Set 4

Please study the following Problems 31-40 by February 8 (Friday).

As in my course on February 1 (Friday), for a finitely generated commutative ring $A$ over $\mathbb{Z}$, the Hasse zeta function of $A$ is defined by

$$\zeta_A(s) = \prod_m (1 - \frac{1}{\sharp(A/m)^{-s}})^{-1}$$

where $m$ ranges over all maximal ideals of $A$.

31. Let $\mathbb{F}_q$ be a finite field consisting of $q$ elements. As an analogue of the formula for Riemann’s zeta function

$$\zeta(s) = \zeta_{\mathbb{Z}}(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

in which $p$ ranges over all prime numbers, prove

$$\zeta_{\mathbb{F}_q[T]}(s) = \sum_f \frac{1}{\sharp(\mathbb{F}_q[T]/(f))^s}$$

where $f$ ranges over all monic polynomials in $\mathbb{F}_q[T]$. By using it, prove that

$$\zeta_{\mathbb{F}_q[T]}(s) = \frac{1}{1 - q^{1-s}}.$$ 

Remark. The analogue of Riemann’s hypothesis for $\zeta_{\mathbb{F}_q[T]}(s)$ is not interesting, for this zeta function has no zero. In homeworks Set 5, you will see that the analogues of Riemann’s hypothesis for $\zeta_A(s)$ are interesting for some friends $A$ of $\mathbb{F}_q[T]$.

Before I present Problem 32, I write something about $\zeta(s)$. For the fact there exist infinitely many prime numbers, Euler gave the following analytic proof by using two presentations (the additive presentation and the product presentation) of $\zeta(s)$. We can prove that when $s > 1$ tends to 1, then $\sum_{n=1}^{\infty} n^{-s}$ tends to $\infty$. If there were only finite number of prime numbers, when $s > 1$ tends to 1, $\prod_p (1 - p^{-s})^{-1}$ should converge to $\prod_p (1 - p^{-1})^{-1}$ and would not tend to $\infty$. Thus we have a contradiction and hence there are infinitely many prime numbers.

32. By using Problem 31 and by using the method of Euler, prove that there are infinitely many irreducible monic polynomials in $\mathbb{F}_q[T]$.

33. For a commutative ring $A$ which is finitely generated over $\mathbb{Z}$, prove that $\zeta_{A[T_1,\ldots,T_n]}(s) = \zeta_A(s-n)$. Here $\zeta_A(s)$ denotes the Hasse zeta function of $A$.

(This is a homework in algebra (not in analysis), and so please do not worry here about the convergence of the infinite product. Precisely speaking, this problem has sense if we already know that the (usually infinite) product $\zeta_A(s)$ converges.
absolutely when $Re(s)$ is sufficiently large.) Suggestion of the proof. Use induction on $n$ and the result on $\zeta_{q,\mathbb{T}}(s)$ in Problem 31.

34. By using Problem 33, prove that for any commutative ring $A$ which is finitely generated over $\mathbb{Z}$, $\zeta_A(s)$ (as a product) converges absolutely when $Re(s)$ is sufficiently large.

Since this is a homework in algebra, it is fine that analytic arguments are not perfectly precise.

Preparation for Problems 35.

A Dirichlet character is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ where $N$ is an integer $\geq 1$. For a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, the Dirichlet $L$-function $L(s, \chi)$ is defined as

$$L(s, \chi) = \sum_n \chi(n)n^{-s}$$

where $n$ ranges over all integers $\geq 1$ which are coprime to $N$ ($\chi(n)$ means $\chi(n \text{ mod } N)$). This infinite series converges when the real part $Re(s)$ of the complex number $s$ is $> 1$. In the case $N = 1$ and $\chi$ is the trivial homomorphism, $L(s, \chi)$ is Riemann’s zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and so Dirichlet $L$-function is a generalization of Riemann zeta function. Like Riemann zeta function, Dirichlet $L$-function has the presentation as a product over prime numbers

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

where $p$ ranges over all prime numbers which do not divide $N$. $L(s, \chi)$ has an analytic continuation to the whole $\mathbb{C}$ as a meromorphic function, and is holomorphic at any $s \neq 1$. (In fact, if $\chi$ is not the trivial homomorphism, then $L(s, \chi)$ is holomorphic also at $s = 1$.)

Example. In the case $N = 1$ and $\chi : (\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3 \text{ mod } 4\} \to \mathbb{C}^\times$ is given by $\chi(1) = 1$ and $\chi(3) = -1$, we have

$$L(s, \chi) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \ldots$$

We consider an analogue of Dirichlet $L$ function for $\mathbb{F}_q[T]$. Let $g \in \mathbb{F}_q[T]$, $g \neq 0$, let $\chi : (\mathbb{F}_q[T]/(g))^\times \to \mathbb{C}^\times$ be a homomorphism, let $c \in \mathbb{C}^\times$, and consider

$$L_c(s, \chi) = \sum_f \chi(f) \cdot c^{\deg(f)} \cdot \#(\mathbb{F}_q[T]/(f))^{-s}$$

where $f$ ranges over all monic polynomials in $\mathbb{F}_q[T]$ which are coprime to $g$, and $\deg$ means the degree. In the case $c = 1$, we will denote $L_c(s, \chi)$ just by $L(s, \chi)$.

