Analysis III
Home Assignment 1

Subhadip Chowdhury

Problem 1.1

1.1.(a)
we can characterize an ellipse by the locus of points such that the sum of distances from the two foci is constant. Thus we have three cases:

Case 1: $k > |a-b|$ The set of points $z \in \mathbb{C}$ such that

$$|z-a| + |z-b| \leq k$$

is the region bounded by the ellipse on complex plane with a and b as foci and length of major axis equal to k.

Case 2: $k = |a-b|$ The set of points $z \in \mathbb{C}$ such that

$$|z-a| + |z-b| \leq k$$

is the line segment $\overline{ab}$.

Case 2: $k < |a-b|$ The set of points $z \in \mathbb{C}$ such that

$$|z-a| + |z-b| \leq k$$

is the empty set by triangle inequality.

1.1.(b)

$$\left|\frac{z-a}{1-\overline{a}z}\right| = \frac{1}{|a|}\left|\frac{z-a}{\frac{a}{|a|^2}-z}\right|$$

Thus

$$|w| \mathrel{\substack{\geq \\ = \\ <}} 1 \iff |z-a| \mathrel{\substack{\geq \\ = \\ <}} |a|\left|z-\frac{a}{|a|^2}\right|$$

By Appollonius' definition of a circle, we then get that $|w|=1$ is the circle with center given by

$$z_0 = \frac{1}{1-|a|^2}a - \frac{|a|^2}{1-|a|^2}\frac{a}{|a|^2} = 0$$

and radius

$$r = \frac{|a|}{1-|a|^2}\left|a - \frac{a}{|a|^2}\right| = 1$$

since $|a| < 1$. Thus

$$\{|w| < 1\} = \{z \in \mathbb{C} : |z| < 1\}$$
$$\{|w| = 1\} = \{z \in \mathbb{C} : |z| = 1\}$$
$$\{|w| > 1\} = \{z \in \mathbb{C} : |z| > 1\}$$

Problem 1.2

Let $z_1, z_2, \cdots, z_n$ be the roots of P. Thus $|z_i| < 1$ for all i and $P(z) = \prod_{j=1}^{n}(z - z_j)$. Hence

$$P^*(z) = z^n \prod_{j=1}^{n}\left(\frac{1}{z} - \overline{z_j}\right) = \prod_{j=1}^{n}(1 - \overline{z_j}z)$$

Suppose z_0 is a root of $P + e^{i\theta}P^*$. Then

$$P(z_0) = -e^{i\theta}P^*(z_0) \Rightarrow |P(z_0)| = |P^*(z_0)| \Rightarrow \prod_{j=1}^{n}\left|\frac{z_0 - z_j}{1 - \overline{z_j}z_0}\right| = 1$$

Suppose $|z_0| > 1$. Then by problem 1.1.(a), we know that $\left|\frac{z_0 - z_j}{1-\overline{z_j}z_0}\right| < 1$ for all j since $|z_j| < 1$. Similarly if $|z_0| < 1$, then $\left|\frac{z_0 - z_j}{1-\overline{z_j}z_0}\right| > 1$ for all j . In either case $\prod_{j=1}^{n}\left|\frac{z_0 - z_j}{1-\overline{z_j}z_0}\right| \neq 1$. So

$$\prod_{j=1}^{n}\left|\frac{z_0 - z_j}{1 - \overline{z_j}z_0}\right| = 1 \Rightarrow |z_0| = 1$$

Thus all roots of $P + P^*$ lie on $\{|z| = 1\}$.

Problem 1.3

Observe that

$$(1 - z)P(z) = \sum_{i=1}^{n}(p_i - p_{i-1})z^i - p_n z^{n+1} + p_0$$

Thus

$$\begin{aligned}
|(1 - z)P(z)| &= \left|p_0 - \sum_{i=1}^{n}(p_{i-1} - p_i)z^i - p_n z^{n+1}\right| \\
&\geq p_0 - \left|\sum_{i=1}^{n}(p_{i-1} - p_i)z^i\right| - |p_n z^{n+1}| \\
&\geq p_0 - \sum_{i=1}^{n}\left|(p_{i-1} - p_i)z^i\right| - |p_n z^{n+1}| \\
&\geq p_0 - \sum_{i=1}^{n}(p_{i-1} - p_i)|z|^i - |p_n||z|^{n+1}
\end{aligned}$$

Now suppose P has a root z_0 such that $|z_0| < 1$. Then by above inequality,

$$0 = |(1-z_0)P(z_0)| \geq p_0 - \sum_{i=1}^{n}(p_{i-1}-p_i)|z_0|^i - p_n|z_0|^{n+1} \geq p_0 - \sum_{i=1}^{n}(p_{i-1}-p_i) - p_n = 0$$

Thus equality must occur in all previous steps when $z = z_0$. Hence $|z_0| = 1$ and z_0^i are collinear with origin for all i. Hence $z_0 = \pm 1$. But $P(\pm 1) \neq 0$. Contradiction!!
Thus all roots of P lie in the region $\{|z| > 1\}$.

