Analysis III
Home Assignment 4

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Problem 4.1

For a lattice $\Lambda = \alpha\mathbb{Z} \oplus \beta\mathbb{Z} \subseteq \mathbb{C}$, let $\lambda = \frac{1}{\alpha}$ where wlog $\alpha \neq 0$. Then Λ is equivalent to the lattice generated by $\{1, \frac{\beta}{\alpha}\}$ which is identical to the lattice generated by $\{1, -\frac{\beta}{\alpha}\}$. Thus Λ is equivalent to the lattice generated by $\{1, \eta\}$ where $\eta = \left|\frac{\beta}{\alpha}\right|$.

Suppose $\{1, \alpha\}$ and $\{1, \beta\}$ generate two equivalent lattices Λ_1 and Λ_2 i.e. there exists $\lambda \in \mathbb{C}^*$ such that $\lambda\Lambda_1 = \Lambda_2$. Hence $\{\lambda, \alpha\lambda\}$ and $\{1, \beta\}$ are two different basis for the same free $\mathbb{Z}$-module, and so there are $a, b, c, d \in \mathbb{Z}$ such that

$$\lambda \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \beta \end{pmatrix} = A \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

and $\det(A) = \pm 1$ since A^{-1} is also an integer matrix. Since $a, b, c, d \in \mathbb{Z}$ and the conjugation automorphism fixes $\mathbb{Z}$, we have

$$\begin{pmatrix} \lambda & \overline{\lambda} \\ \lambda\alpha & \overline{\lambda\alpha} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta & \overline{\beta} \end{pmatrix}$$

Taking determinants we have

$$|\lambda|^2(\overline{\alpha} - \alpha) = \det(A)(\overline{\beta} - \beta)$$

Since $\alpha, \beta \in \mathbb{H}$, we get that $\frac{(\overline{\alpha}-\alpha)}{(\overline{\beta}-\beta}$ is positive. Hence we must have $\det(A) = +1$. Then from the first equality, we have

$$c + d\beta = \alpha(a + b\beta) \Rightarrow \alpha = \frac{c + d\beta}{a + b\beta} \Rightarrow \beta = \frac{-a\alpha + c}{b\alpha - d}$$

with $(-a)(-d) - bc = \det A = 1$. Conversely, if $\{1, \eta_1\}$ is a basis generating a lattice Λ; and we can find integers a, b, c, d with $ad - bc = 1$ then,

$$\eta_2 = \frac{a\eta_1 + b}{c\eta_1 + d} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \eta_1 \\ 1 \end{pmatrix} = (c\eta_1 + d) \begin{pmatrix} \eta_2 \\ 1 \end{pmatrix}$$

and hence $(\eta_1, 1)$ and $((c\eta_1 + d)\eta_2, (c\eta_1 + d)$ form the same lattice. Hence $(\eta_1, 1)$ and $(\eta_2, 1)$ are equivalent.

By the above paragraphs it is evident that each equivalent class of lattices has one with a basis of the form $\{1, \eta\}$ with $\eta \in \mathbb{H}$ and $\{1, \eta_1\}$, $\{1, \eta_2\}$ are equivalent iff η_1 and η_2 are $PSL(2, \mathbb{Z})$ conjugate. Hence there is exactly one representative from each equivalence class of lattices in the fundamental domain of $PSL(2, \mathbb{Z})$ on $\mathbb{H}$. Hence there is a unique $\eta \in R$ such that $\{1, \eta\}$ is equivalent to Λ.

We calculated that the torsion elements of $PSL(2,\mathbb{Z})$ are Id and the conjugates of S and ST where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. So the elliptic torsion elements are conjugates of S and ST. The fixed point of S in R is given by

$$Sz = z \Rightarrow \frac{1}{-z} = z \Rightarrow z = i$$

and the fixed point of ST in R is given by

$$STz = z \Rightarrow \frac{-1}{z+1} = z \Rightarrow z^2 + z + 1 = 0 \Rightarrow z = e^{i\pi/3}$$

The first lattice is the lattice generated by $\{1, i\}$ i.e. the integer lattice in $\mathbb{C}$. The second one corresponds to the lattice generated by $\{1, e^{i\frac{\pi}{3}}\}$.

