REVIEW SHEET #2 SOLUTIONS

MATH 152 SECTION 35, WINTER 2006

Solution 1. (a) By the fundamental theorem of calculus, G is differentiable and $G' = g$. Since g is differentiable, G is twice differentiable.

(b) Since $G'' = g'$ for $x < 1$ we have $G''(x) = g'(x) < 0$, so G is concave down for $x < 1$. Similarly, for $x > 1$ we have $G''(x) = g'(x) > 0$, so G is concave up for $x > 1$.

(c) We have $G'(1) = g(1) = 0$, so 1 is a critical value of G.

(d) We want to show that $G'(x) > 0$ for all $x \neq 1$. But $G' = g$, and the given data tell us that g has a local minimum at $x = 0$, which is its only critical point. If g were zero or negative anywhere else, it would have to have a local maximum somewhere, by Rolle's theorem; thus g must be positive everywhere except at $x = 1$, and so G is increasing everywhere.

(e) The only critical point of G is at $x = 1$, and since G is increasing everywhere, that is not an extremum.

Solution 2. (a) $\displaystyle\int \sin(2\pi x)dx = \frac{-1}{2\pi}\cos(2\pi x) + C$

(b) We let $u = \sin x$; then $du = \cos x dx$, so we have

$$\begin{aligned}\int \sin^3 x \cos x dx &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{\sin^4 x}{4} + C\end{aligned}$$

(c) We let $x = \sin u$; then $dx = \cos u du$. When $x = -1$, we have $u = -\pi/2$, and when $x = 1$, we have $u = \pi/2$. Thus we have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx &= \int_{-\pi/2}^{\pi/2} \frac{\cos u}{\sqrt{1-\sin^2 u}}du \\ &= \int_{-\pi/2}^{\pi/2} \frac{\cos u}{\sqrt{\cos^2 u}}du \\ &= \int_{-\pi/2}^{\pi/2} \frac{\cos u}{\cos u}du \\ &= \int_{-\pi/2}^{\pi/2} du \\ &= u\Big|_{-\pi/2}^{\pi/2} \\ &= \pi\end{aligned}$$

Solution 3. (a) The region is shown below.

The area is in two pieces, so we find the area of each and add the two together. The first is from $\pi/2$ to $5\pi/4$, and the second is from $5\pi/4$ to $3\pi/2$. In the first region, $\sin x$ is larger, while in the second region, $\cos x$ is larger; thus we have

$$\begin{aligned}
A &= \int_{\pi/2}^{5\pi/4} [\sin x - \cos x] dx + \int_{5\pi/4}^{3\pi/2} [\cos x - \sin x] dx \\
&= \Big[-\cos x - \sin x\Big]_{\pi/2}^{5\pi/4} + \Big[\sin x + \cos x\Big]_{5\pi/4}^{3\pi/2} \\
&= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 - 1)\right] + \left[(-1 + 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] \\
&= 2\sqrt{2}
\end{aligned}$$

(b) The region is shown below.

The area is between $x = 0$ and $x = 1$, and $\sqrt[3]{x}$ is always larger than x in this region, so we have

$$\begin{aligned}
A &= \int_0^1 [\sqrt[3]{x} - x] dx \\
&= \left[\frac{3x^{4/3}}{4} - \frac{x^2}{2}\right]_0^1 \\
&= \frac{3}{4} - \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Solution 4. The region Ω_B is shown below.

(a) By the disc method for volumes we have

$$\begin{aligned} V_B &= \pi \int_1^B \left[\frac{1}{x}\right]^2 dx \\ &= \pi \left[\frac{-1}{x}\right]_1^B \\ &= \pi \left(1 - \frac{1}{B}\right) \end{aligned}$$

(b) Clearly, as $B \to \infty$, we have $V_B \to \pi$, which is finite.

Problem 5. The region Ω is shown below.

By the shell method for volumes we have

$$\begin{aligned} V &= 2\pi \int_0^2 x(x^2) dx \\ &= 2\pi \left[\frac{x^4}{4}\right]_0^2 \\ &= 8\pi. \end{aligned}$$

Solution 6. (a) True; this was a theorem stated in class.
(b) True; this is the mean value theorem for integrals.
(c) False; for I to be the integral of f it must be the *unique* number such that $L_f(P) \leq I \leq U_f(P)$ for all partitions P.
(d) True; we stated this in class as well.
(e) True; a continuous function is integrable, and by the fundamental theorem of calculus, its integral is an antiderivative for it.
(f) True; we have $m_i \leq M_i$ for all i in any partition, and hence $L_f(P) \leq U_f(P)$.
(g) False; all we can conclude is that they differ by a constant.

Solution 7. Let $f(x) = \frac{1}{x}$ and consider the partition $P = \{1, \frac{3}{2}, 2\}$ of the interval $[1, 2]$.

(a) There are two intervals in the partition, $[1, 3/2]$ and $[3/2, 2]$. Since $f(x)$ is decreasing, the maximum value on each interval will occur at its left endpoint. Thus we have

$$\begin{aligned} m_1 &= f(1) = 1 \\ m_2 &= f(3/2) = 2/3 \end{aligned}$$

and therefore

$$\begin{aligned} U_f(P) &= m_1 \Delta x_1 + m_2 \Delta x_2 \\ &= 1(3/2 - 1) + (2/3)(2 - 3/2) \\ &= 1/2 + 1/3 \\ &= 5/6. \end{aligned}$$

(b) The upper sum $U_f(P)$ above is an upper bound for the integral $\int_1^2 \frac{1}{x} dx$, which is precisely $\ln 2$. Thus we have

$$\ln 2 \leq 5/6 < 1.$$

It follows by applying exp to both sides that

$$2 < \exp(1) = e.$$

Solution 8. Suppose that $f \neq g$. Then there is some c such that $f(c) \neq g(c)$. Suppose that $f(c) > g(c)$; the other case is similar. Define $h(x) = f(x) - g(x)$; then h is continuous and $h(c) > 0$. Thus, by the first hint, there is an interval $(c - \varepsilon, c + \varepsilon)$ on which $h(x) > 0$.

Let $[a, b] \subset (c - \varepsilon, c + \varepsilon)$; then $[a, b]$ is a closed interval on which $h(x) > 0$. By the second hint, we have $\int_a^b h(x)dx > 0$ and therefore $\int_a^b f(x)dx > \int_a^b g(x)dx$. Dividing by $(b - a)$, we find that the average value of f on $[a, b]$ is greater than the average value of g on $[a, b]$, a contradiction since f and g were assumed to have the same average value on every interval. Thus, we must have $f = g$.