Problem 1.4

1.4.(a) Let $\Phi(x_1, x_2, x_3) = (y_1 + iy_2)$ where $y_1 = \frac{x_1}{1-x_3}$ and $y_2 = \frac{x_2}{1-x_3}$. Now the Euclidean metric on $\mathbb{C}_\infty$ is given by $ds^2 = dy_1^2 + dy_2^2$. We pull it back to a metric on S^2. The pullback of dy_i is of the form

$$\frac{dx_i}{1-x_3} + \frac{x_i}{(1-x_3)^2}dx_3 = \frac{1}{1-x_3}\left(dx_i + \frac{x_i}{1-x_3}dx_3\right)$$

Thus

$$\Phi^*(ds^2) = \frac{1}{(1-x_3)^2}\left(\left(dx_1 + \frac{x_1}{1-x_3}dx_3\right)^2 + \left(dx_2 + \frac{x_2}{1-x_3}dx_3\right)^2\right)$$

But we have

$$x_1^2 + x_2^2 + x_3^2 = 1$$

and

$$x_1dx_1 + x_2dx_2 + x_3dx_3 = 0$$

Hence

$$\begin{aligned}
\Phi^*(ds^2) &= \frac{1}{(1-x_3)^2}\left(dx_1^2 + dx_2^2 + \frac{(1-x_3^2)}{(1-x_3)^2}dx_3^2 + \frac{2dx_3}{1-x_3}(-x_3dx_3)\right) \\
&= \frac{1}{(1-x_3)^2}\left(dx_1^2 + dx_2^2 + \frac{(1-x_3^2) - 2x_3(1-x_3)}{(1-x_3)^2}dx_3^2\right) \\
&= \frac{1}{(1-x_3)^2}\left(dx_1^2 + dx_2^2 + \frac{(1-x_3^2) - 2x_3(1-x_3)}{(1-x_3)^2}dx_3^2\right) \\
&= \frac{1}{(1-x_3)^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right)
\end{aligned}$$

Thus at each point the pullback of the Euclidean metric on $\mathbb{C}_\infty$ is a positive multiple of the Euclidean metric on S^2. Since multiplying distances in a tangent space by a positive constant does not change angles or orientation, the map $\Phi : S^2 \to \mathbb{C}_\infty$ preserves angles and orientation. Thus Φ is conformal.

1.4.(b) Let $z = (y_1 + iy_2) \in \mathbb{C}_\infty$ where $y_i = \frac{x_i}{1-x_3}$ for $i = 1, 2$ for $(x_1, x_2, x_3) \in S^2$. Then clearly, $y_1^2 + y_2^2 = \frac{1+x_3}{1-x_3}$. Thus

$$x_3 = \frac{y_1^2 + y_2^2 - 1}{y_1^2 + y_2^2 + 1} = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Then

$$x_1 = y_1(1-x_3) = \frac{2y_1}{y_1^2+y_2^2+1} = \frac{z+\overline{z}}{|z|^2+1}$$

and

$$x_2 = y_2(1-x_3) = \frac{2y_2}{y_1^2+y_2^2+1} = \frac{z-\overline{z}}{2i(|z|^2+1)}$$

