and B be C^∞ manifolds where A is compact, and let $F : A \times I \to B$ be a C^∞ map. Let $f_t : A \to B$ be defined by $f_t(a) = F(a,t)$ for all $a \in A$ and $t \in [0,1]$. If f_t is an embedding for all $t \in I$, then there is a C^∞ map $G : B \times I \to B$ such that g_t is a diffeomorphism for all $t \in I$, and $f_t = g_t \circ f_0$ for all $t \in I$, where $g_t(b) = G(b,t)$.*

Recall that if A is an arbitrary subset of a C^∞ manifold M, then given a map $f : A \to \mathbb{R}$, we say that it is C^∞ map when there exist open sets $U_\lambda \subset M$ for all $\lambda \in \Lambda$ such that $f|_{A\cap U_\lambda} = f_\lambda|_{A\cap U_\lambda}$ and $W := \bigcup U_\lambda$ contains A. Then $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover of W, so there is a partition of unity subordinate to this cover. Let $\varphi_\lambda : W \to \mathbb{R}$ be subordinate to U_λ.

Consider $\varphi_\lambda|_{U_\lambda} \circ f_\lambda : U_\lambda \to \mathbb{R}$, which has support contained in U_λ, and extends by zero to a C^∞ function on W denoted by $\varphi_\lambda f_\lambda$. If we then define $\widetilde{f} = \sum_{\lambda\in\Lambda} \varphi_\lambda f_\lambda$, then $\widetilde{f}$ is a C^∞ function defined on W that extends f. More generally, if we have a C^∞ bundle

$$\begin{array}{ccc} & \overset{s}{\nearrow} & V \\ & & \downarrow p \\ A & \hookrightarrow & M \end{array}$$

where A is arbitrary, then what we've shown is that it extends to a C^∞ section on an open $W \supset A$.

A variant of this result is that if A is a closed set, then note that $\{U_\lambda \mid \lambda \in \Lambda\} \cup \{M - A\}$ is also an open cover, so we can create a partition of unity $\{\varphi_\lambda \mid \lambda \in \Lambda\} \cup \{\varphi_0\}$. If we define $f_0 : (M - A) \to \mathbb{R}$ to be zero, then let

$$\widetilde{f} = \sum_\lambda \varphi_\lambda f_\lambda + \varphi_0 f_0.$$

Once again, $\widetilde{f} : M \to \mathbb{R}$ and $\widetilde{f}|_A = f$. Finally, if A is compact, then we see that $\widetilde{f}$ can be chosen to have compact support.

Proof of Thom's lemma. WLOG, we can assume that $B \subset \mathbb{R}^N$, so that $F : A \times I \to B$ can be extended to a C^∞ map $F : A \times (-\epsilon, 1+\epsilon) \to B$. This is because we can extend to a map $A \times \mathbb{R} \to \mathbb{R}^N$, and letting U be a tubular neighborhood around B in $\mathbb{R}^N$, we can find an open neighborhood V around $A \times I$ in $A \times \mathbb{R}$ that maps into U, and because A is compact we can take V to be of the form $A \times (-\epsilon, 1+\epsilon)$, and then we can use the retraction from U to B to map everything into B.

$$\begin{array}{ccccc} A \times I & \subset & V & \subset & A \times \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ B & \subset & U & \subset & \mathbb{R}^N \end{array}$$

Because A is compact, we can assume that f_t is an embedding for all $t \in (-\epsilon, 1+\epsilon)$. Define $\widetilde{F} : A \times (-\epsilon, 1+\epsilon) \to B \times (-\epsilon, 1+\epsilon)$ to be the map sending $(a,t) \mapsto (F(a,t), t)$. Then $\widetilde{F}$ sends $(0, \frac{d}{dt})$ to a vector field $(w, \frac{d}{dt})$. Let $C = \widetilde{F}(A \times (-\epsilon, 1+\epsilon))$.

We have that $C \hookrightarrow B \times (-\epsilon, 1+\epsilon)$ is closed and a section w of $p_1^* TB|C$, where $p_1 : B \times (-\epsilon, 1+\epsilon) \to B$. There exists a global C^∞ section $\widetilde{w}$ that extends w. Consider $v = (\widetilde{w}, \frac{d}{dt})$, which is a vector field on $B \times (-\epsilon, 1+\epsilon)$. Let ϕ_t be the flow associated to v.

Fact 1: We know that for all $a \in A$, the map $t \mapsto (f_t(a), t)$ is an integral curve.