Problem 4.2

We will use Einstein Summation notation throughout. By definition for a Euclidean metric g, we have $g(\nabla f, X) = df(X) = X(f)$ for any $X \in \Gamma(TM)$. Thus in local coordinates we get

$$\nabla f = g^{ij}\partial_i d\partial_j$$

Similarly the divergence is defined by

$$(\text{div}(X))\nu = d(i_X \nu)$$

where $\nu = \sqrt{|\det g|}dx^1 \dots dx^n$ is the volume form. Then for $X = X^i\partial_i$, we have

$$\begin{aligned}
(\text{div}X)\sqrt{|\det g|}dx^1 \dots dx^n &= d\left(i_{X^i\partial_i}\sqrt{|\det g|}dx^1 \dots dx^n\right) \\
&= d\left(X^i\sqrt{|\det g|}(-1)^{i-1}dx^1 \dots \widehat{dx^i} \dots dx^n\right) \\
&= \partial_i(X^i\sqrt{|\det g|})dx^1 \dots dx^n
\end{aligned}$$

Thus

$$\text{div}(X) = \frac{1}{\sqrt{|\det g|}}\partial_i(X^i\sqrt{|\det g|})$$

And consequently,

$$\begin{aligned}
\Delta f = \text{div}(\nabla f) &= \text{div}(g^{ij}\partial_i f \partial_j) \\
&= \frac{1}{\sqrt{|\det g|}}\partial_i\left(\sqrt{|\det g|}g^{ij}\partial_i f\right)
\end{aligned}$$

i.e.

$$\Delta = \frac{1}{\sqrt{|\det g|}}\partial_i\left(\sqrt{|\det g|}g^{ij}\partial_i\right)$$

Now suppose we have two metrics $g_1 = \rho g_2$ as in the problem. Suppose the corresponding Laplacians are denoted by Δ_1 and Δ_2. Let $\dim M = d$. Then we can compute

$$\begin{aligned}
\Delta_1 &= \frac{1}{\sqrt{\rho^d |\det g_2|}}\partial_i\left(\sqrt{\rho^d |\det g_2|}\rho^{-1} g_2^{ij}\partial_j\right)\\
&= \frac{\rho^{-d/2}}{\sqrt{|\det g_2|}}\partial_i\left(\rho^{\frac{d}{2}-1}\sqrt{|\det g_2|}g_2^{ij}\partial_j\right)\\
&= \frac{\rho^{-d/2}}{\sqrt{|\det g_2|}}\left(\frac{d}{2}-1\right)\rho^{\frac{d}{2}-2}\frac{\partial\rho}{\partial x_i}\left(\sqrt{|\det g_2|}g_2^{ij}\partial_j\right) + \frac{\rho^{-d/2}}{\sqrt{|\det g_2|}}\rho^{\frac{d}{2}-1}\left(\sqrt{|\det g_2|}g_2^{ij}\partial_j\right)\\
&= \frac{1}{\rho}\Delta_2 + \frac{\rho^{-1}}{\sqrt{|\det g_2|}}\left(\frac{d}{2}-1\right)\frac{\partial\rho}{\partial x_i}\left(\sqrt{|\det g_2|}g_2^{ij}\partial_j\right)
\end{aligned}$$

Thus in the case M is a surface we have $d = 2$, hence

$$\Delta_1 = \frac{1}{\rho}\Delta_2$$

If $\dim(M) = d > 2$, then clearly the two Laplacians are not proportional unless the second term is zero.