Now Euclidean distance between (x_1, x_2, x_3) and (x_1', x_2', x_3') is $(2-2(x_1x_1'+x_2x_2'+x_3x_3'))^{\frac{1}{2}}$. Thus

$$\begin{aligned} d(z,w)^2 &= 2-2\frac{1}{(|z|^2+1)(|w|^2+1)}\left((|z|^2-1)(|w|^2-1)+(z+\overline{z})(w+\overline{w})+(z-\overline{z})(w-\overline{w})\right) \\ &= 2-2\frac{1}{(|z|^2+1)(|w|^2+1)}\left((|z|^2-1)(|w|^2-1)+2zw+2\overline{zw}\right) \\ &= 4\frac{1}{(|z|^2+1)(|w|^2+1)}\left(|z|^2+|w|^2-zw-\overline{zw}\right) \\ &= 4\frac{1}{(|z|^2+1)(|w|^2+1)}(z-\overline{w})(\overline{z}-w) \\ &= \frac{4|z-\overline{w}|^2}{(|z|^2+1)(|w|^2+1)} \\ \Rightarrow d(z,w) &= \frac{2|z-\overline{w}|}{\sqrt{(|z|^2+1)(|w|^2+1)}} \end{aligned}$$

In particular $d(z,\infty)$ = Euclidean distance between $\Phi^{-1}(z)$ and $(0,0,1)$ which is equal to

$$\left(2-2\frac{|z|^2-1}{|z|^2+1}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{|z|^2+1}}$$

1.4.(c) Consider a circle in S^2 given by locus of points $u \in S^2$ such that Euclidean distance of u from a fixed point $u_0 \neq (0,0,1)$ is a constant r. Let $z = \Phi(u)$, $z_0 = \Phi(u_0)$. Thus

$$d(z, z_0) = r$$

We want to show that the image $\{z = \Phi(u) | \text{Euclidean distance of } u \text{ from } u_0 \text{ is } r\}$ is a circle or a line. Now by part (b), we have

$$\begin{aligned} 2|z-\overline{z_0}| &= r\sqrt{(|z|^2+1)(|\overline{z_0}|^2+1)} \\ \Leftrightarrow 4(z-\overline{z_0})(\overline{z}-z_0) &= r^2((|z|^2+1)(|\overline{z_0}|^2+1)) \\ \Leftrightarrow 4(|z|^2+|z_0|^2-zz_0-\overline{zz_0}) &= r^2(|z|^2||\overline{z_0}|^2+1+|z|^2+|\overline{z_0}|^2) \\ \Leftrightarrow Azz'-zz_0-\overline{zz_0}+C &= 0 \end{aligned}$$

for some real numbers A and C. This is an equation of a circle in $\mathbb{C}_\infty$ and equation of a line if $4-r^2|z_0|^2-r^2=0$ i.e. $\frac{2}{\sqrt{|z_0|^2+1}} = r$ i.e. if $d(z_0,\infty) = r$ i.e. $(0,0,1)$ lies on the circle.
If $u_0 = (0,0,1)$, then the equation of the image is given by $d(z,\infty) = r$ i.e. $|z|^2+1 = \textit{constant}$; i.e. a circle. In any case, the stereographic projection of a circle is a circle or a line.

Problem 1.5

☐ Required Mobius transformation is given by

$$z \mapsto 3(z-i)+2$$

which takes $\{|z-i|<1\}$ to $\{|z-2|<3\}$.

☐ Required Mobius transformation is given by the composition of

$$z \mapsto \frac{z+i}{2} \mapsto \frac{1+(z+i)}{1-(z+i)} \mapsto \frac{1+(z+i)}{1-(z+i)}(e^{-i\pi/4}) \mapsto \frac{1+(z+i)}{1-(z+i)}(e^{-i\pi/4})+(1+i)$$

which takes

$$\{|z+i|<2\} \mapsto \{|z|<1\} \mapsto \mathbb{H} \mapsto \{x+y\geq 0\} \mapsto \{x+y\geq 2\}$$

☐ Note that the two circles $\mathcal{C}_1 \equiv \{|z-i|=1\}$ and $\mathcal{C}_2 \equiv \{|z-1|<1\}$ intersect orthogonally at 0 and $(1+i)$. Consider any mobius transformation which takes the origin to the origin and the point $1+i$ to ∞. Now by this mobius transformation, both C_1 and C_2 are taken to two lines l_1 and l_2 which pass through origin. Also $l_1 \perp l_2$ since the transformation preserves angles. By composing with a rotation (which is also a Mobius transformation) if necessary, we may assume wlog, l_1 and l_2 are the X and $Y-$ axis respectively. Thus we have a Mobius transformation which takes interior of C_1 to a half plane, namely one side of $X-axis$ and interior of C_2 to another halfplane, namely one side of $Y-axis$. Composing again with a rotation if necessary, we may assume that the intersection of the interior of C_1 and C_2 is then taken to the first quadrant. Thus there exist a Mobius transformation with the required property.

