Algebra III
Home Assignment 1

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Problem 1

Consider the sequence of homomorphisms

$$f_n : \mathbb{Z}[[x]] \to \mathbb{Z}/p^n\mathbb{Z}, \;\; f_n\left(\sum a_i x^i\right) = \sum_{i<n} a_i p^i \mod p^n$$

Clearly these maps are compatible with the canonical transition homomorphisms

$$\varphi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$$

hence defining the projective limit, we get that there is a unique homomorphism

$$f : \mathbb{Z}[[x]] \to \varprojlim \mathbb{Z}/p^n\mathbb{Z} = Z_p$$

compatible with the f_n's.

Now take any $y \in \mathbb{Z}_p$. Suppose the map $\mathbb{Z}_p \to \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ associates with y the sequence $(y_i)_{i\in\mathbb{N}} \in \mathbb{Z}/p^i\mathbb{Z}$. Let $a_0 = y_1 \in \mathbb{Z}$. Next note that $y_2 \in \mathbb{Z}/p^2\mathbb{Z}$ and $\varphi_2(y_2) = y_1$. So $\exists a_1 \in \mathbb{Z}$ such that $y_2 = y_1 + a_1 p$. Similarly $\varphi_n(y_n) = y_{n-1}$ implies $\exists a_i \in \mathbb{Z}$ such that $y_n = y_{n-1} + a_{n-1}p^{n-1}$. Continuing similarly, we find that $\exists a_i \; \forall i \in \mathbb{N}$ such that $y_{n+1} = y_1 + a_1 p + a_2 p^2 + \ldots + a_n p^n$. Consider the element of $\mathbb{Z}[[x]]$ defined by $\sum_{i=0}^{\infty} a_i x^i$. Then clearly by construction, $f_n\left(\sum_{i=0}^{\infty} a_i x^i\right) = y_n$. So $f\left(\sum_{i=0}^{\infty} a_i x^i\right) = y$. Hence f is surjective.

Next we want to prove that $ker(f)$ is the principal ideal generated by $x - p$. So we have to show that if we have a formal power series $\sum a_i x^i$ such that $\sum_{i<n} a_i p^i \mod p^n = 0$ for all $n \in \mathbb{N}$, then $x - p$ divides the formal power series.

For $n = 1$, we have $a_0 \equiv 0 (\mod p)$. Hence $a_0 = p\alpha_0$ for some $\alpha_0 \in \mathbb{Z}$.
Then for $n = 2$, we have $a_0 + a_1 p \equiv 0 (\mod p^2) \Rightarrow \alpha_0 + a_1 \equiv 0 (\mod p)$. Hence there is some integer α_1 such that $\alpha_0 + a_1 = p\alpha_1$.
Continuing in similar way, we get that for any $n \geq 1$, we have

$$p^n \alpha_{n-1} + a_n p^n = a_0 + a_1 p + \ldots + a_n p^n \equiv 0 (\mod p^{n+1})$$

so that there is an integer α_n with $\alpha_{n-1} + a_n = p\alpha_n$. We can summarize these relations by

$$a_0 = p\alpha_0, \;\; a_n = p\alpha_n - \alpha_{n-1} \;\forall n \geq 1 \qquad (\star)$$

Hence

$$a_0 + a_1 x + a_2 x^2 + \ldots = (p - x)(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots) \Rightarrow (x - p) \Big| \sum a_i x^i$$

Conversely if $\sum a_i x^i \in (x-p)$, then $(\star)$ holds. Thus $\sum_{i<n} a_i p^i \equiv 0 \mod p^n$. Thus we have a ring isomorphism

$$g : \mathbb{Z}[[x]]/(x-p) \overset{\cong}{\to} \mathbb{Z}_p$$

By construction g is clearly a continuous bijection. Now $\mathbb{Z}[[x]]/(x-p) = \varprojlim \mathbb{Z}[x^n]/(p^n, x-p)$, hence it is compact and $\mathbb{Z}_p$ is Hausdorff since the p-adic valuation makes it a metric space. Thus g is a continuous bijection from a compact set to a Hausdorff space, hence it is a homeomorphism. Hence g is an isomorphism of topological rings.

Problem 2

Denote $\overline{\alpha}$ by α_1. Thus we have a simple root α_1 of $\overline{P}$ in $\mathbb{F}_p$ i.e. $P(\alpha_1) \equiv 0 \mod p$. We want to show that $\exists \alpha \in \mathbb{Z}_p$ such that $P(\alpha) = 0$ in $\mathbb{Z}_p$ and $f_1(\alpha) = \alpha_1$ where $f_n(\sum a_i p^i) = \sum_{i<n} a_i p^i$ is the canonical projection map from $\mathbb{Z}_p$ to $\mathbb{Z}/p^n Z$.

