Geom/Top: Homework 1 (due Monday, 10/08/12)

1. Read Farb notes.

2. Read Hatcher, Intro to Chapter 2 and Section 2.1.

3. (Do not hand this in) Let $\phi : A \to B$ and $\psi : B \to A$ be group homomorphisms. Suppose that $\phi \circ \psi = \text{Id}_B$. Prove that $\phi$ is surjective and $\psi$ is injective.

1. Give a $\Delta$-complex structure and use it to compute the homology of the following spaces.
   (a) The annulus/cylinder $S^1 \times [0, 1]$.
   (b) The Mobius band.
   (c) $\mathbb{CP}^2$.
   (d) The closed, connected (oriented) surface $\Sigma_2$ of genus 2.

   **Remark.** Computing for any $g \geq 2$ the simplicial homology of the closed surface $\Sigma_g$ of genus $g$ is possible, but it is rather cumbersome. We will later have various methods for which that computation is very easy.

2. (Reduced homology groups) For various formulas in topology and homological algebra, it is convenient to use a slight variation of homology. Let $X$ be any nonempty $\Delta$-complex. The **reduced complex of simplicial chains** on $X$ is the chain complex $\tilde{C} := \{C_n(X), \partial_n\}$, where we set $C_{-1}(X) = \mathbb{Z}$, we define $\partial_{-1} := 0$, and we define $\partial_0 : C_0(X) \to C_{-1}(X) = \mathbb{Z}$ by

   $\partial_0 \left( \sum_i a_i v_i \right) := \sum_i a_i$

   This is in contrast to the (unreduced) complex chain complex on $X$, where we defined $C_{-1}(X) := 0$. Thus the complex of reduced simplicial chains on $X$ is

   $\cdots \xrightarrow{\partial} C_3(X) \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} \mathbb{Z} \to 0$

   Define the **reduced homology** $\tilde{H}_i(X)$ of $X$ by

   $\tilde{H}_i(X) := H_i(\tilde{C})$

   (a) Let $X$ be any nonempty $\Delta$-complex. Prove that

   $H_0(X) \approx \tilde{H}_0(X) \oplus \mathbb{Z}$

   and that $\tilde{H}_i(X) \approx H_i(X)$ for all $i > 0$. In particular, the effect of using reduced homology of a space is that connected spaces $X$ have $\tilde{H}_0(X) = 0$ instead of $H_0(X) \approx \mathbb{Z}$.
(b) Here is an example of a formula that is cleaner using reduced homology. Let \( X \) and \( Y \) be connected \( \Delta \)-complexes. Pick \( x \in X \) and \( y \in Y \). Denote the wedge \( X \vee Y \) to be the quotient space given by the quotient of the disjoint union of \( X \) and \( Y \) by the relation \( x \sim y \). Prove that

\[
\tilde{H}_i(X \vee Y) = \tilde{H}_i(X) \oplus \tilde{H}_i(Y)
\]

for all \( i \geq 0 \).

3. (Simplicial Maps) Let \( X \) and \( Y \) be \( \Delta \)-complexes. A map \( f : X \to Y \) is called a simplicial map if it maps simplices to simplices (perhaps of lower dimension): for each simplex \( \sigma : [v_0 \cdots v_n] \to X \), there is a simplex \( \tau : [v_{i_0} \cdots v_{i_m}] \to Y \), \( 0 \leq m \leq n \) so that \( f \circ \sigma \) is the composition \( \tau \circ L \), where \( L : [v_0 \cdots v_n] \to [v_{i_0} \cdots v_{i_m}] \) is the linear ("collapse") map of \( [v_0 \cdots v_n] \) that projects it onto its subsimplex \( [v_{i_0} \cdots v_{i_m}] \), taking each \( v_{i+0} \) to itself.

(a) Fix \( n \geq 0 \). For any \( n \)-simplex \( \sigma \), we have \( f(\sigma) \) is an \( m \)-simplex for some \( m \leq n \). Define \( f_\#(\sigma) \) to be 0 if \( m < n \), and to be the element \( f(\sigma) \in C_n(X) \) otherwise. \( f_\# \) has a unique linear extension \( f_\# : C_n(X) \to C_n(Y) \). Prove that \( f_\# \) is a chain map, and so induces homomorphisms \( f_* : H_n(X) \to H_n(Y) \). Note that by the general "chain maps" theorem in homological algebra, the association \( f \mapsto f_* \) is functorial.

(d) Deduce that a simplicial homeomorphism \( f : X \to Y \) induces an isomorphism of simplicial homology groups.

4. (Simplicial complexes) Let \( X \) be a \( \Delta \)-complex, and assume (as usual) that every subsimplex of a simplex in \( X \) is a simplex in \( X \). Prove that the following are equivalent:

(a) Each simplex of \( X \) is uniquely determined by its vertices.