☐ Note that in this case too, the two circles $\{|z-2i|=2\}$ and $\{|z-1|=1\}$ intersect orthogonally at origin and at another point. So by similar methods as in last case, we can find a Mobius transformation with required property.

☐ The two circles $C_1 \equiv \{|z-\sqrt{3}|=2\}$ and $C_2 \equiv \{|z+\sqrt{3}|=2\}$ intersect at i and $-i$. The tangent to C_1 at i is $\sqrt{3}x+y=1$ and tangent to C_2 at i is $-\sqrt{x}+y=1$. Thus the two circles intersect each other at $\pi/6+\pi/6=\pi/3$ angle at two point i and $-i$. Now by a translaation (a Mobius transformation) $z \mapsto z+i$ we can take one of the intersection points, $-i$ to the origin. Next take a Mobius transformation which takes 0 to 0 and $2i$ to ∞. By the same argument as in the last to parts, we can then assume , after a rotation if necessary, that the image of the intersection is taken to $\{z\in\mathbb{C}: 0<arg(z)<\pi/3\}$. Then applying the transformation $z\mapsto z^3$, we can transform above region to the first quadrant. Thus a required Mobius transformation exists.

Problem 1.6

Note that by definition, since $m_j \gneq 0$ for each j and $\sum_{j=1}^{n} m_j = 1$, the point z lies in the interior of the **convex hull** of the points $z_1, z_2, \ldots, z_n$ and it lies on the boundary iff the convex hull is degenerate i.e. all z_i's are collinear. Thus every line l through z separates the n points unless all z_j are collinear.

Problem 1.7

1.7.(a) Write each $z_j = x_j + iy_j$. Then $\sum z_j$ converges $\Rightarrow \sum x_j$ converges. Also each $x_j \geq 0$. Hence $\sum x_j^2 \leq (\sum x_j)^2 < \infty$.
Now $z_j^2 = (x_j^2 - y_j^2) + 2ix_jy_j$. Hence $\sum z_j^2$ converges imply $\sum(x_j^2 - y_j^2)$ converges.
Thus by above two paragraphs $\sum y_j^2$ converges. Then $\sum(x_j^2 + y_j^2)$ converges. In other words $\sum |z_j|^2$ converges.

1.7.(b) True.

Problem 1.8

Let

$$u(x+iy) = \frac{x(1+x^2+y^2)}{1+2(x^2-y^2)+(x^2+y^2)^2}$$

Then

$$\begin{aligned}
u(z) &= \frac{\Re(z)(1+|z|^2)}{1+(z^2+\overline{z}^2)+(|z|^2)^2} \\
&= \frac{\Re(z)(1+|z|^2)}{1+(z^2+\overline{z}^2)+(z^2\overline{z}^2)} \\
&= \frac{\Re(z)(1+|z|^2)}{(1+z^2)(1+\overline{z}^2)} \\
&= \frac{\Re(z)+z\overline{z}\Re(\overline{z})}{(1+z^2)(1+\overline{z}^2)} \\
&= \frac{\Re(z+z\overline{z}\overline{z})}{(1+z^2)(1+\overline{z}^2)} \\
&= \frac{\Re(z(1+\overline{z}^2))}{(1+z^2)(1+\overline{z}^2)} \\
&= \Re\left(\frac{(z(1+\overline{z}^2))}{(1+z^2)(1+\overline{z}^2)}\right) \\
&= \Re\left(\frac{z}{1+z^2}\right)
\end{aligned}$$

Thus we may take $f(z) = \frac{z}{1+z^2}$. Clearly f is holomorphic and $f(0) = 0$.

Problem 1.9

Let the given map be denoted by T. Let $z = re^{i\theta}$. Thus

$$T(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \frac{1}{2}(r+r^{-1})\cos\theta + \frac{i}{2}(r-r^{-1})\sin\theta$$

Note that if $T(re^{i\theta}) = x + iy$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = a(r) = \frac{r+r^{-1}}{2}$ and $b = b(r) = \frac{r^{-1}-r}{2}[\because |r| < 1 \Rightarrow r^{-1} - r > 0]$.

Thus we have an ellipse passing through $(\pm a, 0)$ and $(0, \pm b)$. Note that a and b are strictly decreasing on $(0, 1)$. Also $a(1) = 1, b(1) = 0, a(0) = \infty, b(0) = \infty$. Hence, T is an injective map that takes $\{|z| < 1\}$ onto $\mathbb{C}_\infty \backslash [?1, 1]$.