■■ Note that $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Hence $\partial\overline{\partial} = \frac{1}{4}(\partial_{xx} + \partial_{yy})$. On the other hand note that being harmonic is a local property and it is invariant under a conformal map as proved in first part. Since the metric on φ is defined to be the pullback of the Euclidean metric on $\mathbb{C}$ via conformal maps φ, we get that $\Delta f = 0$ iff $\Delta(f \circ \varphi^{-1}) = 0$ for every chart φ. But for a metric on X of constant curvature we may consider $g = Id$ so that $\Delta = \frac{1}{\sqrt{|\det g|}}\partial_i\left(\sqrt{|\det g|}g^{ij}\partial_i\right) = \partial_{ii}$. Thus

$$\Delta f = 0 \Leftrightarrow \Delta(f\circ\varphi^{-1}) = (\partial_{xx} + \partial_{yy})(f\circ\varphi^{-1}) = 0 \Leftrightarrow \frac{1}{4}(\partial_{xx} + \partial_{yy})(f\circ\varphi^{-1}) = 0 \Leftrightarrow \partial\overline{\partial}(f\circ\varphi^{-1}) = 0$$

Problem 4.3

Given a compact hyperbolic surface X, let $\mathfrak{I}(X)$ be the group of isometries of X. We want to show that $\mathfrak{I}(X)$ is finite.

First we prove that $\mathfrak{I}(X)$ is compact metric space where the metric is the usual one, defined as

$$d(f,g) = \sup_{x\in X} d(f(x), g(x))$$

for isometries $f, g \in \mathfrak{I}(X)$. Let f_n be a sequence of isometries in $\mathfrak{I}(X)$. Since X is compact, it is separable and hence by a diagonal argument, we can find a convergent subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$. Suppose f_{n_i} converge to f. We want to show that $f \in \mathfrak{I}(X)$. Since for each $i > 0$, $d(f_{n_i}(x), f_{n_i}(y)) = d(x, y)$; taking limit as i tends to ∞ we get $d(f(x), f(y)) = d(x, y)$. Thus f is distance preserving. Now considering the sequence of isometries $\{f_{n_i}^{-1}\}$ we find that they have a further subsequence converging to a some g which is also distance preserving. Now for any point $x \in X$, considering a small enough ball around $f(x)$, we can easily see that $f_{n_i}^{-1}(f(x)) \to x \Rightarrow g(f(x)) = x$. Thus $g = f^{-1}$. So f is a surjective distance preserving map on X proving that it is an isometry. Thus $\mathfrak{I}(X)$ is compact.

Next we want to show that $\mathfrak{I}(X)$ is discrete. Given $f \in \mathfrak{I}(X)$, suppose we have a converging sequence $\{g_n\}$ in $\mathfrak{I}(X)$ such that $g_n \to f$ uniformly. Recall that for smooth maps f, g from a compact smooth manifold Z to a compact manifold Y, there exists δ_Y such that $d_Y(f, g) < \delta_Y$ implies f is smoothly homotopic to g. Thus in our case we know that there exists δ_X such that for sufficiently large n, $d_X(g_n, f) < \delta_X$, and hence there exists N such that g_n is smoothly homotopic to f for all $n > N$.

Fix any such g_n homotopic to f via $H(s,t)$ so that $H(0,t) = g_n(t)$ and $H(1,t) = f(t)$. Since H is smooth, the length of the path $\gamma_{t_0} = H(s, t_0) : [0,1] \to X$ is a smooth function of $t_0 \in X$. Since X is compact, this implies $\max_{t \in X}\{length(\gamma_t)\}$ is bounded above.

