Algebra II

HOME ASSIGNMENT 1

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P 1. 

\[ t \in S \Rightarrow f(t) = 0 \text{ for all } f \in I(S) \Rightarrow t \in V(I(S)) \Rightarrow S \subseteq V(I(S)) \]

But \( V(I(S)) \) is a closed set. Thus \( \overline{S} \subseteq V(I(S)) \).

Conversely, assume, by contradiction, \( \overline{S} \neq V(I(S)) \). Suppose \( x \in V(I(S)) \setminus \overline{S} \). Now \( X \) is compact,hausdorff implies \( \exists f \in A \) such that \( f(x) = 1 \) and \( f \) has value 0 at any point of \( \overline{S} \). Thus in particular, \( f \in I(S) \Rightarrow f(x) = 0 \Rightarrow \) CONTRADICT.

P 2. If \( S = V(I) \) for some ideal \( I \) of \( A \), then by definition, \( S \) is closed.

Conversely assume that \( S \) is closed i.e. \( S = \overline{S} \). By prob 1, then \( S = V(I(S)) \). But clearly \( I(S) \) is an ideal of \( A \).

P 3. \( x \in \bigcap_{i=1}^{n} V(f_{i}) \)

\[ \Rightarrow f_{i}(x) = 0, \forall i \in \{1, 2, \ldots, n\} \]

\[ \Rightarrow (\sum_{i=1}^{n} f_{i} \overline{f_{i}})(x) = 0 \]

\[ \Rightarrow \bigcap_{i=1}^{n} V(f_{i}) \subseteq V(\sum_{i=1}^{n} f_{i} \overline{f_{i}}). \]

On the other hand

\[ x \in V(\sum_{i=1}^{n} f_{i} \overline{f_{i}}) = 0 \]

\[ \Rightarrow (\sum_{i=1}^{n} f_{i} \overline{f_{i}})(x) = 0 \]

\[ \Rightarrow \sum_{i=1}^{n} |f_{i}(x)|^{2} = 0 \text{ where } |f_{i}(x)| \text{ denotes the absolute value of the complex no. } f_{i}(x) \]

\[ \Rightarrow |f_{i}(x)| = 0, \forall i \in \{1, 2, \ldots, n\} \]

\[ \Rightarrow x \in V(f_{i}), \forall i \in \{1, 2, \ldots, n\}. \]

Thus \( \bigcap_{i=1}^{n} V(f_{i}) = V(\sum_{i=1}^{n} f_{i} \overline{f_{i}}) \) for \( f_{1}, \ldots, f_{n} \in A \).

Suppose, on contrary to what we have to prove, we have \( V(I) = \emptyset \) but \( I \neq A \). Note that \( f_{i} \in I \) for \( i \in \{1, 2, \ldots, n\} \Rightarrow g = \sum_{i=1}^{n} f_{i} \overline{f_{i}} \in I \) since \( I \) is an ideal.

Thus \( I \neq A \Rightarrow g \) is not invertible in \( A \).

\[ \Rightarrow g(x) = 0 \text{ for some } x \in X. \]

\[ \Rightarrow \bigcap_{i=1}^{n} V(f_{i}) = V(\sum_{i=1}^{n} f_{i} \overline{f_{i}}) \neq \emptyset. \]

In other words, the class of closed sets \( \{V(f)\}_{i \in I} \) has the finite intersection property. But \( X \) is a compact,hausdorff space. Hence we must have \( \bigcap_{i \in I} V(f_{i}) \neq \emptyset \). But clearly from definition, \( \bigcap_{i \in I} V(f_{i}) = V(I) \) which is given to be empty. So we have a contradiction. Hence our initial assumption was wrong and we have \( I = A \).

P 4. We name the given map \( \psi : X \rightarrow \operatorname{max}(A) \) such that \( \psi(p) = \{f \in A | f(p) = 0\} = I(\{p\}). \)

For \( p, q \in X, \psi(p) = \psi(q) \Rightarrow I(\{p\}) = I(\{q\}) \Rightarrow V(I(\{p\})) = V(I(\{q\})) \Rightarrow p = q \) by problem 1. So \( \psi \) is injective.

To show that \( \psi \) is surjective we need to prove that every element in \( \operatorname{max}(A) \) is of the form \( I(\{p\}) \) for some \( p \in X \). By problem 3 we know that \( m \in \operatorname{max}(A) \Rightarrow V(m) \neq \emptyset \). Let \( x \in V(m) \). Then \( I(\{x\}) \supseteq I(V(m)) \supseteq m \).

