# Analysis III
# Home Assignment 2

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## Problem 2.1

Suppose the set of points $\mathcal{S}$ where $\{|f| < c\}$ is not simply connected. Then we can find a simple closed curve $\gamma$ in $\mathcal{S}$ such that $\gamma$ is not null-homotopic. Note that we can break $\gamma$ into union of simple closed curves such that each part is homeomorphic to $S^1$. Then there is at least one part, let it be $\gamma'$ such that $\gamma' \cong S^1$ and $\gamma'$ is not null homotopic in $\mathcal{S}$. Suppose $\mathcal{B}$ denote the interior of the domain bounded by $\gamma'$. Then clearly, there exists $z_0 \in \mathcal{B}$ such that $z_0 \notin \mathcal{S}$. Note that $z \in \Omega$ since $\Omega$ is simply connected. Thus $|f(z_0)| \geq c$. Then we have $f$ holomorphic on $\mathcal{B}$ and a point $z_0 \in \mathcal{B}$ such that $|f(z_0)| > |f(z)|$ for $z \in \partial B = \gamma'$ which is false by the maximum modulus principle. Thus the set of points where $\{|f| < c\}$ is simply connected.

## Problem 2.2

Suppose, there is a continuous branch of $\log z$ on $\Omega$ but 0 and $\infty$ do not belong to the same connected component of $\hat{\mathbb{C}}\backslash\Omega$. Denote the component containing 0 by $\Omega_1$ and $(\hat{\mathbb{C}}\backslash\Omega)\backslash\Omega_1$ by $\Omega_2$. Note that $\Omega_2$ contains $\infty$. Also $\Omega_1$ and $\Omega_2$ are closed and bounded; hence compact. So $dist(\Omega_1, \Omega_2) = d > 0$.

Now we cover $\Omega_1$ with open $\epsilon$ balls $\{B_\epsilon(z)\}$ around each point $z \in \Omega_1$ where $\epsilon = d/3$. By compactness we can choose a finite number of them $B_\epsilon(z_1), B_\epsilon(z_2), \ldots, B_\epsilon(z_n)$ which cover $\Omega_1$. Consider the union

$$D = \bigcup_{i=1}^{n} B_\epsilon(z_i)$$

Let $\gamma = \partial D$. Then note that $\gamma$ is a piecewise smooth curve entirely contained in $\Omega$ and it is not null-homotopic. In particular

$$\int_\gamma \frac{dz}{z} = Ind_\gamma(0) = 1 \neq 0$$

However if a continuous branch of $\log z$ exists then

$$\int_\gamma \frac{dz}{z} = 0$$

Contradiction!! Hence if there is a continuous branch of $\log z$ on $\Omega$ 0 and $\infty$ must belong to the same connected component of $\hat{\mathbb{C}}\backslash\Omega$.

Conversely, suppose 0 and $\infty$ must belong to the same connected component of $\hat{\mathbb{C}}\backslash\Omega$. Then we can draw a curve $\sigma$ joining 0 and $\infty$ on $(\hat{\mathbb{C}})$ which lies entirely in $\hat{\mathbb{C}}\backslash\Omega$. Thus if $\gamma$ is any closed curve in $\Omega$; then it cannot intersect $\sigma$. In particular, $\gamma$ does not go around 0 i.e. $Ind_\gamma(0) = 0$. Thus for any closed curve $\gamma$ in $\Omega$ we have

$$\int_\gamma \frac{dz}{z} = 0$$

Now we fix an $z_0 \in \Omega$ and given any $z \in \Omega$ we can define

$$f(z) = \int_{z_0}^{z} \frac{d\zeta}{\zeta}$$

where the integral is taken over any path connecting from $z_0$ to $z$. Since the integral over a closed simple curve is zero, the function $f$ is well defined. Then clearly $\tilde{f} = f + \log(z_0)$ is a branch of $\log z$ on $\Omega$. So a continuous branch of $\log z$ exists on $\Omega$.

# Problem 2.3

First, Note that if a branch of $\log f$ exists in $\Omega$ then we can define

$$g(z) = e^{\frac{1}{n}\log f}$$

on $\Omega$ to be a branch of the $n^{th}$ root of $f$ for $n \in \mathbb{N}$, since $g^n = f$. Thus there is a branch of $f^{\frac{1}{n}}$ in $\Omega$ for all $n \in \mathbb{N}$.