We know that X is a hyperbolic space; hence $\mathbb{D}$ is a covering space. We choose lifts $\tilde{g}_n, \tilde{f}$ of g_n and f and lift the homotopy H to a homotopy $\tilde{H}$ of $\tilde{g}_n$ and $\tilde{f}$. Then clearly the length of $\tilde{H}(s,.)$ is same as length of $H(s,.)$ and hence $d_{\mathbb{D}}(\tilde{g}_n, \tilde{f})$ is bounded above as well.
Thus we have two isometries $\tilde{g}_n, \tilde{f}$ of $\mathbb{D}$ which are bounded distance apart. Consider a sequence of points $\{x_i\}$ in $\mathbb{D}$ converging to a pint x on the boundary. Then since $d(\tilde{g}_n(x_i), \tilde{f}(x_i))$ is bounded (independent of x_i), taking limit we find that $d(\tilde{g}_n(x), \tilde{f}(x))$ is bounded. But hyperbolic distance between any two points on the boundary is infinite unless they are identical. Thus $\tilde{g}_n$ and $\tilde{f}$ are equal on the boundary of $\mathbb{D}$. Since any isometry of $\mathbb{D}$ can be specified by its action on the boundary (in particular by the fixed points, which determine the axis of isometry) , we get that $g_n \equiv f$ for large enough n. Thus we have proved that f has a neighbourhood in $\mathfrak{I}(X)$ such that it does not contain any other point from $\mathfrak{I}(X)$, i.e $\mathfrak{I}(X)$ is discrete.

Thus $\mathfrak{I}(X)$ is compact and discrete; and hence finite.

Problem 4.4

4.9

(a)

Given any element $g = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \in G$, we have $a\bar{a} - b\bar{b} = 1$, hence $G \subseteq SL(2, \mathbb{C})$. Also it is obvious that for $g, h \in G$, $gh^{-1} \in G$. Hence G is a subgroup of $SL(2, \mathbb{C})$. Now we define a map $T : G \to Aut(\mathbb{D})$ by

$$Tg = \frac{az + \bar{b}}{bz + \bar{a}} = \frac{a^2 z + a\bar{b}}{|a|^2 + baz} = \frac{a^2}{|a|^2} \frac{z + \frac{\bar{b}}{a}}{1 + \frac{\bar{\bar{b}}}{\bar{a}} z}$$

Hence Tg is a rotaion composed with a fractional linear transformation of the form $\frac{z-w}{1-\overline{w}z}$ where $|w| = |b/a| < 1$. Hence $Tg \in Aut(\mathbb{D})$.
So to prove that $G/\pm Id \cong Aut(\mathbb{D})$, all it remains is to check that any automorphism $\varphi \in Aut(\mathbb{D})$ arises via T upto the sign of a, b. Let

$$\varphi^{-1}(0) = -\frac{\bar{b}}{a} \text{ where } |a|^2 - |b|^2 = 1$$

Note that a, b can be uniquely defined this way upto the sign of $\bar{b}/a$. Then

$$T\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} = \frac{a^2}{|a|^2}\varphi$$

Since $\frac{a^2}{|a|^2}$ is also in $Aut(\mathbb{D})$, we are done.
The second method is just mechanical verification , so we skip it.

(b) We give $S^2 \cong \mathbb{C}_\infty$ the metric defined in problem 1.4. Then a conformal homeomorphism T of S^2 is a rigid motion iff it preserves the metric i.e. $d(z,w) = d(Tz, Tw)$. In other words,

$$\frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}} = \frac{2|Tz-Tw|}{\sqrt{(1+|Tz|^2)(1+|Tw|^2)}}$$

Taking $w \to z$, we then have

$$1 + |Tz|^2 = (1+|z|^2)|T'z|$$

Putting $Tz = \frac{az+b}{cz+d}$ we get

$$|cz+d|^2 + |az+b|^2 = (1+|z|^2)|ad-bc|$$

$$\Rightarrow c\overline{d} + a\overline{b} = 0, |c|^2 + |a|^2 = |ad-bc| = |d|^2 + |b|^2 = 1 \Rightarrow \overline{a} = d, c = -\overline{b}$$

as required.