Similarly if we consider $a = \cos\theta$ and $b = \sin\theta$, then the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is a hyperbola orthogonal to the above family of ellipses.

Thus the map T takes $\{|z| = 1\}$ onto the segment $[-1, 1]$. For $\{|z| > 1\}$, note that $S(z) = T(z^{-1})$ has the same properties as T. Thus the mapping properties remain invariant.

Problem 1.10

1.10.(a) Clearly if $a, b, c, d \in \mathbb{R}$, $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Conversely, suppose $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Then there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_\infty$ such that $T(\lambda_1) = 0, T(\lambda_2) = 1, T(\lambda_3) = \infty$. Then we can uniquely define T as

$$T(z) = \frac{az+b}{cz+d} = \frac{(z-\lambda_1)(\lambda_2-\lambda_3)}{(z-\lambda_3)(\lambda_2-\lambda_1)}$$

Clearly then $a, b, c, d \in \mathbb{R}$.

1.10.(b) Suppose T is a Mobius transformation such that $T(\mathbb{T}) = \mathbb{T}$. Suppose S is another Mobius transformation such that $S(\mathbb{R}_\infty) = \mathbb{T}$. Then clearly $F = S^{-1}TS$ is a mobius transformation such that $F(\mathbb{R}_\infty) = \mathbb{R}_\infty$. In fact given any mobius transformation F such that $F(\mathbb{R}_\infty) = \mathbb{R}_\infty$ we know that $T = SFS^{-1}$ is a Mobius transformation such that $T(\mathbb{T}) = \mathbb{T}$. Then we know that $F(z) = \frac{pz+q}{rz+s}$ for some real numbers p, q, r and s. Also we can take $Sz = \frac{z-i}{z+i}$. Then $S^{-1}(z) = i\frac{1+z}{1-z}$. Then T is of the form

$$\begin{aligned} T(z) = S\frac{pi(1-z)+q(1+z)}{ri(1-z)+s(1+z)} &= \frac{pi(1-z)+q(1+z)-i(ri(1-z)+s(1+z))}{pi(1-z)+q(1+z)+i(ri(1-z)+s(1+z))} \\ &= \frac{(-pi+q-r-is)z+(pi+q+r-is)}{(-pi+q+r+is)z+(pi+q-r+is)} \\ &= \frac{z+\frac{(q+r)+i(p-s)}{(q-r)-i(p+s)}}{1+\frac{q+r-i(p-s)}{(q-r)+i(p+s)}z} \times \frac{(q-r)-i(p+s)}{(q-r)+i(p+s)} \\ &= \frac{z-z_0}{1-\overline{z_0}z}\zeta \end{aligned}$$

where $z_0 = -\frac{(q+r)+i(p-s)}{(q-r)-i(p+s)}$ and $\zeta = \frac{(q-r)-i(p+s)}{(q-r)+i(p+s)}$ so that $|\zeta| = 1$. In fact given any ζ $in\mathbb{T}$ and $z_0 \in \mathbb{C}$ we can find p, q, r, s such that above holds. Thus all T such that $T(\mathbb{T}) = \mathbb{T}$ have the form $T(z) = \frac{z-z_0}{1-\overline{z_0}z}\zeta$ for $|\zeta| = 1, z_0 \in \mathbb{C}$.

1.10.(c)

Note that if $T(\mathbb{D}) = \mathbb{D}$, then $T(\mathbb{T}) = \mathbb{T}$. Hence T is of the form $T(z) = \frac{z-z_0}{1-\overline{z_0}z}\zeta$ for $|\zeta| = 1, z_0 \in \mathbb{C}$. However, by $1(b)$, if $|T(z)| < 1$ for all $|z| < 1$, then we must have $|z_0| < 1$. So the required T's are of the form $T(z) = \frac{z-z_0}{1-\overline{z_0}z}\zeta$ for $|\zeta| = 1, |z_0| < 1$.

Problem 1.11

1.11.(a)

Define a new function $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Then we claim that $g \in \mathcal{H}(\mathbb{D})$. Clearly f is analytic and $f(0) = 0$ implies near origin, f has a power series expansion of the form $a_1 z + a_2 z^2 + \ldots$. So we can define $g(z) = a_1 + a_2 z + a_3 z^2 + \ldots$ on a neighbourhood of 0 and hence g is differentiable at 0. Also clearly g is differentiable on $\mathbb{D}\backslash\{0\}$. Thus $g \in \mathcal{H}(\mathbb{D})$.

