Analysis III
Home Assignment 3

Subhadip Chowdhury

Problem 3.1

In each of the following we give the Riemann map from the given domain onto $\mathbb{D}$. The required map in the problem is inverse of these.

3.1.(i)

$$z \longmapsto -z \longmapsto \sqrt{-z} \longmapsto \frac{\sqrt{-z}-i}{\sqrt{-z}+i}$$

$$\mathbb{C}\backslash(-\infty,0] \longmapsto \mathbb{C}\backslash[0,\infty) \longmapsto \{\Im z>0\} \longmapsto \mathbb{D}$$

3.1.(ii)

$$z \longmapsto i\frac{1+z}{1-z} \longmapsto \frac{1+z}{1-z} \longmapsto \frac{\sqrt{\frac{1+z}{1-z}}-i}{\sqrt{\frac{1+z}{1-z}}+i}$$

$$\hat{\mathbb{C}}\backslash[-1,1] \longmapsto \hat{\mathbb{C}}\backslash i[0,\infty] \longmapsto \hat{\mathbb{C}}\backslash[0,\infty] \longmapsto \mathbb{D}$$

3.1.(iii)

$$z \longmapsto \exp(\pi z) \longmapsto \frac{\exp(\pi z)-i}{\exp(\pi z)-i}$$

$$\{0<\Im z<1\} \longmapsto \{\Im z>0\} \longmapsto \mathbb{D}$$

3.1.(iv)

$$z \longmapsto z^2 \longmapsto \frac{z^2-i}{z^2-i}$$

$$\text{First Quadrant} \longmapsto \{\Im z > 0\} \longmapsto \mathbb{D}$$

3.1.(v)

$$z \longmapsto i\frac{1+z}{1-z} \longmapsto -i\left(i\frac{1+z}{1-z}\right) = \frac{1+z}{1-z} \longmapsto \frac{\left(\frac{1+z}{1-z}\right)^2-i}{\left(\frac{1+z}{1-z}\right)^2+i}$$

$$\{0 < \Im z\} \cap \mathbb{D} \longmapsto \text{Second Quadrant} \longmapsto \text{First Quadrant} \longmapsto \mathbb{D}$$

3.1.(vi)

$$z \longmapsto \sqrt{z} \longmapsto \frac{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2-i}{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2+i}$$

$$\mathbb{D}\backslash[0,1) \longmapsto \{\Im z > 0\} \cap \mathbb{D} \longmapsto \mathbb{D}$$

Problem 3.2

The set $\mathcal{S}$ of function f which vanish at $z_0 \in G$; holomorphic in G and are bounded in modulus by 1 is a **closed subset** of the **precompact set** of functions $\{f \in \mathcal{H}(G) : |f(z)| < 1 \quad \forall z \in G\}$. Hence $\mathcal{S}$ is compact. Now

$$f \mapsto \int\int_G |f(x+iy)|dxdy = \int_G |f(z)|dz \wedge d\overline{z}$$

is a continuous function on the compact set $\mathcal{S}$. Hence $\exists f_0 \in \mathcal{S}$ such that the integral is maximized at f_0.

Problem 3.3

We know that there is a biholomorphic Riemann map from the half-plane $\{\Im z > 0\}$ onto the unit disc $\mathbb{D}$. Call the map $g : \mathbb{H} \to \mathbb{D}$. Let $\mathcal{S} = \{f \in \mathcal{H}(G) : f(z) \in \mathbb{H} \quad \forall z \in G\}$. Then $g \circ \mathcal{S} = \{g \circ f : f \in \mathcal{S}\}$ is the set of holomorphic functions from G to $\mathbb{D}$. In particular their modulus is bounded by 1. Hence $g \circ \mathcal{S}$ is a normal family of functions. Hence $g^{-1} \circ g \circ \mathcal{S} = \mathcal{S}$ is also a normal family since g^{-1} is also biholomorphic from $\mathbb{D} \to \mathbb{H}$.

