Geom/Top: Homework 2 (due Monday, 10/15/12)

1. Read Farb notes.
2. Read Hatcher, Section 2.1.
3. (Don't hand in): Check that chain homotopy is an equivalence relation on the set of chain maps.

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1. Some problems on contractibility:

   (a) The **Dunce Hat** is the quotient space obtained from the standard 2-simplex $[v_0v_1v_2]$ by identifying $[v_0v_1]$ with $[v_0v_2]$ with $[v_1v_2]$ (note that the ordering matters!). Prove that the Dunce Hat is contractible.

   (b) Prove that the retract of a contractible space is contractible.

   (c) Prove that S^∞ is contractible.

   (d) Prove that the torus minus one point deformation retracts to $S^1 \vee S^1$.

   (e) Prove that a space Y is contractible if and only if for every space X, any two continuous maps $f, g : X \to Y$ are homotopic. Prove the same theorem but replacing $f : X \to Y$ with $f : Y \to X$.

2. Let $f, g : S^n \to S^n, n > 0$ be continuous maps with the property that $f(x)$ and $g(x)$ are not antipodal for any x. Prove that f and g are homotopic.

3. Hatcher page 19, Problem 20.

4. Let X be a space. The **suspension** SX is defined as the quotient space

$$SX := \frac{X \times [0,1]}{(X \times \{0\}) \sqcup (X \times \{1\})}$$

   (a) Let X be a Δ-complex. Prove that SX is a Δ-complex.

   (b) Prove that $\tilde{H}_i(X) \approx \tilde{H}_{i+1}(SX)$ for all $i \geq 0$.

5. Think of S^1 as the unit circle in the complex plane. Let $f, g : S^1 \to S^1$ be the maps $f(z) = z$ and $g(z) = z^2$.

   (a) Prove that f is not homotopic to g.

   (b) Let $F(z,t) : S^1 \times [0,1] \to S^1$ be defined by $F(z,t) = z^{t+1}$. Why isn't F a homotopy from f to g?

   (c) Classify all continuous maps $f : S^1 \to S^1$ up to homotopy.

6. Let X be any topological space.

   (a) Prove that for any element $c \in H_1(X)$ (singular homology) there is a continuous map $f : S^1 \to X$ so that $c = f_*([S^1])$, where $[S^1]$ is a generator of $H_1(S^1)$.

(b) Let X be a Δ-complex (for simplicity). Prove that for any $c \in H_2(X)$ (use simplicial or singular homology, as you like), there is a closed surface S and a continuous map $f : S \to X$ so that $c = f_*([S])$, where $[S]$ denotes a generator of the cyclic (by previous homework) group $H_2(S)$.

The problem of representing homology classes by manifolds is deep and important. It was studied by Rene Thom in his Fields Medal work (see his paper in Comment. Math. Helv. You just did dimensions $i = 1, 2$, but the problem gets harder for $i > 2$. Amazingly, cycles of dimension $i < 8$ can be represented by manifolds, but in general this is not true in dimension $i \geq 8$. But, Thom proved that it is true in rational homology (to be discussed later) up to a rational multiple.

7. Let X be a Δ-complex and let G be any abelian group. We define the simplicial homology of X **with coefficients in** A as follows. Let $\{C_n(X), \partial_n\}$ denote the complex of simplicial chains on X. The **complex of G-valued simplicial chains** is defined to be the collection of abelian groups $C_n(X; G) := C_n(X) \otimes G$ with boundary homomorphisms

$$\partial_n \otimes \mathrm{Id} : C_n(X) \otimes G \to C_{n-1}(X) \otimes G$$

Thus any element of $C_n(X; G)$ can be written as a finite sum $\sum_\sigma g_\sigma \sigma$ where $g_\sigma \in G$ and the sum is taken over all n-simplices σ of X. It is easy to check that $(\partial \otimes \mathrm{Id})^2 = 0$, so that $\{C_n(X; G), \partial_n\}$ is a chain complex. One defines the simplicial homology of X **with coefficients in** G, denoted $H_i(X; G)$, as the homology of this chain complex.

(a) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Z}/2\mathbf{Z}$.

(b) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Z}/3\mathbf{Z}$.

(c) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Q}$.

Extra credit problems

1. You can still hand in any extra credit problems from Homework 1.

2. For any $n \geq 1$, give a Δ-complex structure on S^n, and compute its simplicial homology groups.