Now note that on a circle of radius $r \in (0, 1]$, $|g(z)| \leq \frac{1}{r}$. Thus applying maximum modulus principle to g to get that $|g(z)| \leq \frac{1}{r}$ for all $z \in \mathbb{D}$ such that $|z| \leq r$. Taking $r \to 1$, we then get that $|g(z)| \leq 1$ on $\mathbb{D}$. In particular

$$|f(z)| \leq |z| \quad \forall z \in \mathbb{D} \text{ and } |f'(0)| \leq 1$$

If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, that means $|g(z_0)| = 1$ for some $z = z_0 \in \mathbb{D}$. Thus there exists $z_0 \in \mathbb{D}$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \mathbb{D}$. Hence again by maximum modulus principle, g is constant. Thus $\exists c \in \mathbb{C}$, such that $f(z) = cz$ and since $|c| = |g(z_0)| = 1$, we get that f is a rotation.

1.11.(b)

Note that for any $a \in \mathbb{D}$, the Mobius transformation $Tz = \frac{z-a}{1-\bar{a}z}$ takes $\mathbb{D}$ bijectively onto itself. In particular it sends a to 0.

Fix any $z_1 \in \mathbb{D}$. Suppose $f(z_1) = w_1$. Then we know that $|w_1| < 1$. Consider the linear transformation $Tz = \frac{z-z_1}{1-\overline{z_1}z}$ and the linear transformation $Sz = \frac{z-w_1}{1-\overline{w_1}z}$. Then define the function $g(z) = Sf(T^{-1}z)$ defined on $\mathbb{D}$. Clearly, g is Holomorphic and $|g| < 1$. Also $g(0) = Sf(z_1) = Sw_1 = 0$. Thus by part (a), we know that $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$ or in other words, $|Sf(z)| \leq |Tz|$ for all $z \in \mathbb{D}$. Hence,

$$\left|\frac{f(z) - f(z_1)}{1 - \overline{f(z_1)}f(z)}\right| \leq \left|\frac{z - z_1}{1 - \overline{z_1}z}\right|$$

In particular,

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| \quad \forall z_1, z_2 \in \mathbb{D}$$

Similarly fix $z_0 \in \mathbb{D}$. Suppose

$$Tz = \frac{z - z_0}{1 - \overline{z_0} z} \quad Sz = \frac{z - f(z_0)}{1 - \overline{f(z_0)} z}$$

Then $g(z) = Sf(T^{-1}(z))$ defined on $\mathbb{D}$ has $g(0) = 0$ and so $|g'(0)| \leq 1$. By chain rule,

$$|g'(0)| = |S'(f(z_0))f'(z_0)(T^{-1})'(0)| \leq 1 \Rightarrow |f'(z_0)||S'(f(z_0))| \leq \frac{1}{|(T^{-1})'(0)|}$$

Now substitute $S'(f(z_0)) = \frac{1}{1-|f(z_0)|^2}$ and $(T^{-1})'(0) = \frac{1}{T'(z_0)} = 1 - |z_0|^2$. Thus we get

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \leq \frac{1}{1 - |z_0|^2} \quad \forall z_0 \in \mathbb{D}$$

If equality occurs in 1.12 or 1.13, that means g is a rotation. Then $f = S^{-1}gT$ is a fractional linear transformation.

Problem 1.12

1.12.(a)

$f \in \mathcal{H}(\Omega)$ implies f is analytic at each point. Hence if $f'(z_0) = 0$ at some point $z = z_0$ then f is n-to-1 around z_0 for some $n \geq 2$. Contradiction! Hence $f'(z) \neq 0$ for all $z \in \Omega$.

By open mapping theorem, $f(\Omega)$ is open. We do not need $1 - 1$ for $f(\Omega)$ to be open; f being non-constant and holomorphic is enough.

By open mapping theorem, f^{-1} is continuous. Now we know that $f'(z) \neq 0$ for all z. Also $f(f^{-1}(z)) = z$ for all $z \in f(\Omega)$. Thus f^{-1} is differentiable and

$$(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$$

Thus f^{-1} is holomorphic.

