1. Suppose we have a homomorphism of short exact sequences of abelian groups:

\[
0 \to A \to B \to C \to 0 \quad \downarrow f \quad \downarrow g \quad \downarrow h \\
0 \to A' \to B' \to C' \to 0
\]

Prove that there is an exact sequence

\[
0 \to \ker(f) \to \ker(g) \to \ker(h) \to \cok(f) \to \cok(g) \to \cok(h) \to 0
\]

2. Let \( X \) be any topological space, and let \( A \subseteq X \) be a nonempty subspace.
   (a) Prove directly (i.e. not using any long exact sequences, but only the definitions), that if \( x \in X \) is any point, then

\[
H_i(X, x) \approx \tilde{H}_i(X) \quad \text{for all } i \geq 0
\]

   (b) Suppose that there is a retraction \( r : X \to A \). Show that

\[
H_i(X) \approx H_i(A) \oplus H_i(X, A)
\]

   for all \( i \geq 0 \).

   (c) Prove that if the inclusion \( i : A \to X \) is a homotopy equivalence then then \( H_i(X, A) = 0 \) for all \( i \geq 0 \).

   (d) Prove that the inclusion \( A \to X \) induces isomorphisms on all homology groups if and only if \( H_i(X, A) = 0 \) for all \( i \geq 0 \).

3. Give an example of pairs of spaces \((X, X_0)\) and \((Y, Y_0)\) where \( H_i(X) \approx H_i(Y) \) and \( H_i(X_0) \approx H_i(Y_0) \) but \( H_i(X, X_0) \neq H_i(Y, Y_0) \).

4. Compute the following examples of relative homology groups of the following pairs \((X, A)\) using the long exact sequence of a pair. Note how the answer compares to the homology of the quotient \( X/A \).
   (a) An annulus relative to its boundary (union of 2 circles).
   (b) A Mobius band relative to its boundary. Give a generator for relative \( H_1 \) in this case.
   (c) \((S^n, S^{n-1})\) for each \( n \geq 0 \).

5. Prove for an \( n \)-manifold \((n \geq 1)\), and any point \( x \in X \), that

\[
H_i(X, X - x) = \begin{cases} 
\mathbb{Z} & i = 0, n \\
0 & i \neq 0, n
\end{cases}
\]
6. Let \( \gamma \subset S^2 \) be a subset homeomorphic to \([0, 1]\), and let \( x \in \gamma \) be any point. Prove that the inclusion of pairs \((S^2 - x, \gamma - x) \rightarrow (S^2, \gamma)\) is not an excision.

7. Check that the “connecting homomorphism” \( \partial : H_n(X, A) \rightarrow H_n(A) \) in the long exact sequence of a pair \((X, A)\), defined purely via homological algebra, has a geometric meaning: if \( c \in Z_n(X, A) \) is a cycle representing an element of \( H_n(X, A) \), then the element \( \partial[c] \in H_{n-1}(A) \) is represented by the cycle \( \partial c \in Z_{n-1}(A) \).

8. Assuming that \( G \) is abelian, what can you say about the isomorphism type of \( G \) if 
   (a) \( 0 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow \mathbb{Z}^m \rightarrow 0 \) is exact?
   (b) \( 0 \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow G \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow 0 \) is exact?

9. Let \( X \) be any space, and let
   \[
   0 \rightarrow G \rightarrow G' \rightarrow G'' \rightarrow 0
   \]
   be a short exact sequence of abelian groups.
   (a) Prove that there is a short exact sequence of chain complexes (with varying coefficients):
   \[
   0 \rightarrow C_n(X; G) \rightarrow C_n(X; G') \rightarrow C_n(X; G'') \rightarrow 0
   \]
   By the Fundamental Theorem of Homological Algebra one gets the induced long exact sequence of homology groups. Denote the associated “connecting homomorphism” by
   \[
   \beta : H_n(X; G'') \rightarrow H_{n-1}(X; G)
   \]
   (b) Compute \( \beta \) when \( X \) is the Klein bottle and the coefficient sequence is
   \[
   0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0
   \]
   (c) Do the same for the coefficients
   \[
   0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0
   \]

**Extra Credit Problems**

1. You can still hand in any extra credit problems from Homeworks 1 and 2.