(b) The intersection of any two simplices in \( X \) is a subsimplex of each.

A \( \Delta \)-complex satisfying any (hence all) of these properties is called a simplicial complex.

5. Prove that any \( \Delta \)-complex is homeomorphic to a simplicial complex.

6. (a) Find a simplicial complex \( X \) that is homeomorphic to the torus, using as few 2-simplices as you can.

(b) Use Euler’s Formula \( V - E + F = 0 \) for the Torus and the definition of simplicial complex to prove your number is minimal. Note that the minimal number is much bigger than the two 2-simplices required for the \( \Delta \)-complex structure on \( T^2 \) given in class.

(c) Do the same for \( \mathbb{R}P^2 \).

7. (Infinite \( \Delta \)-complexes and simplicial complexes) The definition of a (not necessarily finite) \( \Delta \)-complex \( X \) is the same as that for finite \( \Delta \)-complexes except now the set of simplices \( \{ f_\alpha : \Delta_\alpha \to X \}_{\alpha \in I} \) is indexed
by a set $I$ of arbitrary cardinality, and one must specify a topology. We will endow $X$ with the \textbf{weak topology}, where a subset $Y \subseteq X$ is declared to be open if and only if $f^{-1}_\alpha(Y)$ is open in $\Delta_\alpha$ for every $\alpha \in I$. One can define infinite simplicial complexes in the same way.

Now let $X$ be a $\Delta$ complex.

(a) Prove that $X$ is compact if and only if it has finitely many simplices.

(b) Let $Y$ be any topological space. Prove that any map $h : X \to Y$ is continuous if and only if for every simplex $\sigma : \Delta_n \to X$, the composition $h \circ \sigma$ is continuous.

(c) Suppose that $X$ has countably many simplices, is locally finite and that every simplex of $X$ has dimension at most $n < \infty$. Prove that $X$ can be embedded in $\mathbb{R}^{2n+1}$. Prove that $2n + 1$ is sharp.

\section*{Extra credit problems}

1. Let $f(g)$ be the minimal number of triangles in any triangulation of the closed, orientable surface of genus $g$. Find $f(g)$. The correct lower bound is elementary (although the answer is rather strange-looking), and can be deduced from obvious relations among $V, E$ and $F$, together with Euler’s formula $V - E + F = 2 - 2g$. The correct upper bound, where one needs “only” construct an efficient enough triangulation, is significantly more difficult, and wasn’t known until the 1960’s. It would be nice to find a simpler proof. Also, is there an easy way to see that $f$ is a monotone increasing function of $g$?

2. Research problem: Let $h(g, n)$ be the number of triangulations of the closed, orientable surface $S_g$ of genus $g$ which have exactly $n$ triangles. Here we consider two triangulations to be the same if there is a homeomorphism of $S_g$ taking one triangulation to the other. Describe $h(g, n)$ as much as possible. How does it grow as a function of $g$ and $n$ as one is fixed and the other goes to infinity? Compute $h$ for small values of $g$ and $n$.

3. Do triangulations with greater or fewer triangles have bigger or smaller automorphism group? Can you bound the order of the automorphism group of any triangulation on a closed surface of genus $g$, i.e. can you give a bound depending only on $g$ and not on the triangulation?

4. Research Problem: Let $S_g$ be the (connected, compact, boundaryless) genus $g \geq 0$ surface. For any triangulation $T$ of $S_g$, we can get a new triangulation $T'$ by performing an \textit{elementary move}: pick an edge of $T$, remove it so that you get a quadrilateral consisting of the union of the
two triangles intersecting at your edge, then add back the other diagonal to this quadrilateral. This clearly gives a triangulation $T'$ of $S_g$.

Let $\Gamma_g(n)$ be the graph whose vertices are the triangulations of $S_g$ and where two vertices $T, T'$ are connected by an edge when they differ by an elementary move.

GENERAL PROBLEM: What does $\Gamma_g(n)$ look like as a function of $g$ and $n$? For example, $\Gamma_g(n)$ is empty whenever $n < f(g)$, where $f$ is the function defined in Problem 2 above. Here are some sample problems/questions:

- Try to determine $\Gamma_g(n)$ explicitly for all $n \geq 1$ for $g = 0$ and $g = 1$. Same for small $g$ and $n$.
- If $\Gamma_g(n)$ is connected, must $\Gamma_g(n + 1)$ be connected?
- For a given $g$, what is the minimal $N$ so that $\Gamma_g(n)$ is connected for all $n \geq N$ (if such an $N$ exists)?
- Study other graph-theoretic properties of $\Gamma_g(n)$ (many can be found via google, and also via any graph theory book).