Geom/Top: Homework 8 (due Monday, 12/03/12)

1. Read Farb notes.

2. Read along in Hatcher.

1. Hatcher, §1.1, Problems 5, 6, 7, 16(c)(f).

2. Let $X$ be a path-connected space, and let $x \in x$. Let $\phi : \pi_1(X,x) \to H_1(X,\mathbb{Z})$ be the map $\phi([\gamma]) := \gamma_*(S^1)$, where we think of $\gamma : S^1 \to X$. Prove that $\phi$ is surjective, and that the kernel of $\phi$ is precisely the commutator subgroup of $\pi_1(X,x)$.

3. Let $G$ be a path-connected topological group, and suppose that $\pi_1(G) = 0$. Let $\Gamma$ be a discrete normal subgroup of $G$. Prove that $\pi_1(G/\Gamma) \approx \Gamma$. Note that the case $G = \mathbb{R}, \Gamma = \mathbb{Z}$ gives another proof that $\pi_1(S^1) \approx \mathbb{Z}$.

4. Now let $G$ be any connected topological group, and let $\Gamma$ be any discrete normal subgroup. Prove that $\Gamma$ is central in $G$ (i.e. each $g \in \Gamma$ commutes with every element of $G$). Deduce that $\pi_1(G/\Gamma)$ is abelian. In particular this proves that $\pi_1(G)$ is abelian.

5. Let $p : S^n \to \mathbb{R}P^n$ be the standard quotient map sending $v$ to $\{\pm v\}$. Prove by hand, just as in the proof that $\pi_1(S^1) \approx \mathbb{Z}$, that $\pi_1(\mathbb{R}P^2) \approx \mathbb{Z}/2\mathbb{Z}$.