1.12.(b)

Automorphisms of $\mathbb{D}$: Since $f(\mathbb{D}) = \mathbb{D}$ we know that, $|f(z)| < 1$ for all $z \in \mathbb{D}$. Also by composing with a mobius transformation if necessary, we may assume, wlog $f(0) = 0$. Thus $|f'(0)| \leq 1$. But f^{-1} is holomorphic implies $|f'(f^{-1}(0))| \geq 1 \Rightarrow |f'(0)| \geq 1$. Thus $|f'(0)| = 1$ and thus by Schwarz' lemma, f is a rotation. Thus the automorphisms of $\mathbb{D}$ are precisely all the Mobius transformations of the form

$$e^{i\alpha} \frac{z - a}{1 - \bar{a} z} \quad \text{for some } a \in \mathbb{C} \text{ such that } |a| < 1 \text{ and } 0 \leq \alpha < 2\pi$$

Automorphisms of $\mathbb{H}$: Since the transformation $z \mapsto f(z) = \frac{z-i}{z+i}$ takes $\mathbb{H}$ to $\mathbb{D}$ isometrically, we deduce that all the automorphisms of $\mathbb{H}$ look like $S = f^{-1} \circ T \circ f$ where T is an automorphism of $\mathbb{D}$.

Problem 1.13

1.13.(a) Note that since $\mathbb{H}$ and $\mathbb{D}$ with the given metrics are isometric (via $z \mapsto \frac{z-i}{z+i}$) it is enough to prove that for $f : \mathbb{D} \to \mathbb{D}$, f holomorphic, we have

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2)$$

for all $z_1, z_2 \in \mathbb{D}$. Now suppose $\gamma : [0,1] \to \mathbb{D}$ is a geodesic such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Then $f \circ \gamma$ is a path joining $f(z_1)$ and $f(z_2)$ and certainly has more length than the hyperbolic distance between $f(z_1)$ and $f(z_2)$.

$$\begin{aligned} d(f(z_1), f(z_2)) &\leq length(f \circ \gamma) \\ &= \int_{f\circ\gamma} \frac{|dw|}{1 - |w|^2} \\ &= \int_\gamma \frac{|f'(z)||dz|}{1 - |f(z)|^2} \\ &\leq \int_\gamma \frac{|dz|}{1 - |z|^2} \quad \text{[by equation1.13]} \\ &= d(z_1, z_2) \quad \text{[since } \gamma \text{ is a geodesic]} \end{aligned}$$

Thus

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2) \quad \text{for all } z_1, z_2 \in \mathbb{D}$$

1.13.(b) If f is an isometry, by above calculation we find that equality will hold in 1.13; and hence f is a fractional linear transformation. Conversely if f is a fractional linear transformation such that $f(\mathbb{D}) = \mathbb{D}$, then equality holds in 1.13. Thus any orientation preserving isometry from $\mathbb{H}$(resp. $\mathbb{D}$) to itself is an automorphism of $\mathbb{H}$(resp. $\mathbb{D}$). We have characterised all such maps in 1.12(b). All orientation preserving isometry from $\mathbb{H}$ to $\mathbb{D}$ are then of the form $f \circ g \circ h$ where f(resp. h) is an automorphism of $\mathbb{D}$(resp. $\mathbb{H}$) and $g(z) = \frac{z-i}{z+i}$.

1.13.(c) We work with $\mathbb{H}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{1}{(\Im(z))^2} dz d\overline{z}$. Since the metric does not depend on x, we know that the vertical lines $\Re(z) = consstant$ are geodesics in $\mathbb{H}$. Now given any two points p, q on $\mathbb{H}$ we can construct an orientation preserving isometry f which takes them to points $f(p), f(q)$ with equal real parts. Then the vertical line (in Euclidean sense) joining them is the unique geodesic between $f(p)$ and $f(q)$. Then the preimage of the vertical line is the unique geodesic joining p, q; by uniqueness of geodesics and the fact that isometric image of a geodesic is a geodesic. Since orientation preserving isometries are automorphisms and hence conformal, the images of vertical lines(in Euclidean sense) are either other vertical lines or a circular arc intersecting real axis orthogonally at both end points. Thus the geodesics are

- Either of the form $\Im(z) = constant$
- Or $\{z \in \mathbb{C} : |z - t| = r, t \in \mathbb{R}, r \in \mathbb{R}, \Im(z) > 0\}$