But \( m \) is a maximal ideal of \( A \Rightarrow \text{Either } I(\{x\}) = A \text{ or } I(\{x\}) = m \). Clearly \( I(\{x\}) \neq A \) since \( \exists f \in A \) such that \( f(x) \neq 0 \). Thus \( m \) is of the form \( \{f \in A | f(x) = 0\} = \psi(x) \). So \( \psi \) is surjective.
P 5. The map $\rho : X \to \text{hom}_C(A, C)$ is defined by the map $\rho(x) : A \to C$ which has $\ker(\rho(x)) = \psi(x)$ where $\psi$ is defined in problem 4.

First of all we need to check that the map is well-defined i.e. given $m = \psi(p)$, a maximal ideal in $A$, there is a unique $\rho(p)$ such that $\ker(\rho(p)) = m$.

Let $F \in \text{hom}_C(A, C)$ be such that $\ker(F) = m$. Take any $f \in A$. Consider the element $c_p \in A$ which sends every $x \in X$ to the constant $f(p)$. Note that $(f - c_p)(p) = 0$. Then we have

$$(f - c_p) \in m \Rightarrow F(f - c_p) = 0 \Rightarrow F(f) = F(c_p) = F(f(p).1) = f(p),$$

a unique constant in $C$ i.e $F$ depends only on $m$ i.e. given $p \in X$, $F = \rho(p)$ is unique. In fact $(\rho(p))(f) = f(p) \Rightarrow \rho(p) = ev_p$, the evaluation map at $p$. So $\rho$ is well-defined.

Also $\rho(p) = \rho(q)$ for $p, q \in X$ implies $\psi(p) = \psi(q)$. By problem 4, then $p = q$. So $\rho$ is injective.

Thus $A/\ker(F) \cong C$, a field. Hence $\ker(F)$ is an maximal ideal in $A$. By problem 4, then $\ker(F) = m = \psi(p)$ for some $p \in X$, i.e. $\rho(p) = F$.

P 6. By problem 5 we have following bijections:

$$X \xrightarrow{\rho_X} \text{hom}_C(A, C) \quad Y \xrightarrow{\rho_Y} \text{hom}_C(B, C)$$

$$p \longmapsto ev_p \quad q \longmapsto ev_q$$

Given $y \in Y$, consider the following:

$$A \xrightarrow{\varphi} B \xrightarrow{ev_y} C$$

We define $\Phi(y) := \rho_X^{-1}(ev_y \circ \varphi)$.

Note that $\varphi(f)(y) = ev_y(\varphi(f)) = \rho_X(\Phi(y))(f) = ev_{\Phi(y)}(f) = (f \circ \Phi)(y)$. So $\varphi(f) = f \circ \Phi$.

Since $\varphi$ is given and $\rho_X$ is a bijection by problem 5, the map $\Phi$ is well defined and unique.

To prove that $\Phi$ is continuous, it will be enough to show that inverse image of closed sets are closed.

Now, a closed set in $X$ is of the form $S = V(I) = \bigcap_{f \in I} V(f)$ for an ideal $I$ of $A$. We have $\Phi(y) \in V(I)$

$\iff \Phi(y) \in V(f), \forall f \in I$

$\iff ev_y \circ \varphi \in \rho_X(V(f)) = \{ev_x | f(x) = 0\} = \{ev_x | ev_x(f) = 0\}; \forall f \in I$

$\iff ev_y \circ \varphi(f) = 0; \forall f \in I$

$\iff (\varphi(f))(y) = 0; \forall f \in I$

$\iff y \in V(\varphi(f)); \forall f \in I$

$\iff y \in \bigcap_{f \in I} V(\varphi(f))$.

Thus $\Phi^{-1}(V(I)) = \bigcap_{f \in I} V(\varphi(f))$, an arbitrary intersection of closed sets; hence closed.

Thus $\Phi$ is continuous.

P 7. $\Rightarrow$

Suppose $\varphi$ is injective. Suppose $\exists x \in X \setminus \Phi(Y)$. We know that $\Phi$ is continuous. Hence $\Phi(Y)$ is compact image of a compact set, hence itself compact. So $\Phi(Y)$ is a compact subset of $X$, a compact Hausdorff space; hence it is closed. Now by Uryshonn’s lemma, $\exists f \in A$ such that $f(\Phi(y)) = 0$ for all $y \in Y$ and $f(x) = 1$. But $f(\Phi(y)) = 0 \Rightarrow \varphi(f)(y) = 0; \forall y \in Y$. Since $\varphi$ is injective; that means $f \equiv 0$. But $f(x) = 1$. Contradiction! Thus $X = \Phi(Y)$ i.e. $\Phi$ is surjective.