Fact 2: We may assume that $\operatorname{supp}(\widetilde{w}) \xrightarrow{p_2} (-\epsilon, 1+\epsilon)$ is proper. This implies that for all $z \in (-\epsilon, 1+\epsilon)$, the flow $\phi_t(B \times z)$ is defined for all t with $|t| < \delta$, say. In particular, $\phi_t(B \times z)$ is defined for all $z \in I$ and fro all t with $|t| < \delta$.

Fact 3: We have that $\phi_t(B \times z) \subset B \times \{z + t\}$, from which it follows that for all $0 \leq z \leq 1$, ϕ_t is defined on $B \times z$ for all $-z \leq t \leq 1 - z$.

From these facts, we have that $\phi_t|_{B\times 0} \xrightarrow{\cong} B \times t$ is a diffeomorphism for all $0 \leq t \leq 1$. Now define $g_t = \phi_t$ and we are done. $\square$

Lecture 4 (2013--)

Lecture 5 (2013--)

Lecture 6 (2013-02-27)

Let M be a C^∞ manifold and v a C^∞ vector field on M. Let $\phi_t(x) = \gamma_x(t)$ be the integral curve for v with $\gamma_x(0) = x$. Let ω be any object attached to the manifold, such as for example a section of $TM^{\otimes m} \otimes T^*M^{\otimes n}$. Then the Lie derivative of ω with respect to v makes sense:

$$L_v\omega = \frac{d}{dt}\phi_t^*\omega\bigg|_{t=0}$$

In particular, $L_v w$ is defined when w is a vector field.

Proposition. *For all vector fields v, w on M, we have $L_v w = [v, w]$.*

Lemma (Leibniz rule for sections of bundles)**.** *Let v be a vector field.*

1. $L_v(\omega \wedge \eta) = (L_v\omega) \wedge \eta + \omega \wedge L_v(\eta)$, *where ω is a k-forma and η is an ℓ-form*
2. $L_v i_w\theta = i_{L_v(w)}\theta + i_w L_v\theta$¡ *where w is a vector field, and θ is a k-form*
3. $v(\theta(w)) = \theta(L_v(w) + (L_v\theta)(w)$, *where θ is a 1-form (this is just a special case of 2)*

Proof. Let V_1, V_2, V_3 be vector bundles on M, and let B be a bilinear map

$$\begin{array}{ccc} V_1 \times_M V_2 & \xrightarrow{\quad B \quad} & V_3 \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

i.e. $B(x) : V_1(x) \times V_2(x) \to V_3(x)$ is bilinear for all $x \in M$. Let s_t^1, s_t^2 be families of C^∞ sections of V_1 and V_2 respectively, indexed by $t \in (-\epsilon, \epsilon)$. Let p_1 be the projection $p_1 : M \times (-\epsilon, \epsilon) \to M$, so that each s_i is a secton of $p_1^* V_i$. Then

$$\frac{d}{dt}B(s_t^1, s_t^2) = B\left(\frac{d}{dt}\, s_t^1, s_t^2\right) + B\left(s_t^1, \frac{d}{dt}\, s_t^2\right).$$

How will we apply this - we want to choose $s_i = \phi_t^*(?)$.

Let $V_1 = TM$, $V_2 = \Lambda^k T^*M$, $V_3 = \Lambda^{k-1}T^*M$, and let $B(x) : T_xM \times \Lambda^k T_x^*M \to \Lambda^{k1-}T_x^*M$ be defined by $B(x)(\omega, \theta) = i_\omega(\theta)$.

Part 2 is then an application of the Leibniz rule

$$i_{v^*}(\omega \wedge \eta) = i_{v^*}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge i_{v^*}(\eta)$$

where $v^* \in V^*$ and $\omega \in \Lambda^k V$, and 3 is just 2 for $k = 1$.

Given $\theta = df$, where $f : M \to \mathbb{R}$ is a C^∞ map, then

$$\theta(w) = (df)(w) = w(f)$$

That $v(\theta(w)) = v(w(f))$ is just the left side of 3. But

$$\theta(L_v w) = (L_v w)(f),$$

hence

$$L_v(\theta) = L_v(df) = dL_v f = d(v(f)),$$

hence

$$(L_v\theta)(w) = w(v(f)).$$

Now 3 reads as

$$v(w(f)) = w(v(f)) + (L_v w)(f),$$

i.e.