(c) Recall that we proved in Algebra-I home assignments the homomorphism

$$Q : SU(2) \to SO(3)$$

sending

$$T_A \to \Phi^{-1} \circ T_A \circ \Phi$$

where T_A is the linear transformation associated to A and Φ is the stereographic projection; is an onto map with kernel given by $\pm Id$. Hence we further have

$$G \cong SU(2) \cong SO(3)/ \pm I$$

We also proved that the unit quarternion $\cos\theta + \sin\theta(x_1 i + x_2 j + x_3 k)$ where $x_1^2 + x_2^2 + x_3^2 = 1$ is a rotaion of $\mathbb{R}^3$ around the axis (x_1, x_2, x_3) by angle 2θ; when $\mathbb{R}^3$ is thought to be spanned by i, j, k.

3.3

(a) Suppose we choose k such that the real valued harmonic function $\tilde{u}(z) = u(z) - k\log|z|$ has a harmonic conjugate on $\mathcal{A}$. Indeed if v is a harmonic conjugate of $\mathcal{A}$ then

$$\nabla v = (-\tilde{u}_y, \tilde{u}_x) = \left(-u_y + k\frac{y}{x^2+y^2}, u_x - k\frac{x}{x^2+y^2}\right)$$

Then $\Delta v = 0$ implies by Green's theorem,

$$\begin{aligned} 0 = \iint\limits_{|z|\le r} (\tilde{u}_{yy} + \tilde{u}_{xx})dxdy &= -k \oint\limits_{|z|=r} \left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) + \oint\limits_{|z|=r} (-u_y dx + u_x dy) \\ &= -\frac{k}{r^2}\int_0^{2\pi} r^2 d\theta + \oint\limits_{|z|=r} (-u_y dx + u_x dy) \\ \Rightarrow k &= \frac{1}{2\pi} \oint\limits_{|z|=r} (-u_y dx + u_x dy) \end{aligned}$$

So for this choice of $k \in \mathbb{R}$ we can find an analytic function $f \in \mathcal{H}(\mathcal{A})$, such that

$$u(z) - k \log|z| = \Re(f(z)) \quad \forall z \in \mathcal{A}$$

■■ We have from above equation,

$$\int_0^{2\pi} u(re^{i\theta})d\theta - k \int_0^{2\pi} \log|r| d\theta = \Re\left(\int_0^{2\pi} f(re^{i\theta})d\theta\right)$$

$$\Rightarrow \int_0^{2\pi} u(re^{i\theta})d\theta - 2k\pi \log(r) = \Re\left(\int\limits_{|z|=r} f(z)\frac{1}{iz}dz\right)$$

$$\Rightarrow \int_0^{2\pi} u(re^{i\theta})d\theta - k \log r = \Re\left(\frac{1}{2\pi i}\int\limits_{|z|=r} \frac{f(z)}{z}dz\right)$$

which is independent of r by Cauchy integral formula. Hence taking $r \to 0$, we find that LHS is bounded iff $k = 0$. Then putting

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta$$

we see that the mean value theorem is satisfied and thus u extends to a harmonic function on $|z| < r_2$.

(b) By part (a), around each point $w \in \Omega$ there is a small enough neighbourhood $U_w \subseteq \Omega$ such that $u(z) - \log|z - z_0|$ is harmonic and bounded on U_w and hence we can find a holomorphic function $f_w \in \mathcal{H}(U_w)$ such that

$$u(z) - \log|z - z_0| - 0\log|z| = u(z) - \log|z - z_0| = \Re(f_w(z))$$

for $z \in U_w$. Note that if $U_{w_1} \cap U_{w_2} \neq \emptyset$, then by monodromy theorem, $f_{w_1} \equiv f_{w_2}$. Since Ω is simply connected, we can thus find a single holomorphic function $g \in \mathcal{H}(\Omega)$ such that

$$u(z) - \log|z - z_0| = \Re(g(z))$$

Define $f(z) = (z - z_0)e^{g(z)}$. Then note that

$$\log(|f(z)|) = \log|z - z_0| + \log|e^{\Re(g(z))}e^{i\Im(g(z))}| = \log|z - z_0| + \Re(g(z)) = u(z)$$

Observe that by definition f is clearly one-one on some disk around z_0. Thus all the conditions required are satisfied by this choice of f.