Problem 3.4

We know that the given sequence of function $\{f_n\}$ from the normal family has a subsequence $\{f_{n_k}\}$ which converge locally uniformly to f. Suppose we are given that $\{f_n\}$ converge on a the set E having a limit point. Then

$$\lim_{k\to\infty} f_{n_k}(z) = f(z) \quad \forall z \in E$$

Now assume, in contary to what we have to prove, f_n do not converge locally uniformly i.e. suppose $\exists K$, a compact subset of G such that $\{f_n\}$ do not converge uniformly to f on G. Thus $\exists \epsilon > 0$, a subsequence $\{f_{m_i}\}$ and points $z_i \in G$ such that

$$|f_{m_i}(z_i) - f(z_i)| \geq \epsilon \quad \forall i \in \mathbb{N} \tag{$\star$}$$

Now $\{f_{m_i}\}$ itself has a subsequence which converges locally uniformly to an analytic function g. Then by equations $(\star)$ surely $g \neq f$. However g and f agree on all points of E. Then since E has a limit point; looking at zeroes of $f - g$ we find that $f \equiv g$ on G. Contradiction!!

Hence given sequence of holomorphic functions converge locally uniformly in G.

Problem 3.5

Let $\Omega = \{-1 < \Re(z) < 1\}$. Consider the sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ defined by

$$f_n(z) = f(z + in)$$

Let $z = i\lambda \in E = \{\Re(z) = 0\}$, where $\lambda \in \mathbb{R}$. Then

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} f(i(n+\lambda)) = \lim_{y\to\infty} f(iy)$$

exists i.e. the sequence $\{f_n\}$ in $\mathcal{H}(\Omega)$ converge on the set E which has limit points. Also since f is bounded, $\{f_n\}$ is a normal family. Hence by Vitali's theorem (problem 4), we know that f_n converge locally uniformly. In particular consider the compact set $[-r, r] \times [a, b]$ for $a, b \in \mathbb{R}; a < b$. Then the sequence of function f_n converge uniformly on this compact set. But note that for any $(x, y) \in [-r, r] \times [a, b]$,

$$\lim_{n\to\infty} f_n(x + iy) = \lim_{n\to\infty} f(x + i(y + n)) = \lim_{y\to\infty} f(x + iy)$$

Since a, b are arbitrary, we thus have proved that $\lim\limits_{y\to\infty} f(x + iy)$ exists uniformly on $\{-r \leq x \leq r\}$.

Problem 3.6

Call the given sequence of functions $\{f_n\}$. Thus $f_n \in \mathcal{H}(G)$. Define

$$G_N = \{z \in G : |f_n(z)| \leq N \quad \forall n \in \mathbb{N}\}$$

Since $\lim\limits_{n\to\infty} f_n(z)$ exists for all $z \in G$, we get that

$$\bigcup_{i=1}^{\infty} G_i = G$$

Now clearly each set G_N is closed. Consider any nonempty open set U such that $\overline{U} \subseteq G$. Then

$$\bigcup_{i=1}^{\infty} \left(G_i \cap \overline{U}\right) = \overline{U}$$

Hence by Baire category theorem, there exists $N_0 \in \mathbb{N}$ such that $G_i \cap \overline{U}$ is **not** nowhere dense in $\overline{U}$. Thus $\exists$ an open subset $\Delta_U \subseteq G_i \cap \overline{U} \subseteq \overline{U}$ such that $|f_n(z)| \leq N_0$ for all $z \in \Delta_U$. Let

$$G_0 = \bigcup_{\overline{U} \subseteq G} \Delta_U$$

Then note that G_0 is dense and open in G. Since any compact subset K of G can be covered by finitely many Δ_U's we note that the sequence of functions $\{f_n\}$ is locally uniformly bounded. Hence $\{f_n\}$ is a Normal family. Then by problem 4(Vitali's theorem), we have, $\{f_n\}$ converge locally uniformly on G_0 and hence the limit is holomorphic on G_0, an open dense subset of G.

