Geom/Top: Homework 4 (due Monday, 10/29/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Let X be a connected space which is a finite union of polygons $\{P_i\}$ with the property that the intersection of any two polygons is either empty, a common edge, or a common vertex. Let V, E and F denote the total number of vertices, edges, and faces of these polygons (so e.g. a face is precisely the interior of some polygon).

   (a) Prove that $\chi(X) = V - E + F$.

   (b) The space X is called a **regular polytope** if all the P_i have the same number r of edges, if each edge in X lies in exactly two faces, and if each vertex in X lies in some fixed number s of faces. Prove that there are precisely five regular polytopes homeomorphic to S^2.

   (c) What are the possibilities for r and s when X is homeomorphic to the torus T^2?

2. Let X be any finite, connected Δ-complex. Prove that $f_* : H_0(X) \to H_0(X)$ is the identity homomorphism for any continuous map $f : X \to X$.

3. Let Σ_g denote the closed, connected, genus $g \geq 0$ surface.

   (a) For each $g \neq 1$, find a homeomorphism $f : \Sigma_g \to \Sigma_g$ with no fixed point.

   (b) Prove that for $g \neq 1$ any homeomorphism $f : \Sigma_g \to \Sigma_g$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

4. Let G be a path-connected, compact **topological group**. That is, G is a group and also a compact topological space, such that the maps $G \times G \to G$ with $(a, b) \mapsto ab$ and $G \to G$ with $g \to g^{-1}$ are continuous. Assume that G has some Δ-complex structure.

   (a) For any $g \in G$, let $L_g : G \to G$ be "left translation by g", i.e. $L_g(h) = gh$. Prove that L_g is homotopic to the identity.

   (b) Conclude that $\chi(G) = 0$.

   (c) Prove that $S^{2n}, n > 0$ cannot be given the structure of a topological group.

   (d) Prove that the only compact surface which is a topological group is the torus; in particular rule out the Klein bottle. [Warning: I am not 100% sure the Klein bottle part can be solved at this point.]

5. Give an example of a finite Δ-complex X and a continuous self-map $f : X \to X$ such that f has a fixed point but the Lefschetz number $\Lambda(f) = 0$. Thus the converse of the Lefschetz Fixed Point Theorem does not hold.

6. Vector fields $\{V_i\}$ on S^n are **linearly independent** if for each $z \in S^n$ the vectors $\{V_i(z)\}$ are linearly independent in the vector space TS^n_z.

   (a) Recall that in class we found a nonvanishing vector field on S^{2n+1}. Adapt this construction to give 3 linearly independent nonvanishing vector fields on S^{4n+3}.

   (b) Construct 7 linearly independent vector fields on S^7.

   (c) Generalize these constructions to produce the maximal number of possible linearly independent vector fields on S^n for each n, where the upper bound is given by Adams's Theorem.

7. Think of S^n as the set of unit vectors v in $\mathbf{R}^{n+1}$. Consider the question: when does there exist a continuous map $f : S^m \to S^n$ satisfying $f(-v) = -f(v)$, that is, preserving the property of points being antipodal to each other. Note that when $m < n$ this is trivial: just let $f : S^m \to S^n$ be a standard inclusion.

   Theorem (Borsuk-Ulam): *When $m > n$ there does not exist any continuous map $f : S^m \to S^n$ satisfying $f(-v) = -f(v)$ for all $v \in S^m$.*

   Assume for now this theorem.

   (a) Prove that any continuous map $f : S^n \to \mathbf{R}^n$ there exists $v \in S^n$ with $f(v) = f(-v)$. This statement is the more frequently stated "Borsuk-Ulam Theorem". Deduce that S^n does not embed in $\mathbf{R}^n$.

   (b) Use (a) to prove invariance of dimension: $\mathbf{R}^m \approx \mathbf{R}^n$ implies $m = n$.

   (c) Prove the following: Let $\{X_1, \ldots, X_{n+1}\}$ be a covering of $S^n, n > 0$ by closed sets. Then some X_i contains a pair of antipodal points. [Hint: Consider the map $f : S^n \to \mathbf{R}^n$ given by

   $$f(v) = (d(v, X_1), \ldots, d(v, X_n))$$

   where $d(v, X_i)$ denotes the closest distance in S^n from v to some any point in the set X_i.]

8. Recall the Fundamental Theorem of Algebra: Every polynomial $P(x) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_i \in \mathbf{C}$ has a zero in $\mathbf{C}$. Prove this theorem as follows: Let S_r denote the circle of radius r in $\mathbf{C}$. Suppose P has no zero inside S_r. Then we can think of the restriction $f = P|_{S_r}$ of P to S_r as a continuous map $f : S_r \to \mathbf{C} - 0$.

   (a) Prove that $f_* : H_1(S_r) \to H_1(\mathbf{C} - 0)$ is trivial.

   (b) Prove that for r sufficiently large, f is homotopic to to the map $z \mapsto z^n$. [Hint: Let $F_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$.]

   (c) Derive a contradiction.

Extra Credit Problems

1. Let X be any finite Δ-complex with $\chi(X) \neq 0$. Prove that any homeomorphism $f : X \to X$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

2. Prove that if M is a closed manifold that admits a nonvanishing vector field, then $\chi(M) = 0$.