Geom/Top: Homework 4 (due Monday, 10/29/12)

1. Read Farb notes.

2. Read along in Hatcher.

1. Let $X$ be a connected space which is a finite union of polygons $\{P_i\}$ with the property that the intersection of any two polygons is either empty, a common edge, or a common vertex. Let $V, E$ and $F$ denote the total number of vertices, edges, and faces of these polygons (so e.g. a face is precisely the interior of some polygon).
   (a) Prove that $\chi(X) = V - E + F$.
   (b) The space $X$ is called a regular polytope if all the $P_i$ have the same number $r$ of edges, if each edge in $X$ lies in exactly two faces, and if each vertex in $X$ lies in some fixed number $s$ of faces. Prove that there are precisely five regular polytopes homeomorphic to $S^2$.
   (c) What are the possibilities for $r$ and $s$ when $X$ is homeomorphic to the torus $T^2$?

2. Let $X$ be any finite, connected $\Delta$-complex. Prove that $f_* : H_0(X) \rightarrow H_0(X)$ is the identity homomorphism for any continuous map $f : X \rightarrow X$.

3. Let $\Sigma_g$ denote the closed, connected, genus $g \geq 0$ surface.
   (a) For each $g \neq 1$, find a homeomorphism $f : \Sigma_g \rightarrow \Sigma_g$ with no fixed point.
   (b) Prove that for $g \neq 1$ any homeomorphism $f : \Sigma_g \rightarrow \Sigma_g$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

4. Let $G$ be a path-connected, compact topological group. That is, $G$ is a group and also a compact topological space, such that the maps $G \times G \rightarrow G$ with $(a, b) \mapsto ab$ and $G \rightarrow G$ with $g \mapsto g^{-1}$ are continuous. Assume that $G$ has some $\Delta$-complex structure.
   (a) For any $g \in G$, let $L_g : G \rightarrow G$ be “left translation by $g$”, i.e. $L_g(h) = gh$. Prove that $L_g$ is homotopic to the identity.
   (b) Conclude that $\chi(G) = 0$.
   (c) Prove that $S^{2n}, n > 0$ cannot be given the structure of a topological group.
   (d) Prove that the only compact surface which is a topological group is the torus; in particular rule out the Klein bottle. [Warning: I am not 100% sure the Klein bottle part can be solved at this point.]

5. Give an example of a finite $\Delta$-complex $X$ and a continuous self-map $f : X \rightarrow X$ such that $f$ has a fixed point but the Lefschetz number $\Lambda(f) = 0$. Thus the converse of the Lefschetz Fixed Point Theorem does not hold.
6. Vector fields \( \{ V_i \} \) on \( S^n \) are **linearly independent** if for each \( z \in S^n \) the vectors \( \{ V_i(z) \} \) are linearly independent in the vector space \( TS^n \).

(a) Recall that in class we found a nonvanishing vector field on \( S^{2n+1} \). Adapt this construction to give \( 3 \) linearly independent nonvanishing vector fields on \( S^{4n+3} \).

(b) Construct \( 7 \) linearly independent vector fields on \( S^7 \).

(c) Generalize these constructions to produce the maximal number of possible linearly independent vector fields on \( S^n \) for each \( n \), where the upper bound is given by Adams's Theorem.

7. Think of \( S^n \) as the set of unit vectors \( v \) in \( \mathbb{R}^{n+1} \). Consider the question: when does there exist a continuous map \( f : S^m \to S^n \) satisfying \( f(-v) = -f(v) \), that is, preserving the property of points being antipodal to each other. Note that when \( m < n \) this is trivial: just let \( f : S^m \to S^n \) be a standard inclusion.

**Theorem (Borsuk-Ulam):** When \( m > n \) there does not exist any continuous map \( f : S^m \to S^n \) satisfying \( f(-v) = -f(v) \) for all \( v \in S^m \). Assume for now this theorem.

(a) Prove that any continuous map \( f : S^n \to \mathbb{R}^n \) there exists \( v \in S^n \) with \( f(v) = f(-v) \). This statement is the more frequently stated "Borsuk-Ulam Theorem". Deduce that \( S^n \) does not embed in \( \mathbb{R}^n \).

(b) Use (a) to prove invariance of dimension: \( \mathbb{R}^m \approx \mathbb{R}^n \) implies \( m = n \).

(c) Prove the following: Let \( \{ X_1, \ldots, X_{n+1} \} \) be a covering of \( S^n, n > 0 \) by closed sets. Then some \( X_i \) contains a pair of antipodal points. [Hint: Consider the map \( f : S^n \to \mathbb{R}^n \) given by

\[
f(v) = (d(v, X_1), \ldots, d(v, X_n))
\]

where \( d(v, X_i) \) denotes the closest distance in \( S^n \) from \( v \) to some any point in the set \( X_i \).]

8. Recall the Fundamental Theorem of Algebra: Every polynomial \( P(x) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) with \( a_i \in \mathbb{C} \) has a zero in \( \mathbb{C} \). Prove this theorem as follows: Let \( S_r \) denote the circle of radius \( r \) in \( \mathbb{C} \). Suppose \( P \) has no zero inside \( S_r \). Then we can think of the restriction \( f = P|_{S_r} \) of \( P \) to \( S_r \) as a continuous map \( f : S_r \to \mathbb{C} - \{0\} \).

(a) Prove that \( f_* : H_1(S_r) \to H_1(\mathbb{C} - \{0\}) \) is trivial.

(b) Prove that for \( r \) sufficiently large, \( f \) is homotopic to to the map \( z \mapsto z^n \). [Hint: Let \( F_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0) \).]

(c) Derive a contradiction.

**Extra Credit Problems**
1. Let $X$ be any finite $\Delta$-complex with $\chi(X) \neq 0$. Prove that any homeomorphism $f : X \to X$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

2. Prove that if $M$ is a closed manifold that admits a nonvanishing vector field, then $\chi(M) = 0$. 