Definition: Let $A$ be a not necessarily commutative ring. An $A$-module $M$ is said to be of finite length $n$ if, for any chain of $A$-submodules $0 \neq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_m = M$, we have $m \leq n$ and, moreover, there exists a chain as above with $m = n$.

1. Let $M$ be an $A$-module of finite length and let $u : M \to M$ be an $A$-module morphism. Put

$$\text{Image}(u^\infty) := \bigcap_{k=1}^\infty \text{Image}(u^k), \quad \text{resp.} \quad \text{Ker}(u^\infty) = \bigcup_{k=1}^\infty \text{Ker}(u^k).$$

Show that $\text{Image}(u^\infty)$ and $\text{Ker}(u^\infty)$ are $A$-submodules in $M$ and, moreover, we have

$$M = \text{Image}(u^\infty) \oplus \text{Ker}(u^\infty).$$

2. Fix a ring $A$ and an idempotent $e \in A$. The subset $eAe \subset A$ is a ring with unit $e$ (thus, $eAe$ is not a subring of $A$, according to our definitions). Prove the following:
   (i) For any $A$-module $M$, there is a natural $eAe$-module structure on the subgroup $eM \subset M$ (here $eM = \{em \mid m \in M\}$). Furthermore, for any $A$-modules $M$ and $N$ there is a natural morphism of additive groups $f : \text{Hom}_A(M, N) \to \text{Hom}_{eAe}(eM, eN)$;
   (ii) Multiplication on the right gives an algebra isomorphism $(eAe)^{op} \cong \text{Hom}_A(Ae, Ae)$;
   (iii) There is an algebra isomorphism $A^{op} \cong \text{Hom}_{eAe}(Ae, Ae)$;
   (iv) For any $A$-modules $M, N$ the map $f$ in (i) is a bijection.

3. (i) Find all group homomorphisms $f : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$.
   Find all continuous group homomorphisms $f$ in the following cases:
   (ii) $f : (\mathbb{R}, +) \to (\mathbb{R}, +); \quad$ (iii) $f : (\mathbb{R}^n, +) \to (\mathbb{R}, +);$
   (iv) $f : (\mathbb{R}, +) \to S^1; \quad$ (v) $f : S^1 \to S^1.$
   Here, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, the unit circle viewed as a group with respect to the operation of multiplication of complex numbers.

4. Recall the setting of Problem 2 of Assignment 1. The function $N$ gives a metric on the vector space $A := \text{End}_C V$. For $r > 0$, let $B_r := \{a \in A \mid N(a) \leq r\}$ be a ball in $A$ of radius $r$. Show that
   (i) The series $e^a := \text{Id} + a + \frac{1}{2!} a^2 + \frac{1}{3!} a^3 + \ldots$ converges (absolutely) on $B_r$ to a continuous function $A \to \text{GL}(V)$, $a \mapsto e^a$.
   (ii) For any fixed $r < 1$, the series $\log(\text{Id} + a) := a - \frac{1}{2} a^2 + \frac{1}{3} a^3 - \ldots$ converges absolutely to a continuous function $B_r \to A$, $a \mapsto \log(\text{Id} + a)$.
   (iii) One has $\log(e^a) = a$, resp., $e^{\log(\text{Id} + a)} = \text{Id} + a$, for any $a \in M_n(\mathbb{C})$ such that $N(a)$ is sufficiently small, say $N(a) < 1/10$.

5. (i) Show that any matrix $g \in \text{GL}_n(\mathbb{C})$ can be written in the form $g = e^a$ for some $a \in M_n(\mathbb{C})$.
   (ii) For fixed $g$, consider the following system of two equations on the matrix $x$:
   $$\begin{cases} e^x = g \\ \text{tr} x = 0. \end{cases}$$
   Give an example of a matrix $g \in \text{SL}_2(\mathbb{C})$ such that the above system has no solution $x \in M_2(\mathbb{C})$.
   Give an example of a matrix $g \in \text{SL}_2(\mathbb{R})$ such that the above system has a solution $x \in M_2(\mathbb{C})$ but has no solution $x \in M_2(\mathbb{R})$. 

Assignment 3: due Friday, October 19
6. (i) Prove that, for any $a \in M_n(\mathbb{R})$, the map $f : (\mathbb{R},+) \to GL_n(\mathbb{R}), \ t \mapsto e^{t \cdot a}$ is a continuous group morphism.

(ii) Prove that any continuous group morphism $f : (\mathbb{R},+) \to GL_n(\mathbb{R})$ has the form $f(t) = e^{t \cdot a}$, for some fixed $a \in M_n(\mathbb{R})$.

7. Fix $n > 1$ and identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Let $dx$ be the standard euclidean Lebesgue measure on the vector space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Find a continuous function $f : GL_n(\mathbb{R}) \to \mathbb{R}_>$ such that the measure $f(x)\ dx$ is a left invariant measure on the group $G = GL_n(\mathbb{R})$.

8. Let $G \to GL(V)$ be a finite dimensional irreducible representation of a finite group $G$ in a complex vector space $V$. Let $\beta_1, \beta_2 : V \times V \to \mathbb{C}$ be a pair of nonzero hermitian (not necessarily positive definite) $G$-invariant forms on $V$. Prove that there exists a nonzero constant $c \in \mathbb{R}$ such that one has $\beta_2(v_1,v_2) = c \cdot \beta_1(v_1,v_2)$, for any $v_1, v_2 \in V$.

9. Let $X \subset \mathbb{R}^n$ be a compact set. Let $C_{\text{cont}}(X)$ be the algebra of continuous functions $f : X \to \mathbb{C}$, with pointwise operations. Define a metric on $C_{\text{cont}}(X)$ by $\text{dist}(f,g) := \max_{x \in X} |f(x) - g(x)|$, for any $f, g \in C_{\text{cont}}(X)$. An ideal $I \subset C_{\text{cont}}(X)$ is called a closed ideal if $I$ is a closed subset of $C_{\text{cont}}(X)$ viewed as a metric space.

(i) Show that the closure $\overline{I}$ of a proper ideal $I \subset C_{\text{cont}}(X)$, is a proper ideal again (in particular, one has $\overline{I} \neq C_{\text{cont}}(X)$). Deduce that any maximal ideal is closed.

(ii) Show that, for any subset $Y \subset X$, the set $I_Y := \{ f \in C_{\text{cont}}(X) \mid f(y) = 0 \ \forall y \in Y \}$ is a closed ideal in $C_{\text{cont}}(X)$. Prove that, for any $x \in X$, the ideal $I_{\{x\}}$ is maximal.

Let $X := \{ x \in \mathbb{R} \mid 0 \leq x \leq 2 \}$ be a closed segment.

(iii) Give an example of a nonclosed ideal in $C_{\text{cont}}(X)$.

(iv) Prove that the ideal $I_{\{1\}}$ is not generated by any finite collection of elements of $C_{\text{cont}}(X)$; in particular, it is not a principal ideal.

(v) Show that any closed ideal $I$ in $C_{\text{cont}}(X)$ has the form $I_Y$ for some closed subset $Y \subset X$.

10. For each $c \in \mathbb{C}$, analyze the existence of nonzero two-sided ideals $J \subset A_c$ in the $\mathbb{C}$-algebra $A_c := \mathbb{C}\langle x,y \rangle / I_c$ where $I_c$ is a two-sided ideal generated by the element $xy - yx - c \cdot x - 1$. 