Geom/Top: Homework 2 (due Monday, 10/15/12)

1. Suppose we have a homomorphism of short exact sequences of abelian groups:

$$\begin{array}{ccccc} 0 \to & A \to & B \to & C \to & 0 \\ & \downarrow f & \downarrow g & \downarrow h & \\ 0 \to & A' \to & B' \to & C' \to & 0 \end{array}$$

   Prove that there is an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \mathrm{cok}(f) \to \mathrm{cok}(g) \to \mathrm{cok}(h) \to 0$$

2. Let X be any topological space, and let $A \subseteq X$ be a nonempty subspace.

   (a) Prove directly (i.e. not using any long exact sequences, but only the definitions), that if $x \in X$ is any point, then

$$H_i(X, x) \approx \tilde{H}_i(X) \quad \text{for all } i \geq 0$$

   (b) Suppose that there is a retraction $r : X \to A$. Show that

$$H_i(X) \approx H_i(A) \oplus H_i(X, A)$$

   for all $i \geq 0$.

   (c) Prove that if the inclusion $i : A \to X$ is a homotopy equivalence then then $H_i(X, A) = 0$ for all $i \geq 0$.

   (d) Prove that the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_i(X, A) = 0$ for all $i \geq 0$.

3. Give an example of pairs of spaces (X, X_0) and (Y, Y_0) where $H_i(X) \approx H_i(Y)$ and $H_i(X_0) \approx H_i(Y_0)$ but $H_i(X, X_0) \not\approx H_i(Y, Y_0)$.

4. Compute the following examples of relative homology groups of the following pairs (X, A) using the long exact sequence of a pair. Note how the answer compares to the homology of the quotient X/A.

   (a) An annulus relative to its boundary (union of 2 circles).

   (b) A Mobius band relative to its boundary. Give a generator for relative H_1 in this case.

   (c) (S^n, S^{n-1}) for each $n \geq 0$.

5. Prove for an n-manifold ($n \geq 1$), and any point $x \in X$, that

$$H_i(X, X - x) = \begin{cases} \mathbf{Z} & i = 0, n \\ 0 & i \neq 0, n \end{cases}$$

6. Let $\gamma \subset S^2$ be a subset homeomorphic to $[0,1]$, and let $x \in \gamma$ be any point. Prove that the inclusion of pairs $(S^2 - x, \gamma - x) \to (S^2, \gamma)$ is not an excision.

7. Check that the "connecting homomorphism" $\partial : H_n(X, A) \to H_n(A)$ in the long exact sequence of a pair (X, A), defined purely via homological algebra, has a geometric meaning: if $c \in Z_n(X, A)$ is a cycle representing an element of $H_n(X, A)$, then the element $\partial[c] \in H_{n-1}(A)$ is represented by the cycle $\partial c \in Z_{n-1}(A)$.

8. Assuming that G is abelian, what can you say about the isomorphism type of G if

   (a) $0 \to \mathbf{Z}^n \to G \to \mathbf{Z}^m \to 0$ is exact?

   (b) $0 \to \mathbf{Z}/4\mathbf{Z} \to G \to \mathbf{Z}/4\mathbf{Z} \to 0$ is exact?

9. Let X be any space, and let

$$0 \to G \to G' \to G^{''} \to 0$$

   be a short exact sequence of abelian groups.

   (a) Prove that there is a short exact sequence of chain complexes (with varying coefficients):

$$0 \to C_n(X; G) \to C_n(X; G') \to C_n(X; G^{''} \to 0$$

   By the Fundamental Theorem of Homological Algebra one gets the induced long exact sequence of homology groups. Denote the associated "connecting homomorphism" by

$$\beta : H_n(X; G^{''}) \to H_{n-1}(X; G)$$

   (b) Compute β when X is the Klein bottle and the coefficient sequence is

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

   (c) Do the same for the coefficients

$$0 \to \mathbf{Z}/2\mathbf{Z}/ \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

Extra Credit Problems

1. You can still hand in any extra credit problems from Homeworks 1 and 2.