Conversely suppose a branch of $f^{1/n}$ exists in $\Omega$ for each $n \in \mathbb{N}$. Then let $g = f^{\frac{1}{n}}$ be a branch of $n^{th}$ root of $f$ in $\Omega$ for some $n \in \mathbb{N}$. Then,

$$g^n = f \Rightarrow f' = ng^{n-1}g' \Rightarrow \frac{f'}{f} = n\frac{g'}{g}$$

since $f \neq 0$ on $\Omega$ implies $g \neq 0$ on $\Omega$. Also by assumption, since $f$ is holomorphic, so is $g$. Now for any simple closed curve $\gamma$ in $\Omega$,

$$\int_\gamma \frac{g'}{g} dz = \int_{f\circ\gamma} \frac{dw}{w} = ind_{f\circ\gamma}(0) \in \mathbb{Z}$$

So

$$\int_\gamma \frac{f'}{f} \in n\mathbb{Z}$$

Note that above result is true for for all $n \in \mathbb{N}$. Hence

$$\int_\gamma \frac{f'}{f} = 0$$

Thus we have proved that for any simple closed curve $\gamma$ in $\Omega$ we have

$$\int_\gamma \frac{f'}{f} = 0$$

Now we fix an $z_0 \in \Omega$ and given any $z \in \Omega$ we can define

$$h(z) = \int_{z_0}^{z} \frac{f'}{f}$$

where the integral is taken over any path connecting from $z_0$ to $z$. Since the integral over a closed simple curve is zero, the function $h$ is well defined. Then clearly $\tilde{h} = h + \log(f(z_0))$ is a branch of $\log f$ on $\Omega$. Thus existence of branch of $f^{1/n}$ for all $n \in \mathbb{N}$ implies the existence of branch of $\log f$.

# Problem 2.4

Suppose $\{z_i\}_{i\in I}$ be the zeros of $f$ in $\mathbb{D}$ and $\{b_j\}_{j\in J}$ be the zeros of $f$ on $\partial\mathbb{D}$ [both counted with repetitions according to their multiplicities]. We want to prove that $\frac{1}{2\pi}$ times the net change experienced by $arg(f)$ in one counterclockwise circuit around $\partial\mathbb{D}$ equals $|I| + \frac{1}{2}|J|$.

Note that since $f$ is holomorphic in a neighbourhood of $\overline{\mathbb{D}}$; the roots of $f$ are discrete and hence $\exists r_j > 0$ such that

- $f$ is holomorphic on $B_{r_j}(b_j)$ for each $j \in J$
- $B_{r_j}(b_j)$ does not contain any other zero of $f$.

Let

$$\epsilon = \min_{j\in J}\{r_j\}$$

Let $C_j$ denote the arc of $\partial B_\epsilon(b_j)$ which is outside $\mathbb{D}$ and let $\gamma_\epsilon$ be the part of $\partial\mathbb{D}$ which is outside $\bigcup_{j\in J} B_\epsilon(b_j)$. Thus

$$\gamma := \gamma_\epsilon \cup \bigcup_{j\in J} C_j = \partial\left(\mathbb{D} \cup \bigcup_{j\in J} B_\epsilon(b_j)\right)$$

Let $\Omega = \mathbb{D} \cup \bigcup_{j\in J} B_\epsilon(b_j)$.

Thus we have simple closed curve $\gamma = \partial\Omega$ such that $f$ is holomorphic on $\Omega$ and $f$ has no roots on $\gamma$. Then by applying the classical Argument formula on $f$ w.r.t. $\gamma$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = |I| + |J| \Rightarrow |I| + |J| = \frac{1}{2\pi i}\int_{\gamma_\epsilon} \frac{f'}{f} + \frac{1}{2\pi i}\sum_{j\in J}\int_{C_j}\frac{f'}{f}$$

Now let us look at $\int_{C_j}\frac{f'}{f}$ for a specific $j$. Note that wlog we may assume $\epsilon$ is small enough so that

$$f(z) = (z-b_j)^{m_j} g(z)$$

around $b_j$, in particular on $B_\epsilon(b_j)$, where $m_j$ is the multiplicity of $b_j$ as a root of $f$. Thus $g(z) \neq 0$ on $B_\epsilon(b_j)$. Then

$$\frac{f'}{f} = \frac{m_j}{z-b_j} + \frac{g'}{g};\ \text{on } C_j$$

Note that we can write

$$C_j = \{b + \epsilon e^{i\theta} : -\tau \le \theta \le \pi + \tau\}$$

for some $\tau$ such that $\tau \to 0$ as $\epsilon \to 0$. Then

$$\int_{C_j}\frac{f'}{f} = m_j\int_{-\tau}^{\pi+\tau} \epsilon^{-1}e^{-i\theta}\epsilon i e^{i\theta} d\theta + \int_{C_j}\frac{g'}{g} = im_j(\pi + 2\tau) + \int_{C_j}\frac{g'}{g}$$