3.4

(a) Since w =constant implies v =constant, v can thought of as a function of w. Let $v = \varphi(w)$. Then

$$0 = \Delta(v) = \varphi''(w)\|\nabla w\|^2 \Rightarrow \varphi''(w) = 0 \Rightarrow v = \varphi(w) = aw + b$$

for some $a, b \in \mathbb{C}$. Here $a \neq 0$ since $\nabla v \neq 0$.

(b) Suppose u' is the harmonic conjugate of u in a simply connected disc around z_0 in Ω. Then by Cauchy Riemann equations, we know that the level curves of u and u' are perpendicular to each other. WLOG we may assume $u'(z_0) = 0$. Now again by Cauchy Riemann equations we know that

$$\nabla u \neq \mathbf{0} \Leftrightarrow \nabla u' \neq \mathbf{0}$$

Hence we get that the level curves of v and u' coincide and both are non-degenerate. Hence by part (a), we can find $a, b \in \mathbb{C}, a \neq 0$ such that

$$v = au' + b \Rightarrow v(z_0) = b \Rightarrow v(z) = au'(z) + v(z_0)$$

Since $\|\nabla u(z_0)\| = \|\nabla u'(z_0)\|$, taking gradient on both sides we get, $\|\nabla v(z_0)\| = |a|\|\nabla u(z_0)\| \Rightarrow a = \pm 1$. Hence $v = \pm u' + b$ implying $u \pm iv$ is harmonic in Ω.

3.5

(i) Suppose $u(z) > M$ for some $z \in \Omega$. Thus $L = \sup\limits_{\Omega} u$ is obtained in Ω. Note that clearly for all $z \in \partial\Omega$, $u(z) \leq M < L \Rightarrow \sup\limits_{\Omega} u$ is achieved in Ω°. Suppose $z_0 \in \Omega^\circ$ is such that $u(z_0) = L$. There exists $\epsilon > 0$ such that $B(z_0, \epsilon) \subseteq \Omega$. Then considering the boundary of $B(z_0, \epsilon)$ we get by SMVP,

$$L = u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \epsilon e^{i\theta}) d\theta \leq L$$

Hence equality holds everywhere and so $u \equiv L$ on $B(z_0, \epsilon)$. Thus the set $U = \{z \in \Omega : u(z) = L\}$ is open. On the other hand clearly by definition the set is closed since u is continuous. Hence $U = \Omega$ implying that

$$\sup_{\zeta \in \partial\Omega} \limsup_{z \to \zeta, z \in \Omega} u(z) = L > M$$

which is a contradiction. Hence $u(z) \leq M$ for all $z \in \Omega$.

Note that if equality occurs i.e. for some $z \in \Omega$, $u(z) = M$, then also by the same argument as above u must be locally constant and hence by connectedness it is globally constant.

(ii) Note that $u - h \in \mathfrak{sh}(K)$ since h satisfies MVP. Hence applying part (i) to $u - h$ we find that

$$\sup_{z \in \partial K} (u - h)(z) \leq 0 \Rightarrow u - h \leq 0 \forall z \in K$$

Also by part (i), equality at any point in K implies equality at all points in K.

(iii) $u_i \in \mathfrak{sh}(\Omega) \Rightarrow u_i$ satisfies SMVP $\Rightarrow$

$$\max(u_1, \ldots, u_N)(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(z_0 + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \max(u_1, \ldots, u_N)(z_0 + re^{i\theta}) d\theta$$

where $\max(u_1(z_0), \ldots, u_N(z_0)) = u_n(z_0)$. Since $\max(u_1, \ldots, u_N)$ is also continuous, it is subharmonic. The proof that $\sum_{j=1}^N c_j u_j$ is continuous and satisfies SMVP is trivial. Hence $\sum_{j=1}^N c_j u_j \in \mathfrak{sh}(\Omega)$.

