1. Let $A$ be a finitely generated $\mathbb{C}$-algebra and let $a \in A$ be a nonalgebraic element. Show that there are uncountably many $\lambda \in \mathbb{C}$ such that the element $a - \lambda$ is not a zero divisor but, at the same time, it is not invertible. (Thus, as opposed to the case of finite dimensional algebras, ‘most’ of noninvertible elements of $A$ are not zero-divisors!)

2. Let $A$ be a central simple $k$-algebra. Prove that:
   
   (i) Any two $A$-modules of the same dimension over $k$ are isomorphic (as $A$-modules).
   (ii) Any $k$-linear algebra automorphism $\phi: A \to A$ is an inner automorphism, i.e., there exists an invertible element $u \in A$ such that $\phi(a) = u \cdot a \cdot u^{-1}$ for all $a \in A$.
   
   [Hint for (ii): Given $\phi$, define an $A$-action on the vector space $A$ by the formula $a(a') := \phi(a) \cdot a'$, $\forall a, a' \in A$. This action gives $A$ the structure of a left $A$-module. Deduce statement (ii) by observing that the constructed $A$-module must be a rank 1 free $A$-module, by (i).]

3. Let $A$ and $B$ be finitely generated $\mathbb{C}$-algebras. Show that for any simple modules $M$ and $N$, over $A$ and $B$ respectively, the $A \otimes \mathbb{C} B$-module $M \otimes \mathbb{C} N$ is simple. (Recall that the action of $A \otimes \mathbb{C} B$ on $M \otimes \mathbb{C} N$ is defined by the formula $(a \otimes b)(m \otimes n) := am \otimes bn$, for all $a, b \in B$, $m \in M$, $n \in N$.)

4. Let $A \subset \mathbb{Q}$ be a subring formed by all rational numbers of the form $p/q$, where $p, q$ are integers such that $\gcd(p, q) = 1$ and $q$ is odd. Let $B$ be the ring of $2 \times 2$-matrices of the form $\begin{pmatrix} a & u \\ 0 & v \end{pmatrix}$, $a \in A$, $u, v \in \mathbb{Q}$. Find $J(A)$ and prove that $\cap_{n \geq 1} J(B)^n \neq 0$ using the inclusion:

   $$J(B) \supseteq \begin{pmatrix} J(A) & \mathbb{Q} \\ 0 & 0 \end{pmatrix}.$$

5. Associated with an arbitrary direct sum $E = \oplus_{i \geq 0} E_i$, of finite dimensional vector spaces $E_i$, there is a formal power series $P_E$, with nonnegative integer coefficients, called Poincaré series:

   $$P_E(t) = \sum_{i=0}^{\infty} \dim E_i \cdot t^i.$$

   Given a vector subspace $E \subset k[x_1, \ldots, x_n]$, we put $E_i := E \cap k^i[x_1, \ldots, x_n]$, $i = 0, 1, \ldots$. We say that $E$ is a graded subspace of $k[x_1, \ldots, x_n]$ if the natural inclusion $\sum_{i \geq 0} E_i \subseteq E$ is an equality. In that case, we have a direct sum decomposition $E = \oplus_{i \geq 0} E_i$.

   Let $A \subset k[x_1, \ldots, x_n]$ be a subalgebra, which is also a graded subspace $A = \oplus_{i \geq 0} A_i$. One can write $A = A_0 \oplus A_{>0}$, where we have $A_0 = k^0[x_1, \ldots, x_n] = k$ and $A_{>0} := \oplus_{i \geq 1} A_i$ is a graded ideal of $A$, called augmentation ideal.

   (i) Show that the ideal $A_{>0}$ is finitely generated (as an ideal) iff $A$ is finitely generated as a $k$-algebra.

   (ii) Let $I := k[x_1, \ldots, x_n] \cdot A_{>0}$ be an ideal of the algebra $k[x_1, \ldots, x_n]$ generated by the set $A_{>0}$. It is clear that $I$ is a graded subspace of $k[x_1, \ldots, x_n]$ and one has a vector space decomposition $k[x_1, \ldots, x_n]/I = \oplus_{i \geq 0} (k^i[x_1, \ldots, x_n]/I_i)$. Let $H \subset k[x_1, \ldots, x_n]$ be a graded vector subspace such that $k[x_1, \ldots, x_n] = I \oplus H$. Show that the following three properties are equivalent:

   (1) $k[x_1, \ldots, x_n]$ is free as an $A$-module;
   (2) The map $A \otimes_k H \to k[x_1, \ldots, x_n]$ induced by multiplication in the algebra $k[x_1, \ldots, x_n]$ is a vector space isomorphism;
   (3) One has an equality: $P_{k[x_1, \ldots, x_n]}(t) = P_A(t) \cdot P_{k[x_1, \ldots, x_n]/I}(t)$, of formal power series.
6. (i) Find closed formulas for Poincaré series $P_{C[x_1,\ldots,x_n]}$ and $P_{C[x_1,\ldots,x_n]}^S_n$.

Let $\text{Harm}(C^n, S_n) \subset C[x_1,\ldots,x_n]$ be the space of $S_n$-harmonic polynomials on $C^n$ (with respect to the natural representation $S_n \rightarrow \text{GL}(C^n)$ via permutation matrices).

(ii) Use Problem 3 of Assignment 6 and Problem 10 of Assignment 7 to show that the map

$$C[x_1,\ldots,x_n]S_n \otimes \text{Harm}(C^n, S_n) \rightarrow C[x_1,\ldots,x_n],$$

induced by multiplication, is a vector space isomorphism.

(iii) Find the Poincaré series $P_{\text{Harm}(C^n, S_n)}$. Deduce that $\text{Harm}(C^n, S_n) = 0$ for all $i > \frac{n(n-1)}{2}$ and that the Vandermonde polynomial $D_n$ is, up to a constant factor, the only homogeneous harmonic polynomial of degree $\frac{n(n-1)}{2}$.

(iv) Show that $\dim \text{Harm}(C^n, S_n) = n!$.

7. Let $A$ be a Banach $C$-algebra with norm $N(-)$ and let $a \in A$.

(i) Show that the set $\text{spec } a$ is a closed subset of the disc $\{z \in C \mid |z| \leq N(a)\}$.

(ii) Prove a more precise formula:

$$\max_{z \in \text{spec } a} |z| = \limsup_{n \to \infty} N(a^n)^{\frac{1}{n}}.$$ 

[Hint: find the radius of convergence of the Taylor expansion of the function $z \mapsto (1 - z \cdot a)^{-1}$.]"