Problem 3.7

****

Problem 3.8

Note that the holomorphic map

$$f : (0, a) \times \mathbb{R} \quad f(t, s) = e^{t + 2\pi i x}$$

is a holomorphic universal covering map of the annulus $\{z : 1 < |z| < e^a\}$ where $z \mapsto z + 2\pi i n$ for $n \in \mathbb{N}$ are the deck transformations. Now the open strip $(0, a) \times \mathbb{R} \equiv \{z : 0 < \Re(z) < a\}$ is conformally equivalent to $\mathbb{H}$ via the biholomorphic map

$$z \mapsto e^{\frac{iz\pi}{a}}$$

Thus we can consider $\mathbb{H}$ as the Universal covering space for the annulus $\{z : 1 < |z| < e^a\}$ with the deck transformations given by

$$z \mapsto e^{\frac{-i\pi 2\pi i n}{a}} z = e^{\frac{2\pi^2 n}{a}} z$$

Note that since the automorphism group of $\mathbb{H}$ is $PSL(2, \mathbb{R})$, the elememt $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $PSL(2, \mathbb{R})$ which acts as multiplication by $e^{\frac{2\pi^2 n}{a}}$ must have $ad - bc = 1$ and $\frac{a}{d} = e^{\frac{2\pi^2 n}{a}}$; implying $a = e^{\frac{\pi^2 n}{a}}, d = e^{-\frac{\pi^2 n}{a}}$. Thus the group of deck transformation is generated by

$$\begin{pmatrix} e^{\frac{\pi^2}{a}} & 0 \\ 0 & e^{-\frac{\pi^2}{a}} \end{pmatrix}$$

in other words we can write

$$A_r = \{z : 1 < |z| < r\} \cong \mathbb{H}/\Gamma_r$$

where

$$\Gamma_r = \left\langle \begin{pmatrix} e^{\frac{\pi^2}{\log r}} & 0 \\ 0 & e^{-\frac{\pi^2}{\log r}} \end{pmatrix} \right\rangle$$

Now we know that given a simply connected space Y and two subgroups G_1, G_2 of $Homeo(Y)$ defining covering space actions on Y; the orbit spaces Y/G_1 and Y/G_2 are homeomorphic iff G_1 and G_2 are conjugate subgroups of $Homeo(Y)$. So if two annuli A_r and A_s are biholoformphically equivalent then Γ_r and Γ_s are conjugate subgroups of $PSL(2,\mathbb{R})$. Thus

$$\begin{pmatrix} e^{\frac{\pi^2}{\log r}} & 0 \\ 0 & e^{-\frac{\pi^2}{\log r}} \end{pmatrix} = g \begin{pmatrix} e^{\frac{\pi^2}{\log s}} & 0 \\ 0 & e^{-\frac{\pi^2}{\log s}} \end{pmatrix} g^{-1}$$

for some $g \in PSL(2,\mathbb{R})$. But then

$$Trace \begin{pmatrix} e^{\frac{\pi^2}{\log r}} & 0 \\ 0 & e^{-\frac{\pi^2}{\log r}} \end{pmatrix} = Trace \begin{pmatrix} e^{\frac{\pi^2}{\log s}} & 0 \\ 0 & e^{-\frac{\pi^2}{\log s}} \end{pmatrix} \Rightarrow r = s$$

Thus there is no biholomorphic equivalence between A_r and A_s if $r \neq s$.

Problem 3.9

Consider any element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$. WLOG we may assume $a \geq 0$. Let $\{e_1, e_2\}$ be the standard Euclidean basis of $\mathbb{R}^2$. Then since $\det g \neq 0$, we know that $\{ge_1, ge_2\}$ forms a basis of $\mathbb{R}^2$.

Suppose ge_1 makes an angle θ with positive $x-axis$. We apply the rotation $\rho_{-\theta}$ a clockwise rotation by $-\theta$ to ge_1 to get a multiple of e_1. Note that

$$\|\rho_{-\theta} ge_1\| = \|ge_1\| = \sqrt{a^2 + c^2} = r(\text{, let})$$

Then multiplying $\rho_{-\theta} ge_1$ by $\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix}$ we get that

$$\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix} \rho_{-\theta} ge_1 = e_1$$

Denote the linear transformation $\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix} \rho_{-\theta} g$ by T. Now we look at what T does to e_2. Note that since $\det g$ is positive, $\{ge_1, ge_2\}$ is oriented the same way as $\{e_1, e_2\}$. So ge_2 is in the upper half plane. Hence after rotating by $-\theta$, multiplying by $\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix}$ which just changes its magnitude; Te_2 is in the upper half plane.

Observe that $\det T = 1$. Thus the area of the parallelogram with sides Te_1 and Te_2 is equal to the area of the unit square which is one. Hence the $y-$coordinate of Te_2 is equal to 1.

