Geom/Top: Homework 1 (due Monday, 10/08/12)

1. Read Farb notes.
2. Read Hatcher, Intro to Chapter 2 and Section 2.1.
3. (Do not hand this in) Let $\phi : A \to B$ and $\psi : B \to A$ be group homomorphisms. Suppose that $\phi \circ \psi = \mathrm{Id}_B$. Prove that ϕ is surjective and ψ is injective.

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1. Give a Δ-complex structure and use it to compute the homology of the following spaces.

   (a) The annulus/cylinder $S^1 \times [0,1]$.

   (b) The Mobius band.

   (c) $\mathbf{RP}^2$.

   (d) The closed, connected (oriented) surface Σ_2 of genus 2.

   Remark. Computing for any $g \geq 2$ the simplicial homology of the closed surface Σ_g of genus g is possible, but it is rather cumbersome. We will later have various methods for which that computation is very easy.

2. (**Reduced homology groups**) For various formulas in topology and homological algebra, it is convenient to use a slight variation of homology. Let X be any nonempty Δ-complex. The **reduced complex of simplicial chains** on X is the chain complex $\tilde{\mathcal{C}} := \{C_n(X), \partial_n\}$, where we set $C_{-1}(X) = \mathbf{Z}$, we define $\partial_{-1} := 0$, and we define $\partial_0 : C_0(X) \to C_{-1}(X) = \mathbf{Z}$ by

$$\partial_0(\sum_i a_i v_i) := \sum_i a_i$$

   This is in contrast to the (unreduced) complex chain complex on X, where we defined $C_{-1}(X) := 0$. Thus the complex of reduced simplicial chains on X is

$$\cdots \stackrel{\partial_4}{\to} C_3(X) \stackrel{\partial_3}{\to} C_2(X) \stackrel{\partial_2}{\to} C_1(X) \stackrel{\partial_1}{\to} C_0(X) \stackrel{\partial_0}{\to} \mathbf{Z} \to 0$$

   Define the **reduced homology** $\tilde{H}_i(X)$ **of** X by

$$\tilde{H}_i(X) := H_i(\tilde{\mathcal{C}})$$

   (a) Let X be any nonempty Δ-complex. Prove that

$$H_0(X) \approx \tilde{H}_0(X) \oplus \mathbf{Z}$$

   and that $\tilde{H}_i(X) \approx H_i(X)$ for all $i > 0$. In particular, the effect of using reduced homology of a space is that connected spaces X have $\tilde{H}_0(X) = 0$ instead of $H_0(X) \approx \mathbf{Z}$.

(b) Here is an example of a formula that is cleaner using reduced homology. Let X and Y be connected Δ-complexes. Pick $x \in X$ and $y \in Y$. Denote the **wedge** $X \vee Y$ to be the quotient space given by the quotient of the disjoint union of X and Y by the relation $x \sim y$. Prove that

$$\tilde{H}_i(X \vee Y) = \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$$

for all $i \geq 0$.

3. (**Simplicial Maps**) Let X and Y be Δ-complexes. A map $f : X \to Y$ is called a **simplicial map** if it maps simplices to simplices (perhaps of lower dimension): for each simplex $\sigma : [v_0 \cdots v_n] \to X$, there is a simplex $\tau : [v_{i_0} \cdots v_{i_m}] \to Y, 0 \leq m \leq n$ so that $f \circ \sigma$ is the composition $\tau \circ L$, where $L : [v_0 \cdots v_n] \to [v_{i_0} \cdots v_{i_m}]$ is the linear ("collapse") map of $[v_0 \cdots v_n]$ that projects it onto its subsimplex $[v_{i_0} \cdots v_{i_m}]$, taking each v_{i+0} to itself.

   (a) Fix $n \geq 0$. For any n-simplex σ, we have $f(\sigma)$ is an m-simplex for some $m \leq n$. Define $f_\#(\sigma)$ to be 0 if $m < n$, and to be the element $f(\sigma) \in C_n(X)$ otherwise. $f_\#$ has a unique linear extension $f_\# : C_n(X) \to C_n(Y)$. Prove that $f_\#$ is a chain map, and so induces homomorphisms $f_* : H_n(X) \to H_n(Y)$. Note that by the general "chain maps" theorem in homological algebra, the association $f \mapsto f_*$ is functorial.

   (d) Deduce that a simplicial homeomorphism $f : X \to Y$ induces an isomorphism of simplicial homology groups.