Next note that since $g$ is nonvanishing on $B_\epsilon(b_j)$, $\frac{g'}{g}$ is holomorphic on this domain and hence bounded. But as $\epsilon \to 0$, $length(C_j) \to 0$. Hence, as $\epsilon \to 0$,

$$\int_{C_j}\frac{g'}{g} \to 0$$

Thus

$$\lim_{\epsilon\to 0}\int_{C_j}\frac{f'}{f} = i\pi m_j$$

Also clearly as $\epsilon \to 0$, $\gamma_\epsilon \to \partial\mathbb{D}$, so that the net change experienced by $arg(f)$ in one counterclockwise circuit around $\partial\mathbb{D}$, denoted by $\Theta$, equals $\lim\limits_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{f'}{f}$. Thus by above,

$$\begin{aligned}
\lim_{\epsilon\to 0}(|I| + |J|) =& \lim_{\epsilon\to 0}\left(\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{f'}{f} + \frac{1}{2\pi i}\sum_{j\in J}\int_{C_j}\frac{f'}{f}\right) \\
=& \frac{1}{2\pi i}\Theta + \frac{1}{2\pi i}\sum_{j\in J} m_j\pi i \\
=& \frac{1}{2\pi i}\Theta + \frac{1}{2\pi i}\pi i|J| \\
\Rightarrow& \frac{1}{2\pi i}\Theta = |I| + \frac{1}{2}|J|
\end{aligned}$$

# Problem 2.5

By Maximum modulus principle we know that $|f|$ attains its maximum on the boundary of the annulus $A$. Hence $|f(z)| \le 1 \quad \forall z \in A$; implying that $f(A) \subseteq \mathbb{D}$, the unit disc. Now suppose $f$ does not have any root in $A$. Then we can apply the maximum modulus principle to $\frac{1}{f}$ to get that $|\frac{1}{f(z)}| \le 1 \quad \forall z \in A$. Then we must have $|f(z)| = 1 \quad \forall z \in \overline{A}$ which then implies (by Maximum modulus principle) that $f$ must be a constant function. Contradiction!! Hence $f$ has atleast one zero in $A$.

Next note that we are given $f(\partial A) \subseteq$ the unit circle in $\mathbb{C}$. We know that if $\psi : S^1 \to S^1$ is not onto then $deg(\psi) = 0$ i.e. $\psi$ is null-homotopic. Let $\gamma_r$ denote the inner boundary of $A$ and let $\gamma_R$ denote the outer boundary of $A$ ($R > r$). Then we know that $f : \gamma_r(\cong S^1) \to S^1$ and $f : \gamma_R(\cong S^1) \to S^1$. If none of the image $f(\gamma_r)$ or $f(\gamma_R)$ is the full $S^1$; i.e. if $f$ is not surjective restricted to $\gamma_r$ or $\gamma_R$; then clearly $f \circ \gamma_r$ and $f \circ \gamma_R$ are both null-homotopic. Hence we then have

$$\int_{\gamma_R} \frac{f'}{f} = 0 = \int_{\gamma_r} \frac{f'}{f}$$

On the other hand considering the boundary of $A$ as $\partial A = \gamma_R - \gamma_r$ where $\gamma_R$ is oriented anticlockwise and $\gamma_r$ is oriented clockwise, we have

$$\text{number of roots of } f \text{ inside } A = \int_{\partial A} \frac{f'}{f} = \int_{\gamma_R} \frac{f'}{f} - \int_{\gamma_r} \frac{f'}{f} = 0$$

But that contradicts the result in the first paragraph. Hence at least one of $f \circ \gamma_R$ and $f \circ \gamma_r$ is equal to $S^1$.

Next note that

$$Ind_{f\circ\partial A}(w) = \int_{\partial A} \frac{f'(z)dz}{f(z) - w}$$

is constant on each connected component. In particular,

$$Ind_{f\circ\partial A}(0) = Ind_{f\circ\partial A}(w)$$

for all $|w| < 1$ i.e. $w \in \mathbb{D}$. But if wlog, we assume $f \circ \gamma_R = \partial\mathbb{D}$ then, there exists $z_0 \in f(\gamma_r)$ such that $|z_0| = 1$ and $z_0 \in f \circ \gamma_R$ i.e. $f$ takes the value $z_0$ at least twice on $\overline{A}$. Then $\exists w \in \mathbb{D}$ which is in a small enough neighbourhood of $z_0$ such that by continuity, $f$ takes the value $w$ at least twice. Thus $Ind_{f\circ\partial A}(w) \geq 2 \Rightarrow Ind_{f\circ\partial A}(0) \geq 2$. Thus $f$ has atleast two roots in $A$.