We have the presentation $L_c(s, \chi) = \prod_h (1 - \chi(h) \cdot c^{\deg(h)} \cdot \#(\mathbb{F}_q[T]/(h))^{-s})^{-1}$ as product, where $h$ ranges over all monic irreducible polynomials in $\mathbb{F}_q[T]$ which do not divide $g$. 

7
35. In the above (**), consider the case $g = T$ and $\chi$ is not the trivial homomorphism. Prove $L_c(s, \chi) = 1$.

Suggestion for the proof. Prove that if $d \geq 1$, $\sum_{a_1, \ldots, a_d \in \mathbb{F}_q} \chi(T^d + a_1 T^{d-1} + \cdots + a_d) = 0$.

The fact there are infinitely many prime numbers $p$ such that $p \equiv 3 \mod 4$ is proved as follows by using Dirichlet $L$-function. Consider the Dirichlet $L$-function $L(s, \chi)$ where $\chi$ is as in the above Example before Problem 35. If there were only finitely many prime numbers $p$ such that $p \equiv 3 \mod 4$, in the product presentation of $L(s, \chi)$, almost all factors $(1 - \chi(p)p^{-s})^{-1}$ (called the Euler factor at $p$) should be $(1 - p^{-s})^{-1}$, that is, $\zeta(s)$ and $L(s, \chi)$ would have the same Euler factors at almost all $p$. Since $\zeta(s)$ diverges to $\infty$ when $s > 1$ tends to 1, $L(s, \chi)$ should diverge to $\infty$ when $s > 1$ tends to 1. But it can be seen that when $s > 1$ and $s \to 1$, $L(s, \chi) = 1 - 1/3^s + 1/5^s - 1/7^s + 1/9^s - 1/11^s + \cdots$ converges to $1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \cdots = \pi/4 < \infty$. Contradiction. Hence there are infinitely many prime numbers $p$ such that $p \equiv 3 \mod 4$. (The fact there are infinitely many prime numbers such that $p \equiv 1 \mod 4$ can be also proved by using this $L(s, \chi)$.)

36. By using Problem 35, prove that there are infinitely many irreducible monic polynomials $f \in \mathbb{F}_3[T]$ whose constant term is $2 \in \mathbb{F}_3$.

37. Prove that in the above (**), if $g$ is of degree $n$ and $\chi$ is not the trivial homomorphism, $L_c(s, \chi)$ is a polynomial of $q^{-s}$ of degree $< n$.

38. By using the formula $(\frac{-1}{p}) = (-1)^{(p-1)/2}$ ($p$ is a prime number $\neq 2$) which appears in the story of quadratic reciprocity law and by considering maximal ideals of $\mathbb{Z}[i]/(p) = \mathbb{Z}[T]/(T^2 + 1, p) = \mathbb{F}_p[T]/(T^2 + 1)$ for each prime number $p$, prove that

$$\zeta_{\mathbb{Z}[i]}(s) = \zeta(s)L(s, \chi)$$

where $\zeta(s)$ is Riemann zeta function and $\chi$ is as in the above Example before Problem 35.

Preparation for Problem 39, 40.

The quadratic reciprocity law has the following analogue for the polynomial ring $\mathbb{F}_q[T]$ over a finite field $\mathbb{F}_q$ of $q$ elements whose characteristic is not 2. This is one example of the mysterious analogies between numbers and polynomials.

Let $f, g \in \mathbb{F}_q[T]$ be irreducible monic polynomials and assume $f \neq g$. Then

$$\left( \frac{g}{f} \right)_T = \left( \frac{f}{g} \right)_T \cdot (-1)^{\frac{\deg(f)}{2} \deg(g)}.$$

Here for $a \in \mathbb{F}_q[T]$ which is not divisible by $f$, $(\frac{a}{f})$ is defined to be 1 if the image of $a$ in the field $\mathbb{F}_q[T]/(f)$ is $r^2$ for some element $r$ of $\mathbb{F}_q[T]/(f)$, and is defined to be $-1$ otherwise. (We have $(\frac{a}{f}) = (\frac{b}{f}) (\frac{b}{f})$ for $a, b \in \mathbb{F}_q[T]$ which are not divisible by $f$.)
39. By using the above analogue of quadratic reciprocity law, prove that
\[ \zeta_{F_5[T, \sqrt{T^3+1}]}(s) = \zeta_{F_5[T]}(s)L(s, \chi), \]
where \( \chi \) is the homomorphism \((F_5[T]/(T^3+1))^\times \to \mathbb{C}^\times \) defined by
\[ \chi(f \mod T^3 + 1) = \left(\frac{f}{T + 1}\right)\left(\frac{f}{T^2 - T + 1}\right). \]

40. Let \( F_q \) be a finite field of characteristic \( \neq 2 \) of order \( q \), let \( g \) be a monic polynomial in \( F_q[T] \) of degree \( n \geq 1 \) which is not divisible by \( h^2 \) for any element \( h \in F_q[T] \) of degree \( \geq 1 \), and let \( A = F_q[T, \sqrt{g}] \). By using Problem 37 and the above analogue of quadratic reciprocity law, prove that the Hasse zeta function of \( A \) has the shape
\[ \zeta_A(s) = \zeta_{F_q[T]}(s)L_c(s, \chi) \]
for some homomorphism \((F_q[T]/(g))^\times \to \{\pm 1\} \subset \mathbb{C}^\times \) and for some \( c \in \{\pm 1\} \), and prove that
\[ \zeta_A(s) = \frac{a \text{ polynomial of } q^{-s} \text{ of degree } < n}{1 - q^{1-s}}. \]