$\Leftarrow$

Suppose $\Phi$ is surjective. Then $\phi(f) = \phi(g) \Rightarrow f \circ \Phi = g \circ \Phi$. Given $x \in X$, then there exists $y \in Y$ such that $\Phi(y) = x$. Then $f \circ \Phi(y) = g \circ \Phi(y) \Rightarrow f(x) = g(x)$. Thus $f = g$, i.e. $\phi$ is injective.
P 8. Suppose \( \varphi \) is surjective. Take \( y_1, y_2 \in Y \) such that \( \Phi(y_1) = \Phi(y_2) \). Thus \( f(\Phi(y_1)) = f(\Phi(y_2)) \)
\( \Rightarrow \varphi(f)(y_1) = \varphi(f)(y_2) ; \forall f \in A \)
\( \Rightarrow g(y_1) = g(y_2) ; \forall g \in B \), since \( \varphi \) is surjective.
\( \Rightarrow y_1 = y_2 \). i.e. \( \Phi \) is injective.

Suppose \( \Phi \) is injective. Take any \( g \in B \). Thus \( g : Y \to C \). For \( x \in \Phi(Y) \), define \( f(x) = g(\Phi^{-1}(x)) \). Note that \( \Phi \) is a bijection from a compact space to a Hausdorff space which is onto its image. Thus \( \Phi \) is a homeomorphism between \( Y \) and \( \Phi(Y) \). Then \( f \) is a continuous function on \( \Phi(Y) \), since both \( \Phi^{-1} \) and \( g \) are continuous. Hence it can be extended to a \( C \)-valued continuous function \( f' \) on \( X \). Thus given \( g \in B \), \( \exists f' \in A \) such that \( (\varphi(f'))(y) = f'((\Phi(y)) = f(\Phi(y)) = g(y) \) for all \( y \in Y \). Thus \( \varphi \) is surjective.

P 9. If \( X \) is connected then \( f \in A \Rightarrow f(X) \) is connected. Also \( f^2(x) = f(x) \Rightarrow f(x) = 0 \) or \( 1 \). Thus \( f(X) \) is a connected subset of \( \{0, 1\} \). Hence either \( f = 0 \) or \( f = 1 \). So there is no \( f \in A \) such that \( f^2 = f, f \neq 0, f \neq 1 \).

Suppose on contrary to what we have to prove \( X \) is not connected. Then for any connected component \( X_1 \subseteq X \)
we can define \( f \in A \) by \( f(x) = \begin{cases} 1 & \text{for } x \in X_1 \\ 0 & \text{o.w.} \end{cases} \) Then \( f^2 = f \) and \( f \neq 1, f \neq 0 \). Contradiction!
Thus \( X \) is connected.

P 10. For \( X \) a finite set \( X = \{x_1, \ldots, x_n\} \), we have \( A = \{f|f : X \to C \text{ is continuous}\} = \{f|f(x_i) \in C, \forall i \in \{1, 2, \ldots, n\}\} \approx C^n \), as a commutative ring. Then any ideal of \( A \) is of the form \( I_1 \times I_2 \times \ldots \times I_n \) where \( I_k \) is an ideal of \( C \), hence either \( \{0\} \) or \( C \). Also given \( S \subseteq X, I(S) = \{f|f(x_i) = 0 \text{ for } i \in S; f(x_i) \in C \text{ o.w.} \} \)
\( = I_1 \times \ldots I_n \) where \( I_k = \begin{cases} \{0\} & \text{for } k \in S \\ C & \text{ o.w.} \end{cases} \)
Thus given any ideal of \( A \), it is of the form \( I(S) \) for some subset \( S \)
of \( X \). i.e. the map \( S \mapsto I(S) \) is surjective.
Also \( I(S_1) = I(S_2) \Rightarrow V(I(S_1)) = V(I(S_2)) \Rightarrow S_1 = S_2 \), since finite sets are closed. Thus the map \( S \mapsto I(S) \)
is injective too.
So \( S \mapsto I(S) \) is a bijection from all subsets of \( X \) to set of all ideals of \( A \).

Note that \( V(I(S)) = S \) for all subsets \( S \) of \( X \). Also the map \( S \mapsto I(S) \) is a bijection onto the ideals of \( A \).
Hence \( J = I(S) \) mapping to \( S = V(J) \), i.e. \( J \mapsto V(J) \) is the inverse map of the above bijection.