For $n \geq 1$, suppose we have an element $\alpha_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that $P(\alpha_n) \equiv 0 \mod p^n$ and $P'(\alpha_n) \not\equiv 0 \mod p$. We look for an element $\alpha_{n+1} = \alpha_n + tp^n \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ such that $P(\alpha_{n+1}) \equiv 0 \mod p^{n+1}$. Note that by expanding out the polynomial P we can write

$$P(\alpha_n + tp^n) = P(\alpha_n) + tp^n P'(\alpha_n) + O(p^{2n})$$

Reducing both sides modulo p^{n+1}, we see that for $P(\alpha_{n+1}) \equiv 0 \mod p^{n+1}$ to hold, we need

$$0 \equiv P(\alpha_n + tp^n) = P(\alpha_n) + tp^n P'(\alpha_n) \mod (p^{n+1})$$

since $2n \geq n+1$. Now $P(\alpha_n) = zp^n$ for some integer z. So,

$$0 \equiv (z + tP'(\alpha_n))p^n \mod p^{n+1} \Rightarrow 0 \equiv z + tP'(\alpha_n) \mod p$$

So putting

$$t = -\frac{z}{P'(\alpha_n)} = -\frac{P(\alpha_n)}{p^n P'(\alpha_n)}$$

we get an explicit formula:

$$\alpha_{n+1} = \alpha_n - \frac{P(\alpha_n)}{P'(\alpha_n)}$$

Since $P'(\alpha_n)$ is nonzero modulo p and hence p^n, it has a unique inverse. Hence α_{n+1} is unique by construction. Also note that by the first order Taylor expansion of P' at α_n, we have

$$\begin{aligned} P'(\alpha_{n+1}) &= P'(\alpha_n + (\alpha_{n+1} - \alpha_n)) \\ &= P'(\alpha_n) + (\alpha_{n+1} - \alpha_n).(\text{terms involving second and higher derivatives of P}) \\ &= P'(\alpha_n) + (\alpha_{n+1} - \alpha_n).s \ \ (\text{,let}) \\ &= u + tsp^n \end{aligned}$$

where u is a unit in $\mathbb{Z}/p\mathbb{Z}$. Thus clearly $P'(\alpha_{n+1}) \not\equiv 0 \mod p$.

Thus by Induction principle, for all $n \in \mathbb{N}$, we have constructed a series of element $\alpha_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that $\varphi_n(\alpha_n) = \alpha_{n-1}$ where $\varphi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection homomorphism and $P(\alpha_n) \equiv 0 \mod p^n$. Then taking projective limit we can find an element $\alpha \in \mathbb{Z}_p$ such that $f_n(\alpha) = \alpha_n$ for all $n \in \mathbb{N}$ and $P(\alpha) = 0$ in $\mathbb{Z}_p$. Thus we have proved that P a zero $\alpha \in Z_p$ of which, image in $\mathbb{F}_p$ is $\alpha_1 = \overline{\alpha}$.

Problem 3

3.1 By the definition of topology on $\mathbb{A}_{fin}$, if $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is open in $\mathbb{A}_{fin}$, then its intersection with $n^{-1}\hat{\mathbb{Z}}$ must be open in $n^{-1}\hat{\mathbb{Z}}$ for all $n \in \mathbb{N}$. In particular for $n = 1$, we must have

$$\prod_{p\in\mathcal{P}} p\mathbb{Z}_p \bigcap \hat{\mathbb{Z}} \text{ is open in } \hat{\mathbb{Z}}$$

Now

$$\prod_{p\in\mathcal{P}} p\mathbb{Z}_p \bigcap \hat{\mathbb{Z}} = \prod_{p\in\mathcal{P}} p\mathbb{Z}_p \bigcap \prod_{p\in\mathcal{P}} \mathbb{Z}_p = \prod_{p\in\mathcal{P}} p\mathbb{Z}_p$$

But the open subsets of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ of the form

$$\prod_{p\in S} \Omega_p \times \prod_{p\notin S} \mathbb{Z}_p$$

where $S \subseteq \mathcal{P}$ is a finite set; forms a basis of open subsets of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$. Then any open subset of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ has infinitely many coordinates equal to $\mathbb{Z}_p$. Hence $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ cannot be an open subset of $\hat{\mathbb{Z}}$ since $p\mathbb{Z}_p \neq \mathbb{Z}_p$ and consequently, $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is **not an open subset of** $\mathbb{A}_{fin}$.

