Geom/Top: Homework 7 (due Monday, 11/26/12)

1. Read Farb notes.
2. Read along in Hatcher.

---

1. Recall that a *bilinear pairing* of groups or vector spaces is a linear map $V \times W \to \mathbf{R}$ which is *nondegenerate* in the sense that for each nonzero $v \in V$, the map $v \mapsto < v, - >$ is an isomorphism $V \to W^*$.

   Give another proof of (an equivalent form of) Poincaré Duality by proving that cup product gives a nondegenerate pairing

   $$H^i(M) \times H_c^{n-i} \to H_c^n(M)$$

   so that $H_c^{n-i} \approx (H^i)^*$.

2. Let S_g denote the closed, orientable surface of genus g. Prove that if $g < h$ then any continuous map $f : S_g \to S_h$ has degree 0. [Hint: Measure degree via the action on $H^2(S_g; \mathbf{Z})$. Prove that there exists $a \in H^1(S_h; \mathbf{Z})$ with $f^*a = 0$. On the other show that there exists b with $a \cup b \neq 0$.]

3. Let M be a compact, connected, nonorientable 3-manifold. Prove that $H_1(M)$ is infinite.

4. Look up Poincaré Duality for manifolds with boundary. Prove that if M is the boundary of some compact connected manifold, then $\chi(M)$ is even. Give an example of a closed manifold that is not the boundary of any compact manifold.