$$(L_v w)(f) = v(w(f)) - w(v(f)) = [v,w](f)$$

for all C^∞ maps $f : M \to \mathbb{R}$. ☐

Corollary (Special case of Cartan's formula)**.** *Let ω be a 1-form, and let v_1 and v_2 be vector fields. Then*

$$d\omega(v_1, v_2) = v_1(\omega(v_2)) - v_2(\omega(v_1)) - \omega([v_1, v_2])$$

Remark. Note that we can identify $\Lambda^k T_x^* M$ with $(\Lambda^k T_x M)^*$ as follows: given $\omega \in \Lambda^k T_x^* M$, we define

$$\omega(v_1, v_2, \ldots, v_k) = i_{v_k} i_{v_{k-1}} \cdots i_{v_1} \omega \in \Lambda^k T_x^* M \in \mathbb{R}$$

for $v_1, \ldots, v_k \in T_x M$.

Proof. We have that $L_v = i_v d + d i_v$. Thus,

$$i_{v_1} d\omega = L_{v_1}\omega - d(i_{v_1}\omega),$$

so that

$$\begin{aligned} d\omega(v_1, v_2) &= (i_{v_1} d\omega) v_2 = (L_{v_1}\omega) v_2 - v_2(\omega(v_1)) \\ &= L_{v_1}(\omega(v_2)) - \omega(L_{v_1} v_2) - v_2(\omega(v_1)) \\ &= v_1(\omega(v_2)) - \omega([v_1, v_2]) - v_2(\omega(v_1)). \end{aligned}$$

☐

Remark. We defined

$$L_v\omega = \frac{d}{dt}\,\phi_t^*\omega\bigg|_{t=0}.$$

It is more generally true that

$$\frac{d}{dt}\,\phi_t^*(\omega)\bigg|_{t=t_0} = \phi_{t_0}^*(L_v\omega).$$

Note that we haven't said what kind of thing ω is; it only makes sense for certain natural bundles. But this works in particular when ω is some vector field w. Then ϕ_t is the flow associated to v; also, let ψ_s be the flow associated to w. Then

$$[v, w] = 0 \iff L_v(w) = 0 \iff \frac{d}{dt}(\phi_t^* w) = 0 \text{ for all } t \iff \phi_t^* w = w \text{ for all } t.$$

Assume that $[v, w] = 0$, so that $\phi_t^* w = w$, and thus ϕ_t(integral curve of w) is an integral curve of w. This is equivalent to saying that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. Thus, we have established the following:

Corollary. $[v, w] = 0 \iff \phi_t \circ \psi_s = \psi_s \circ \phi_t$

Theorem (Ehresmann's theorem)**.** *Let $f : X \to Y$ be a proper submersion. Then f is a C^∞ fiber bundle.*

We will give a second proof of this using flows.

Proof. Let $X \hookrightarrow \mathbb{R}^m$ be an embedding of X in Euclidean space. Thus, for any $x \in X$, $T_x X$ gets an inner product. We have a short exact sequence

$$0 \longrightarrow T_x f^{-1}(f(x)) \longrightarrow T_x X \xrightarrow{f'(x)} T_{f(x)} Y \longrightarrow 0$$

where we have used that f is a submersion. Let $W(x) = T_x f^{-1}(f(x))^{\perp}$, so that we get a subbundle W of TX such that $f'(x) : W(x) \xrightarrow{\cong} T_{f(x)} Y$.

Assume that $Y = (-1,1)^n \subset \mathbb{R}^n$. Thne $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ are vector fields on Y, i.e. sections of TY, and so we get corresponding sections $w_1, \dots, w_n$ of W such that $f'(x)(w_i) = \frac{\partial}{\partial y_i}$ for all i. Note that even though the $\frac{\partial}{\partial y_i}$ all commute with each other, we need not have that the w_i all commute with each other.

Let ϕ_t^i denote the flow associated to w_i. One sees that for any compact $K \subseteq X$, there is an $\epsilon > 0$ such that $\phi_{t_1}^1 \cdots \phi_{t_n}^n(x)$ are defined for all $x \in K$ and $t_i \in (-\epsilon, \epsilon)$.

Let $K = f^{-1}(0)$, which is compact because f is proper. Then we have a commutative diagram

$$\begin{array}{ccc} K \times (-\epsilon, \epsilon)^n & \xrightarrow{h} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ (-\epsilon, \epsilon)^n & \hookrightarrow & (-1,1)^n \end{array}$$

and h induces isomorphisms on tangent spaces at $K \times 0$, so it must do so in a neighborhood of $K \times 0$. Because p_2 is proper, it follows that by shrinking ϵ if necessary, we may assume that h induces isomorphisms on tangent spaces everywhere, and that h is one-to-one. Then h is then a diffeomorphism onto its image U, which is open in X. We want to show that $U = X$; thus, let $F = X \setminus U$. Then F is closed in X, and because f is proper, we have that $f(F)$ is a closed set (we're using Hausdorffness here). Then $F \cap f^{-1}(0) = \varnothing$, because $0 \notin f(F)$, and now replace $(-1,1)^n$ by the complement of $f(F)$. $\square$

We can now state a refinement of Ehresmann's theorem.