Suppose $Te_2 = (x, 1)$. Then applying the linear transformation $S = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$, we note that

$$STe_1 = e_1 \quad \text{and} \quad STe_2 = e_2$$

Hence $ST = Id$. Thus

$$\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{r} & 0\\ 0 & r\end{pmatrix}\rho_{-\theta} g = Id$$

$$\Rightarrow g = \rho_\theta \begin{pmatrix}r & 0\\ 0 & \frac{1}{r}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}e^u & 0\\ 0 & e^{-u}\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$$

where $t = -\theta = -\arccos\frac{a}{\sqrt{a^2+c^2}}$, $u = \ln\sqrt{a^2+c^2}$ and $v = x = \frac{ab+cd}{a^2+c^2}$ are all real numbers.

Problem 3.10

□ We will prove that in fact any covering space of a Riemann Surface is a Riemann surface. Consider an universal cover $\tilde{S}$ of a Riemann surface S and let $\pi : \tilde{S} \to S$ be a covering map. Then given any point $\tilde{p} \in \tilde{S}$, let $p = \pi(\tilde{p})$. Then can find an open neighbourhood U_p around p in S such that restriction of E to each component of $\pi^{-1}(U_p)$ is a homeomorphism. Let $\tilde{U_p}$ be the component which contains $\tilde{p}$. Thus we have an open neighbourhood $\tilde{U_p}$ of $\tilde{p}$ homeomorphic to an open neighbourhood U_p of p. Shrinking U_p to a smaller neighbourhood of p if necessary, we can then find a coordinate chart $\varphi : U_p \to \mathbb{C}$. Then we claim that we can use $\varphi \circ \pi : \tilde{U_p} \to \mathbb{C}$ as a coordinate chart for making $\tilde{S}$ a Riemann Surface.

Clearly by considering preimage of an open neighbourhood of $\varphi(\pi(\tilde{p}))$ in $\varphi(\pi(\tilde{U_p}))$ we may assume that $\varphi \circ \pi$ is a homeomorphism from a neighbourhood $\tilde{U_p}$ of $\tilde{p}$ to the unit disc $\mathbb{D}$. Now let $a, b \in \tilde{S}$ be two distinct points in $\tilde{S}$. Then we can find charts $\psi_a \equiv \varphi_a \circ \pi : U_a \to \mathbb{D}$ and $\psi_b \equiv \varphi_b \circ \pi : U_b \to \mathbb{D}$ so that on $\psi_a(U_a) \cap \psi_b(U_b)$ the map $\psi_a \circ \psi_b^{-1} = \varphi_a \circ \varphi_b^{-1}$ is analytic since S is a Riemann surface. Thus completing this collection of charts to a maximal one, we get that $\tilde{S}$ is a Riemann Surface.

□ Thus we have proved that the Universal cover of a Riemann surface is a Riemann surface. Since the universal cover is simply connected, by Uniformization theorem, we can further say that the universal cover of any Riemann surface is conformally equivalent to either $\hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$.

Note that $\hat{\mathbb{C}}$ is compact. Hence if the universal cover of a Riemann surface is conformally equivalent to $\hat{\mathbb{C}}$, then S must be compact.

Now the Riemann surface $S = \hat{\mathbb{C}} \backslash \{z_0, z_1, z_2\}$ is not compact. Hence $\tilde{S}$ is not conformally equivalent to $\hat{\mathbb{C}}$. The only other possibilities are then $\mathbb{C}$ or $\mathbb{D}$. We claim the following

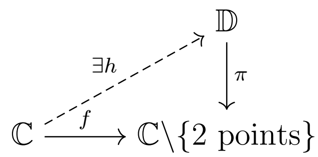
Claim: Suppose that S is a Riemann surface which has a non-abelian fundamental group. Then there is no complex analytic covering map of the form $\pi : \mathbb{C} \to S$.