(iv) By Jensen's inequality,

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi\circ h(z_0+re^{i\theta})d\theta \geq \varphi\left(\int_0^{2\pi}h(z_0+re^{i\theta})d\theta\right)=\varphi(h(z_0))$$

since h is harmonic. Hence $\varphi\circ h\in\mathfrak{sh}(\Omega)$.By the same argument applied to $v\in\mathfrak{sh}(\Omega)$, we also get $\varphi\circ v\in\mathfrak{sh}(\Omega)$.

(v) For any point $z_0\in\Omega$, consider a small enough disc $K=B(z_0,\epsilon)$ of radius ϵ around z_0 in Ω so that u is bounded on $\overline{K}$ and take any harmonic $h\in C(\overline{K})$ with $u\leq h$ on ∂K. Then

$$0\leq\frac{1}{2\pi}\int_0^{2\pi}(u-h)(z_0+re^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}u(z_0+re^{i\theta})d\theta-h(z_0)$$

$$\Rightarrow u(z_0)\leq h(z_0)\leq\int_0^{2\pi}u(z_0+re^{i\theta})d\theta$$

So $u\in\mathfrak{sh}(\Omega)$.

(vi) Note that by the proof of (i), it is evident that local SMVP implies that the maximum principle. Hence (ii) follows. Then by (v), we get that u is subharmonic.

(vii) Suppose we have a harmonic function h on the open disc $D=\{0\leq|z|<r_2\}$ with $h=u$ on ∂D. Then by (ii), we have $u\leq h$ on D. In particular,

$$\int_0^{2\pi}u(z_0+r_1e^{i\theta})d\theta\leq\int_0^{2\pi}h(z_0+r_1e^{i\theta})d\theta=h(z_0)=\int_0^{2\pi}h(z_0+r_2e^{i\theta})d\theta=\int_0^{2\pi}u(z_0+r_2e^{i\theta})d\theta$$

It remains to prove that for any $z_0\in\Omega$ and $0<r<dist(z_0,\partial\Omega)$ we have

$$\int_0^{2\pi}u(z_0+re^{i\theta})d\theta>-\infty$$

Suppose on the contrary, there is some $r=r_0$ such that the integral is not finite. Then by monotonicity, for all $0<r<r_0$, we have,

$$\int_0^{2\pi}u(z_0+re^{i\theta})d\theta=-\infty\quad\forall\quad 0<r<r_0\Rightarrow\iint\limits_{|z-z_0|<r_0}u(x,y)dxdy=-\infty$$

Now $u\in\mathfrak{sh}(\Omega)\Rightarrow u\not\equiv-\infty$. Note that this implies that given any $r>0$ and any $z_1\in\Omega$, there is at least one $z\in B(z_1,r)$ such that $u(z)\neq-\infty$ since otherwise, by SMVP $u(z_1)=-\infty$ and hence $u\equiv-\infty$ on $B(z_1,r)$ or in particular on Ω which is not the case. Thus the points $\{z\in\Omega:u(z)>-\infty\}$ is dense in Ω. It follows from SMVP that there exists w aritrarily close to z_0 such that

$$-\infty<u(w)\int_0^{2\pi}u(w+re^{i\theta})d\theta\quad\forall r\in(0,dist(w,\partial\Omega))$$

Choose r_1 such that $B(w,r_1)\subseteq B(z_0,r_0)$ and $r_1<dist(w,\partial\Omega)$. Then we have

$$-\infty=\iint\limits_{|z-z_0|<r_0}u(x,y)dxdy\geq\iint\limits_{|z-w|<r_1}u(x,y)dxdy>-\infty$$

Contradiction!!