4. (**Simplicial complexes**) Let X be a Δ-complex, and assume (as usual) that every subsimplex of a simplex in X is a simplex in X. Prove that the following are equivalent:

   (a) Each simplex of X is uniquely determined by its vertices.

   (b) The intersection of any two simplices in X is a subsimplex of each.

   A Δ-complex satisfying any (hence all) of these properties is called a **simplicial complex**.

5. Prove that any Δ-complex is homeomorphic to a simplicial complex.

6. (a) Find a simplicial complex X that is homeomorphic to the torus, using as few 2-simplices as you can.

   (b) Use Euler's Formula $V - E + F = 0$ for the Torus and the definition of simplicial complex to prove your number is minimal. Note that the minimal number is much bigger than the two 2-simplices required for the Δ-complex structure on T^2 given in class.

   (c) Do the same for $\mathbf{RP}^2$.

7. (**Infinite Δ-complexes and simplicial complexes**) The definition of a (not necessarily finite) Δ-complex X is the same as that for finite Δ-complexes except now the set of simplices $\{f_\alpha : \Delta_\alpha \to X\}_{\alpha \in I}$ is indexed

by a set I of arbitrary cardinality, and one must specify a topology. We will endow X with the **weak topology**, where a subset $Y \subseteq X$ is declared to be open if and only if $f_\alpha^{-1}(Y)$ is open in Δ_α for every $\alpha \in I$. One can define infinite simplicial complexes in the same way.

Now let X be a Δ complex.

(a) Prove that X is compact if and only if it has finitely many simplices.

(b) Let Y be any topological space. Prove that any map $h : X \to Y$ is continuous if and only if for every simplex $\sigma : \Delta_n \to X$, the composition $h \circ \sigma$ is continuous.

(c) Suppose that X has countably many simplices, is locally finite and that every simplex of X has dimension at most $n < \infty$. Prove that X can be embedded in $\mathbf{R}^{2n+1}$. Prove that $2n+1$ is sharp.

Extra credit problems

1. Let $f(g)$ be the minimal number of triangles in any triangulation of the closed, orientable surface of genus g. Find $f(g)$. The correct lower bound is elementary (although the answer is rather strange-looking), and can be deduced from obvious relations among V, E and F, together with Euler's formula $V - E + F = 2 - 2g$. The correct upper bound, where one needs "only" construct an efficient enough triangulation, , is significantly more difficult, and wasn't known until the 1960's. It would be nice to find a simpler proof. Also, is there an easy way to see that f is a monotone increasing function of g?

2. Research problem: Let $h(g, n)$ be the number of triangulations of the closed, orientable surface S_g of genus g which have exactly n triangles. Here we consider two triangulations to be the same if there is a homeomorphism of S_g taking one triangulation to the other. Describe $h(g, n)$ as much as possible. How does it grow as a function of g and n as one is fixed and the other goes to infinity? Compute h for small values of g and n.

3. Do triangulations with greater or fewer triangles have bigger or smaller automorphism group? Can you bound the order of the automorphism group of any triangulation on a closed surface of genus g, i.e. can you give a bound depending only on g and not on the triangulation?

4. Research Problem: Let S_g be the (connected, compact, boundaryless) genus $g \geq 0$ surface. For any triangulation T of S_g, we can get a new triangulation T' by performing an *elementary move*: pick an edge of T, remove it so that you get a quadralateral consisting of the union of the

two triangles intersecting at your edge, then add back the other diagonal to this quadrilateral. This clearly gives a triangulation T' of S_g.

Let $\Gamma_g(n)$ be the graph whose vertices are the triangulations of S_g and where two vertices T, T' are connected by an edge when they differ by an elementary move.

GENERAL PROBLEM: What does $\Gamma_g(n)$ look like as a function of g and n? For example, $\Gamma_g(n)$ is empty whenever $n < f(g)$, where f is the function defined in Problem 2 above. Here are some sample problems/questions:

- Try to determine $\Gamma_g(n)$ explicitly for all $n \geq 1$ for $g = 0$ and $g = 1$. Same for small g and n.
- If $\Gamma_g(n)$ is connected, must $\Gamma_g(n+1)$ be connected?
- For a given g, what is the minimal N so that $\Gamma_g(n)$ is connected for all $n \geq N$ (if such an N exists)?
- Study other graph-theoretic properties of $\Gamma_g(n)$ (many can be found via google, and also via any graph theory book).