Problem 1.14

Consider the mobius transformation $Tz = \frac{z-1}{z+1}$. Let $Ta = w$. Consider the Mobius transformation $Sz = \frac{z-w}{1-\overline{w}z}$. Let $g = S \circ T \circ f$. Thus $g \in \mathcal{H}(\mathbb{D})$ and $g(\mathbb{D}) \subseteq \mathbb{D} \Rightarrow |g(z)| < 1$ for all $z \in \mathbb{D}$. Thus by Schwarz' lemma, we have $|g'(0)| \leq 1$. Thus

$$1 \geq |g'(0)| = |S'(T(f(0)))||T'(f(0))||f'(0)| = |S'(w)||T'(a)||f'(0)| = \frac{1}{1-|w|^2}\frac{2}{(a+1)^2}|f'(0)|$$

$$\Rightarrow |f'(0)| \leq \frac{(a+1)^2}{2}(1-|w|^2) = \frac{(a+1)^2}{2}\left(1 - \frac{(a-1)^2}{(a+1)^2}\right) = 2a$$

If equality occurs in above, that means g is a rotation. Thus f is of the form

$$f(z) = T^{-1} \circ S^{-1} \circ g(z) = T^{-1} \circ S^{-1}(e^{i\theta}z) = a\frac{1+e^{i\theta}z}{1-e^{i\theta}z}$$

Problem 1.15

Using Morera's theorem, it is enough to prove that:

If f is complex differentiable in Ω, for any triangular path T in Ω, we have

$$\oint_T f(z)dz = 0$$

Let $T = [a, b, c, a]$ and Δ be the closed set formed by T and its inside. Thus $T = \partial\Delta$. Now using the midpoints of the sides of Δ we can form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ of equal area. Give the boundaries of Δ_i appropiate orientation such that $\partial\Delta_i = T_i$ and $T = \sum_{i=1}^4 T_i$. Then

$$\oint_T f = \sum_{i=1}^4 \oint_{T_i} f$$

Thus there exists i such that

$$\left|\oint_T f\right| \leq 4\left|\oint_{T_i} f\right|$$

Call the triangle $\Delta = \Delta^{(1)}$ and $\Delta_i = \Delta^{(2)}$. Replacing $\Delta^{(1)}$ by $\Delta^{(2)}$ we can continue above process to find a sequence of triangles $\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \Delta^{(3)} \ldots$ with boundaries $T^{(1)}, T^{(2)}, T^{(3)}, \ldots$ such that

$$\left|\oint_{T^{(i)}} f\right| \leq 4\left|\oint_{T^{(i+1)}} f\right|$$

$$l(T^{(i+1)}) = \frac{1}{2}l(T^{(i)})$$

$$\mathtt{diam}(\Delta^{(i+1)}) = \frac{1}{2}\mathtt{diam}(\Delta^{(i)})$$

Thus

$$\left|\oint_T f\right| \le 4^n \left|\oint_{T^{(n)}} f\right|$$
$$l(T^{(n)}) = \frac{1}{2^n} l(T)$$
$$\mathtt{diam}(\Delta^{(n)}) = \frac{1}{2^n}\mathtt{diam}(\Delta)$$

By Cantor's intersection theorem, $\bigcup_{i=1}^{\infty} \Delta^{(i)} = \{z_0\}$ for some point $z_0 \in \Omega$. Since f is complex differentiable at z_0, we know that given $\epsilon > 0$ there exists $\delta > 0$ such that for all $z \in \Omega$ such that $|z - z_0| < \delta$ we have,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \epsilon |z - z_0|$$

Now there exists $N \in \mathbb{N}$ such that $\frac{1}{2^n}\mathtt{diam}(\Delta) < \delta$ for all $n \ge N$. Then for all $z \in T^{(n)}$, we have $|z - z_0| < \delta$. Also by Cauchy's integral formula, $0 = \int_{T^{(n)}} dz = \int_{T^{(n)}} z dz$. So,

$$\begin{aligned}
\left|\oint_{T^{(n)}} f dz\right| &= \left|\oint_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)] dz\right| \\
&\le \epsilon \oint_{T^{(n)}} |z - z_0||dz| \\
&\le \epsilon \mathtt{diam} \Delta^{(n)} l(T^{(n)}) \\
\Rightarrow \left|\oint_T f dz\right| &\le 4^n \frac{1}{4^n} \epsilon \mathtt{diam}(\Delta) l(T) = \epsilon \mathtt{diam}(\Delta) l(T)
\end{aligned}$$

Thus taking $\epsilon \to 0$, we get $\oint_T f = 0$. Then by Morera's theorem it follows that $f \in \mathcal{H}(\Omega)$.