(viii) Suppose $u \in \mathfrak{sh}(\Omega)$. Note that by (vii), we have

$$0 \leq \frac{d}{dr}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = \frac{1}{r}\oint_{|z-z_0|=r} \frac{du}{dr} r d\theta = \frac{1}{r}\oint_{|z-z_0|=r} \frac{du}{dn} ds = \frac{1}{r}\iint_{|z-z_0|\leq r} \Delta u dx dy$$

where the last equality follows from divergence theorem. Since z_0 and r are arbitrary, it follows that $\Delta u \geq 0$. Conversely if $\Delta u \geq 0$, then

$$\frac{d}{dr}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \geq 0 \Rightarrow \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \geq \lim_{r\to 0^+}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0) \Rightarrow u \in \mathfrak{sh}(\Omega)$$

(x) Suppose $u \in C(\Omega)$ satisfies MVP. Then u also satisfies SMVP. Hence u is subharmonic. Given $z_0 \in \Omega$, let $D = B(z_0, r) \subseteq \Omega$ Now consider a harmonic function h such that $h = u$ on ∂D. Then by MVP we get $u(z_0) = h(z_0)$. But then by the equality condition of (ii), we have $u = h$ on D. Thus u is harmonic.

Suppose $u_n \to u$ uniformly on compact subsetes of Ω. Take any point $z_0 \in \Omega$ and let $D = B(z_0, r) \subseteq \Omega$. Then given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $-\epsilon < u(z) - u_n(z) < \epsilon$ for all $z \in \overline{D}$. Then we have

$$\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \leq \epsilon + \int_0^{2\pi} u_n(z_0 + re^{i\theta})d\theta = \epsilon + u_n(z_0)$$

Taking limit we have,

$$\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \leq \epsilon + u(z_0)$$

Similarly considering $-u_n \to -u$ we get

$$\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \geq -\epsilon + u(z_0)$$

Combining both inequalities and taking $\epsilon \to 0$, we get u satisfies MVP. Then by above, we get that u is harmonic.

Problem 4.5

Recall that for the Weierstrass function $\wp$ doubly periodic with respect to a lattice Λ, $\wp$ is naturally defined on the complex torus $T = \mathbb{C}/\Lambda$. This torus may be embedded in the complex projective plane by means of the map

$$z \mapsto (1, \wp(z), \wp'(z))$$

. We know there are constants g_2, g_3 such that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Note that this relation is in the form of an elliptic curve. Thus we get an elliptic curve corresponding to a complex torus. Note that above map is a group isomorphism, carrying the natural group structure of the torus into the projective plane. It is also an isomorphism of Riemann surfaces, so topologically, a given elliptic curve looks like a torus.

If the lattice Λ is related by multiplication by a non-zero complex number c to a lattice $c\Lambda$, then the corresponding curves are isomorphic since scaling does not change the curve. Thus we get a bijection between

$$\text{Lattices/Equivalence defined in Problem 1} \leftrightarrow \text{Elliptic curves/Isomorphism class}$$

We also proved that there is an isomorphism between

$$\text{Lattices/Equivalence defined in Problem 1} \leftrightarrow \mathbb{H}/PSL(2,\mathbb{Z})$$

Thus we have an isomorphism

$$\text{Elliptic curves/Isomorphism class} \leftrightarrow \mathbb{H}/PSL(2,\mathbb{Z})$$

Now recall that given any lattice Λ there is $\eta \in \mathbb{H}$ such that $\{1, \eta\}$ generates a lattice in the same equivalence class. We define the meromorphic function $j : \mathbb{H} \to \mathbb{C}$ which is invariant under action of $PSL(2,\mathbb{Z})$ by

$$j(\tau) = 1728\frac{g_2^3}{\Delta}$$

where the modular discriminant Δ is

$$\Delta = g_2^3 - 27g_3^2$$

where g_2, g_3 correspond to Λ via $\wp$. The function defined above is called the $j-$invariant. It can be shown that The function $j(\tau)$ when restricted to the fundamental region of $PSL(2,\mathbb{Z})$ takes on every value in the complex numbers exactly once. Thus using the bijection of the fundamental region and Elliptic curves/Isomorphism class, we find that the $j-$invariant gives a bijection between isomorphism classes of elliptic curves over $\mathbb{C}$ and the complex numbers.