Proof: Suppose $\exists$ a covering map $\pi : \mathbb{C} \to S$. Let G be the fundamental group of S. Then G acts on $\mathbb{C}$ as the deck transformation group. Hence each $g \in G$ must act as a linear map on $\mathbb{C}$. Also, g does not fix any point of $\mathbb{C}$ because a deck transformation is the identity if it fixes even one point. The only linear maps with this property are the translation. In short g is a translations. But any two translations commute and hence G is abelian. This contradiction shows that π does not exist. ■

Now note that $\hat{\mathbb{C}} \backslash \{z_0, z_1, z_2\}$ is homeomorphic to $\mathbb{C} \backslash \{2 \text{ points}\}$. Hence $\pi_1\left(\hat{\mathbb{C}} \backslash \{z_0, z_1, z_2\}\right) = \pi_1(S^1 \wedge S^1) = \mathbb{Z} * \mathbb{Z}$; which is nonabelian. Hence $\tilde{S}$ cannot be $\mathbb{C}$. Thus we have proved that $\tilde{S}$ is conformally equivalent to $\mathbb{D}$.

Problem 3.11

We can think of $f : \mathbb{C} \to \mathbb{C}$ as $f : \mathbb{C} \to \mathbb{C}\backslash\{2 \text{ points}\}$ since f omits two values. Now we know that the univerasal cover of $\mathbb{C}\backslash\{2 \text{ points}\}$ is conformally equivalent to $\mathbb{D}$. Thus we have a covering map $\pi : \mathbb{D} \to \mathbb{C}\backslash\{2 \text{ points}\}$.



Thus we can lift f to a holomorphic map $h : \mathbb{C} \to \mathbb{D}$ such that above diagram commutes. But then h is a bounded holomorphic entire function. Hence by Liouville's theorem h must be constant. Then $f = \pi \circ h$ is constant as well. Thus any holomorphic map from $\mathbb{C}$ to $\mathbb{C}$ which omits two values must be constant.

Problem 3.12

3.12.(a) Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be two elements of $PSL(2, \mathbb{Z})$. We first prove that S, T generate $PSL(2, \mathbb{Z})$.

Let $G = \langle S, T \rangle \subseteq PSL(2, \mathbb{Z})$ be the subgroup generated by S and T. We first look at the action of S and T^n on any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting by multiplication on left. Note that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \text{ and } T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}$$

Now start with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$. Cosider the following operations on $g_1 = g$. If $c \neq 0$ and $|a| \geq |c|$, then applying division algorithm we can find $q, r \in \mathbb{Z}$ such that $a = cq + r$ with $0 \leq r < |c|$. Then applying T^{-q} to g we get that the abosolute value of the upper left entry is now less than the upper left value of lower left entry. Then we apply S to $T^{-q}g$. If the lower left entry of $ST^{-q}g$ is still nonzero, then we know that the abosolute value of the upper left entry is now bigger than the upper left value of lower left entry. Call this new matrix g_2. We apply same operations on g_2 as before. Continuing this way observe that the lower left term of g_i is a strictly decreasing series and hence the series of operation stops at some point and we get a matrix g_k with 0 as its lower left entry. Now $g_k \in PSL(2, \mathbb{Z})$ then implies it is of the form

$$g_k = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

for some $m \in \mathbb{Z}$. But clearly $g_k = T^m$. Hence $Rg = T^m$ for some $R \in G$. Hence $g = R^{-1}T^m \in G$. Thus

$$PSL(2, \mathbb{Z}) = \langle S, T \rangle$$

We finish the proof by noting that $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = TST$ and so $S = T^{-1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}T^{-1}$. Hence

$$G \subseteq \left\langle T, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\rangle \subseteq G \Rightarrow PSL(2,\mathbb{Z}) = \left\langle \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle.$$

3.12.(c) Let $\Omega = \{z \in \mathbb{C} : |\Re(z)| \le \frac{1}{2}, |z| \ge 1\}$. Let $g \in PSL(2,\mathbb{Z}) = \langle S, T\rangle$ where S and T are as above. Thus $Sz = -\frac{1}{z}$ and $Tz = z+1$. Then note that

$$\Im(gz) = \frac{\Im(z)}{|cz+d|^2}.$$

Since c, d are integers, the numbers of pairs (c,d) such that $|cz+d|$ is less than a given number is finite. This shows that there exists $g \in PSL(2,\mathbb{Z})$ such that $\Im(gz)$ is maximum. Now choose an integer n such that $T^n gz$ has real part between $-\frac{1}{2}$ and $\frac{1}{2}$. We claim that the element $z' = T^n gz$ belongs to Ω.