Next we claim that $\mathbb{A}_{fin}$ is a topological ring. Clearly addition is a continuous function from $\mathbb{A}_{fin} \times \mathbb{A}_{fin} \to \mathbb{A}_{fin}$. To show that the multiplication operation $\mathbb{A}_{fin} \times \mathbb{A}_{fin} \to \mathbb{A}_{fin}$ is continuous it is enough to prove that multiplication by an element of $\mathbb{A}_{fin}$ from $n^{-1}\hat{\mathbb{Z}}$ to $m^{-1}\hat{\mathbb{Z}}$ is a continuous map, which is true.
Thus, in particular, multiplication by $(p)_{p\in\mathcal{P}} \in \mathbb{A}_{fin}$ is a continuous function.

Note that $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ is compact by Tychonoff's theorem and hence, the fact that $\hat{\mathbb{Z}}$ continuously injects into $\mathbb{A}_{fin}$ implies it is a compact subset of $\mathbb{A}_{fin}$. Then the continuous image of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ under the multiplication by the element $(p)_{p\in\mathcal{P}}$ is also a compact set. Thus $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ **is a compact subset of** $\mathbb{A}_{fin}$.

Another way of seeing this is to note that we can prove that the topology on $\mathbb{A}_{fin}$ is equivalent to the product topology. Then by Tychonoff theorem, we can directly say that $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is a compact subset of $\mathbb{A}_{fin}$.

Note that $\hat{\mathbb{Z}}$ is product of Hausdorff spaces, and hence Hausdorff. Thus $n^1\hat{\mathbb{Z}}$ is Hausdorff for each n. But given any two points in $\mathbb{A}_{fin}$, we can find $N \in \mathbb{N}$ such that they are elements of $N^{-1}\hat{\mathbb{Z}}$. Thus $\mathbb{A}_{fin}$ is a Hausdorff space. Then $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is a compact subset of a Hausdorff space **and hence a closed subset of** $\mathbb{A}_{fin}$.

Another way of seeing this is to note that the complement of $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ in $\hat{\mathbb{Z}}$ is

$$\bigcup_{p\in\mathcal{P}} \left(\left\{ z \in \mathbb{Z}_p : |z|_p > \frac{1}{p} \right\} \times \prod_{p\neq q\in\mathcal{P}} \mathbb{Z}_q \right)$$

which is open and hence $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is a closed subset of $\hat{\mathbb{Z}}$. But $\hat{\mathbb{Z}}$ is a closed subset of $\mathbb{A}_{fin}$. Hence $\prod_{p\in\mathcal{P}} p\mathbb{Z}_p$ is a closed subset of $\mathbb{A}_{fin}$.

3.2 Note that $\hat{\mathbb{Z}}$ is an open subset of $\mathbb{A}_{fin}$. It is also closed since its complement $\bigcup_{\substack{n\in\mathbb{N}\\ n>1}} n^{-1}\hat{\mathbb{Z}}$ is open. Now the given set $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$ is the inverse image of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ under the multiplication map by $(p)_{p\in\mathcal{P}} \in \mathbb{A}_{fin}$, which is continuous. Hence $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$ **is an open and closed subset of** $\mathbb{A}_{fin}$

Suppose $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$ is compact subset of $\mathbb{A}_{fin}$. Note that $n^{-1}\hat{\mathbb{Z}}$ is open in $\mathbb{A}_{fin}$ for each $n \in \mathbb{N}$, by definition of the topology on $\mathbb{A}_{fin}$. Now clearly, the collection of these open sets

$$\bigcup_{n\in\mathbb{N}} n^{-1}\hat{Z}$$

forms an open cover of $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$; since $\mathbb{A}_{fin} = \bigcup_{n\in\mathbb{N}} n^{-1}\hat{Z}$. Then by compactness, there is a finite subcollection from above collection of open sets which covers $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$. Suppose

$$\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p \subseteq \bigcup_{i=1}^{k} n_i^{-1}\hat{Z}$$

But clearly above result cannot be true. Since for any $p >> \max\{n_1, \ldots, n_k\}$ we have

$$(1, 1, 1, \ldots, 1, \underbrace{\frac{1}{p}}_{p^{th}\text{ term}}, 1, \ldots) \in \mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p \quad \text{but} \quad (1, 1, 1, \ldots, 1, \underbrace{\frac{1}{p}}_{p^{th}\text{ term}}, 1, \ldots) \notin n_i^{-1}\hat{\mathbb{Z}}$$

Contradiction!! Hence $\mathbb{A}_{fin} \cap \prod_{p\in\mathcal{P}} p^{-1}\mathbb{Z}_p$ **is not a compact subset of** $\mathbb{A}_{fin}$.