## Problem 2.6

Define a function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = a_0 + a_1 z + \ldots + a_n z^n$$

Clearly $F \in \mathcal{H}(\mathbb{C})$. Writing $z \in \mathbb{C}$ in polar coordinates we get that

$$f(\theta) = F(1.e^{i\theta}) \quad \forall \theta \in \mathbb{R}$$

Suppose for all $\theta \in \mathbb{R}$, we have $|f(\theta)| \leq |a_0| = |F(0)|$. Thus $|F(z)| \leq |F(0)|$ for all $z \in \mathbb{C}$ such that $|z| = 1$. But by the maximum modulus principle, the maximum of $|F|$ on $\overline{\mathbb{D}}$ must occur on the boundary. Hence $\exists z_0 \in \partial\mathbb{D}$ with $|z_0| = 1$ such that $|F(z_0)| \geq |F(z)|$ for all $z \in \mathbb{D}$. Then we have $|F(0)| \geq |F(z)|$ for all $z \in \mathbb{D}$ which contradicts maximum modulus principle unless $F$ is a constant function. But $F$ is clearly not a constant function. Thus our assumption was wrong. So there exists $\theta \in \mathbb{R}$ such that $|f(\theta)| > |a_0|$.

## Problem 2.7

Suppose $z_1, z_2, \ldots, z_n$ be the roots of $f$. Then

$$f(z) = \prod_{i=1}^{n}(z - z_i) \Rightarrow \frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i} = \overline{\sum_{i=1}^{n} \frac{z - z_i}{|z - z_i|^2}}$$

If $z_i$ is root of $f'$ for any $i$, then clearly it is contained in the closed convex hull of $z_i$'s. So suppose $f'(z_i) \neq 0$ for all $i$. Then for any root $w$ of $f'$, we have $f(w) \neq 0, |w - z_i| \neq 0$ and hence,

$$0 = \overline{\sum_{i=1}^{n} \frac{w - z_i}{|w - z_i|^2}} \Rightarrow \sum_{i=1}^{n} \frac{z_i}{|w - z_i|^2} = \sum_{i=1}^{n} \frac{w}{|w - z_i|^2} = w \sum_{i=1}^{n} \frac{1}{|w - z_i|^2} \Rightarrow w = \sum_{i=1}^{n} \lambda_i z_i$$

where

$$\lambda_i = \frac{\frac{1}{|w - z_i|^2}}{\sum_{i=1}^{n} \frac{1}{|w - z_i|^2}}$$

Thus we have $w = \sum_{i=1}^{n} \lambda_i z_i$ with $0 \leq \lambda_i \leq 1$ for all $i$ and $\sum_{i=1}^{n} \lambda_i = 1$. Hence $w$ is contained in the closed convex hull of $z_i$'s.

## Problem 2.8

When $|z| = 1$, we have

$$|f(z) - (5z^3)| = |z^5 + z^2 + z + 1| \leq |z|^5 + |z|^2 + |z| + |1| = 4 < 5 = |5z^3|$$

Hence by Rouche's theorem, $f(z)$ and $5z^3$ have the same number of zeroes in $|z| < 1$. Thus $f(z)$ has three zeroes inside $|z| < 1$.

On the other hand, when $|z| = 2 + \epsilon$ where $\epsilon > 0$, we have

$$|f(z) - 5z^3| = |z^5 + z^2 + z + 1| \leq |z|^5 + |z|^2 + |z| + 1 = (2 + \epsilon)^5 + (2 + \epsilon)^2 + (2 + \epsilon) + 1 = 39 + \epsilon g(\epsilon)$$

and

$$|5z^3| = 5(2 + \epsilon)^3 = 40 + \epsilon h(\epsilon)$$

where $g$ and $h$ are integer polynomial in $\epsilon$ defined by above equations. Then for sufficiently small $\epsilon$, we can make $\epsilon(g(\epsilon) - h(\epsilon) < 1$, so that

$$|f(z) - 5z^3| < |5z^3|$$

Thus again by Rouche's theorem, $f$ and $5z^3$ have same number of zeroes inside $|z| < 2 + \epsilon$ for sufficiently small $\epsilon > 0$. Thus $f$ has 3 zeros inside $|z| \leq 2$.

Combining above two results, $f$ has no zero in the annulus $1 \leq |z| \leq 2$.