Indeed it suffices to prove that $|z'| \ge 1$. But if $|z'| < 1$, then $-\frac{1}{z'}$ would have an imaginary part strictly larger that $\Im(z)$ which is not possible by construction.

Thus we have found an element $g' = T^n g \in PSL(2,\mathbb{Z})$ such that given any $z \in \mathbb{C}$, there exists $z' \in \Omega$ with $z' = g'z$. Now suppose we have an element $z \in \Omega$ and $g \in PSL(2,\mathbb{Z})$ such that $gz \in \Omega$. Wlog we may assume $\Im(gz) \ge \Im(z)$ i.e. $|cz+d| \le 1$. This is clearly not possible if $|c| \ge 2$. Wlog we may assume $c \ge 0$. Thus $c = 0$ or 1.

If $c = 0$, then $d = \pm 1$, since $|d| \le 1$ and $\det g \ne 0$. Thus $gz = z \pm b$. Since $\Re(z)$ and $\Re(gz)$ are both between $-\frac{1}{2}$ ad$\frac{1}{2}$, this implies either $b = 0$ and $g = I$ or $b = \pm 1$ in which case one of $\Re(z)$ and $\Re(gz)$ is $-\frac{1}{2}$ and the other $\frac{1}{2}$.

If $c = 1$, the fact that $|z+d| \le 1$ implies $d = 0$ except if $z = e^{\frac{2\pi i}{3}}$ (resp. $e^{\frac{\pi i}{3}}$) in which case we have $d = 0, 1$ (resp. $d = 0, -1$). The case $d = 0$ gives $|z| \le 1$ hence $|z| = l$; on the other hand, $ad - bc = 1$ implies $b = -l$, hence $gz = a - 1/z$ and the first part of the discussion proves that $a = 0$ except if $\Re(z) = \pm\frac{1}{2}$ i.e. if $z = e^{\frac{2\pi i}{3}}$ or $e^{\frac{\pi i}{3}}$ in which case we have $a = 0, -1$ or $a = 0, 1$. The case $z = e^{\frac{2\pi i}{3}}, d = 1$ gives $a - b = 1$ and $ge^{\frac{\pi i}{3}} = a + e^{\frac{2\pi i}{3}}$, hence $a = 0, 1$; we argue similarly when $z = e^{\frac{\pi i}{3}}, d = -1$.

Thus we can take

$$D = \{z \in \mathbb{C} : |\Re(z)| < \frac{1}{2}, |z| > 1\} \cup \{|z| = 1, -\frac{1}{2} \le \Re(z) \le 0\} \cup \{\Im(z) > 0, |z| \ge 1, \Re(z) = -\frac{1}{2}$$

to be the fundamental domain for $PSL(2,\mathbb{Z})$.

3.12.(b) Let $g \in PSL(2,\mathbb{Z})$ be an element of finite order n. Suppose the minimal polynomial of g is $m(X)$ and the characteristic polynomial $ch(X)$. Then $g$is a 2×2 matrix $\Rightarrow \deg ch \le 2 \Rightarrow \deg m \le 2$. But $m(X) | X^n - 1 \Rightarrow m(X)$ is squarefree. Thus $m(X)$ can be one of the following

$$X+1, X^2+1, X^2+X+1$$

Since we are in $PSL(2,\mathbb{Z})$, we are working with characteristic 2. Note that if g and h are two elements of $PSL(2,\mathbb{Z})$ of finite order and have the same minimal polynomial, then they must be conjugate. Now the matrices of $PSL(2,\mathbb{Z})$ which have the above minimal polynomial are I, S, ST. Thus all the finite order elements are I or conjugates of S, ST.

Problem 3.13

First of all note that curvature of a surface remains constant under conformal mapping. Now given any compact Riemann surface S its universal cover $\tilde{S}$ is conformally equivalent to either $\hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$. We know that we can produce a Riemannian metric of constant curvature 1 on $\hat{\mathbb{C}} \cong S^2$. Similarly we can give $\mathbb{C}$ a Riemannian metric of constant curvature 1 and $\mathbb{D} \cong \mathbb{H}$ a metric of constant curvature -1. Then if we define the metric of $\tilde{S}$ to be the pullback metric from $\hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$; we can get a Riemannian metric of constant curvature on $\tilde{S}$. Note that by our construction of the analytic structure on $\tilde{S}$ in problem 10, the covering map is an analytic conformal map. Thus pushing forward the Riemannian metric on $\tilde{S}$ to S we can define a Riemannian metric of constant curvature on S.