Theorem. *Let $f : X \to Y$ be a proper submersion, and let $A \subseteq X$ be a closed C^∞ submanifold. Assume also that $f|_A : A \to Y$ is a submersion. Then $f : (X, A) \to Y$ is a fiber-bundle pair.*

2

Lecture 7 (2013-03-01)

Last time, we were discussing the Ehresmann theorem for fiber bundles of pairs. There was just one thing left to prove.

In the notation of the last lecture, we had C^∞ manifolds X and Y, a closed C^∞ submanifold $A \subseteq X$, and a C^∞ map $f : X \to Y$ such that both f and $f|_A$ are submersions. (Note that for the Ehresmann theorem, we would assume properness, but for now we just want to extract the subbundle W which did not need that hypothesis.)

Proposition. *There exists a subbundle $W \subset TX$ such that*

(i) For all $x \in X$, the derivative $f'(x)|_{W(x)} : W(x) \to T_{f(x)}Y$ is an isomorphism.

(ii) For all $x \in A$, we have $W(x) \subset TA$ (both interpreted as subspaces of T_xX).

This proposition implies the Ehresmann theorem for pairs.

The secret code phrase here is that

$$H^1(\text{any sheaf of modules over the sheaf of } C^\infty \text{ functions}) = 0$$

Proof. For the first step, note that the problem makes sense on any open $U \subset X$, so it will suffice to show that W exists locally, i.e. that for all $x \in X$, there is a neighborhood $U(x)$ where the theorem holds.

If $x \notin A$, then we're done, so suppose that $x \in A$. WLOG, we can take $X = \mathbb{R}^n$, $A = \{x \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}$, and $f : X \to Y$ the map $f(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$ where $r \leq m$. In this case, we can just take W to be the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}$.

Now to Step 2; we want to provide an algebraic description of W. This is essential. We can't add subbundles, but we can add / do other linear things to sections of bundles.

For each $x \in X$, let $s(x)$ be the inverse of the isomorphism described in (i); in other words, we want to demonstrate the existence of a map of bundles $s : f^*TY \to TX$ such that

(i') $f'(x) \circ s(x) : T_{f(x)}Y \to T_{f(x)}Y$ is the identity for all $x \in X$

(ii') For all $x \in A$, we have $s(x)(T_{f(x)}Y) \subseteq T_xA$.

Step 3: Suppose that s_1 and s_2, both maps $f^*TY \to TX$, satisfy conditions (i') and (ii'). Then $h = s_2 - s_1 : f^*TY \to TX$ satisfies

(i'') $f'(x) \circ h(x) = 0$ for all $x \in X$

(ii'') $h(x)(T_{f(x)}Y) \subseteq T_xA$ for all $x \in A$

so that

$$Z = \{h : f^*TY \to TX \mid \text{(i'') and (ii'') hold}\}$$

is a module over the ring of C^∞ functions on X. Note that this is a characterization; in other words, if s_1 satisfies (i') and (ii'), then $s_1 + h$ satisfies them if and only if $h \in Z$.

As a corollary of Step 3, we see that if $s_1, \ldots, s_m$ are as in Step 2, and $\varphi_1, \ldots, \varphi_m : W \to \mathbb{R}$ are a C^∞ partition of unity (so that $\sum \varphi_i = 1$), then $\sum \varphi_i s_i$ also satisfies the conditions of step 2, because

$$\sum \varphi_i s_i = \underbrace{\sum \varphi_i (s_i - s_1)}_{\in Z} + \underbrace{\left(\sum \varphi_i\right)}_{=1} s_1.$$

Now we come to the proof of the proposition itself. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover equipped with $s_\lambda : f^*TY|_{U_\lambda} \to TX|_{U_\lambda}$ all satisfying (i') and (ii'). There is a partition of unity φ_λ subordinate to U_λ; then $\varphi_\lambda s_\lambda$ (originally defined only on U_λ) can be extended by 0 to a C^∞ map $\varphi_\lambda s_\lambda : f^*TY \to TX$. Now let $s = \sum \varphi_\lambda s_\lambda : f^*TY \to TX$; the corollary above implies that s satisfies (i') and (ii'). □

Existence of inner products on vector bundles

Given a C^∞ vector bundle $f : V \to M$, we want to construct a map $B : V \times_M V \to \mathbb{R}$ such that $B : V(x) \times V(x) \to \mathbb{R}$ is a positive definite, symmetric, bilinear form.