Clearly this metric is not unique since we can renormalize it and still keep the same curvature. In fact under any conformal equivalence of S, we can give S a new metric by pulling back the original one and keeping constant curvature.

Suppose the given Riemann surface S has genus g and constant curvature K. Then Euler characteristic of S is equal to $2-2g$. By Gauss-Bonnet formula, we know that

$$\int_S K d(area) = 2\pi\chi(S)$$

Thus $\texttt{sign}(K) = \texttt{sign}(2-2g)$. In particular we have three cases:

- If $g=0$, then the curvature is positive.
- If $g=1$, then the curvature is zero.
- If $g>1$, then the curvature is negative.

Problem 3.14

We claim that such an enumeration exists with the stronger property that $E_n = T_1 + T_2 + \ldots + T_n$ is simply connected. We will prove the claim inducting on n. The base case is trivially true. So suppose we have found an enumeration $T_1, T_2, \ldots, T_n$ of n triangles in $\mathcal{T}$ such that E_n is simply connected. Thus ∂E_n is a polygon with sides as edges of some of the triangles $T_1, \ldots, T_n$. Suppose there are k vertices $v_1, v_2, \ldots, v_k$ on the boundary. Note that v_i is a vertex of some $T_j (1 \le j \le n)$ for each $i = 1, \ldots, k$. Now consider any triangle $T \in \mathcal{T}$ different from $T_1, T_2, \ldots, T_n$, which shares at least one edge with E_n. Wlog, we may assume $\overline{v_1 v_2}$ is an edge of T. We can have following cases:

- If $\overline{v_2 v_3}$ is also an edge of T, then we can name T to be T_{n+1} and we can check that it satisfies given properties.
- If T shares only one edge with E_n there are two possibilities:

  - The vertex opposite $\overline{v_1v_2}$ in T is not equal to any of $v_3, v_4, \ldots, v_k$. Then again we can rename T to be T_{n+1} and it satisfies given properties.
  - The vertex opposite $\overline{v_1v_2}$ in T belongs to $\{v_3, v_4, \ldots, v_k\}$. Suppose $T = \triangle v_1v_2v_i$. Then note that $3 \leq i \leq k$. Since E_n is simply connected and $\mathcal{T}$ is a triangulation, one of the regions bounded by the two polygonal paths [$v_2v_3\ldots v_i$ in E_n and the edge $\overline{v_iv_2}$] OR [$v_1v_kv_{k-1}\ldots v_i$ in E_n and the edge $\overline{v_iv_1}$] is bounded , simply connected and homeomorphic to $\mathbb{D}$. Wlog let it be the first one.

    Then consider the edge $\overline{v_2v_3}$ of E_n. Similar to the cases described above, if we can't find a triangle $T' \in \mathcal{T}$ sharing two edges $\overline{v_2v_3}$ and $\overline{v_3v_4}$ with E_n OR a triangle sharing only one edge $\overline{v_2v_3}$ but no vertex with E_n; then we can find $j \in \{4, \ldots, i\}$ such that $T' = \triangle v_2v_3v_j$. Note that the fact that T' does not intersect T forces $j \leq i$.

    Continuing this way, and assuming at each step we can't get a triangle sharing two edges or only one edge(no vertex) with E_n we arrive at last at the following situation:

    We have a triangle S outside E_n which shares an edge $\overline{v_dv_{d+1}}$ with E_n and a vertex v_{d+2} with E_n. Here the indices are clearly modulo k. Then since $\mathcal{T}$ is a triangulation we must have a triangle S' outside E_n which shares the edge $\overline{v_dv_{d+1}}$ with E_n. Then vertex of S' opposite to $\overline{v_dv_{d+1}}$ cannot be in E_n since there is cannot be a path of length zero in a triangulation. Thus we can name S' to be T_{n+1}.

Thus in any case, we can always find T_{n+1} so that the induction hypothesis is satisfied. Note that by construction E_{n+1} is simply connected. Hence by Induction principle, an enumeration with given properties exist.