If W is a vector space, and $B : W \times W \to \mathbb{R}$ is symmetric and bilinear, we say that B is positive semi-definite if $B(w, w) \geq 0$ for all $w \in W$, and positive definite if it is positive semi-definite and $B(w, w) = 0$ implies $w = 0$.

Proof. Step 1. Assume that $V|_U$ is a trivial bundle, i.e. there exist sections $s_1, \ldots, s_k$ of $V|_U$ such that $s_1(x), \ldots, s_k(x)$ form a basis for $V(x)$ for all $x \in U$.

Define $B_U(s_i(x), s_j(x)) = \delta_{ij}(x)$. Given an open cover $\mathcal{U}$, and a partition of unity φ_U subordinate to $\mathcal{U}$, then $\sum \varphi_U B_U$ is a symmetric bilinear positive semi-definite form. But for any $x \in X$, if $v \in V(x)$ is non-zero, then there is some U such that $\varphi_U(x) > 0$, so that $x \in U$ and moreover $B_U(v, v) > 0$, hence $B(v, v) \geq \varphi_U(x)B_U(v, v) > 0$. Thus, this is in fact positive definite. □

Existence of connections on a vector bundle

A good reference for this is Milnor's *Morse Theory*.

Let $p : V \to M$ be a C^∞ vector bundle. A connection is essentially a way of taking a derivative of a section s of a vector bundle v with respect to a vector field on M.

Suppose that $x \in U$ and that $V|_U$ is trivial, and that $s_1, \ldots, s_k$ are sections of $V|_U$ that give a basis for $V(x)$ for each $x \in U$. For any $v \in T_xM$, we define

$$v\left(\sum f_i s_i\right) = \sum v(f_i)s_i.$$

A connection, or a covariant derivative, ∇ on V is a map taking in a vector field v on M, and a section s of V, and outputting $\nabla_v s$, another section of V. We also require that a connection satisfy certain properties: for any C^∞ map $f : M \to \mathbb{R}$,

1. $\nabla_v(s_1 + s_2) = \nabla_v(s_1) + \nabla_v(s_2)$
2. $\nabla_v(fs) = v(f)s + f\nabla_v(s)$ (this is the Leibniz rule)
3. $\nabla_{fv}(s) = f\nabla_v(s)$

We could have stated this definition sheaf-theoretically, which is after all necessary to do it on analytic manifolds, but for C^∞ manifolds, they are equivalent.

We want to show that any C^∞ vector bundle $V \to M$ has a connection.

The argument is the same as we've been doing. Step 1 is to show that they exist locally (this is just the trivial connection). Step 2 is to take two connections ∇^1, ∇^2 and define h via $\nabla^2 = \nabla^1 + h$, i.e. $\nabla^2_v(s) = \nabla^1_v(s) + h_v s$ for all sections s, and note that h satisfies three properties: h is additive in s,

$$h_v(fs) = fh_v(s)$$

for all C^∞ functions f, and $h_{fv}(s) = fh_v(s)$.

Then, if ∇^1 is a connection and $\nabla^2 = \nabla^1 + h$, then ∇^2 is a connection if and only if h satisfies the above three properties. The collection of all such h can be thought of being comprised of precisely the sections of $\mathrm{Hom}(TM, \mathrm{End}(V))$, which is a module over C^∞ functions $M \to \mathbb{R}$.

We then conclude by using a partition of unity and noting that $\sum \varphi_U \nabla_U$ gives a connection.

Let's examine connections in a basic case; let M be an open interval (a,b). By the properties of a connection, all we have to look at is $\nabla_{\frac{d}{dt}}(s)$. In particular, what is

$$\{\text{sections } s : M \to V \mid \nabla_{\frac{d}{dt}}(s) = 0\} \quad ?$$

We know that V is trivial because we're working on an interval; choose a specific trivialization, so that we will think of sections as maps $s : (a,b) \to \mathbb{R}^k$. Define vectors of C^∞ functions m_i by

$$\nabla_{\frac{d}{dt}}(e_i) = m_i,$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

the 1 being in the ith position. Then