# Some remarks inspired by the $C^0$ Zimmer program

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to Bob Zimmer, with friendship, gratitude and admiration.

In the mid 80's when I just arrived in Chicago, Bob Zimmer was lecturing on his recent work showing how perturbations of isometric actions of Kazhdan groups were dynamically simple (preserve a measurable Riemannian metric) [Zi1]. The heady mixture of the beauty of the results, the audacity of the vision (this is really 0% of what should be true…[1]), and Bob's charisma and charm attracted me so that I still cannot but help think about the problem of what large discrete groups of Cat-isomorphisms of a manifold can conceivably look like.

Alas, my ardor has not been well rewarded in this pursuit. In this paper I would like to make some comments and raise some questions about the case of Cat = Top, that is, the group of homeomorphisms of manifolds. Some of these remarks are rather old, but I will give some details if I am not aware of a discussion elsewhere – while a few are new (at least to me). I hope that this paper spurs more exploration of the fascinating $C^0$ aspect of this nexus of problems.

Although this note is short, it has a bit of structure. The beginning is about groups that don't act, the middle is about lattices and hyperbolic groups and quite tame actions, and the end has an ergodic aspect and relates directly to a paper of Lashof and Zimmer. There is also an appendix about deforming homotopy actions to actions at the cost of stabilization.

Besides Bob Zimmer who is the obvious inspiration for this paper, I would also like to thank Kevin Whyte, and the 3Fs: Benson Farb, Steve Ferry, and David Fisher for many conversations that have strongly influenced my perspective. Courtney Thatcher's thesis on cyclic group actions on products of spheres has some overlap with the perspective in section 4.

## 1. Some groups that don't act.

**Proposition 1**. The infinite special linear group SL(**Z**) does not act topologically, nontrivially, on any compact manifold, or indeed on any manifold whose homology with coefficients in a field of positive characteristic is finitely generated.

[Zi2] only mentions the compact case. The more general case is proven simultaneously (and is necessary for the proof, because one must analyze complements of

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[1] Maybe now we are above the 0% mark, at least in this direction, thanks to the wonderful papers [Be] and [FiM].

fixed sets as well as fixed sets in the coming argument), and is motivated by the recent paper [ABJLMS], as is proposition 2 below (see also [FiS] that was also written very recently and has some quite deep further observations about the volume preserving case).

Proof. The proof is based on Smith theory considerations. A p-group acting on a mod p-homology manifold has fixed set that is also a mod p-homology manifold of lower dimension. Moreover, the sum of the betti numbers of the fixed sets can be at most as large for the fixed set as for the original manifold. As a formal consequence by considerations of the open strata, if $(Z/p)^k$ is acting effectively, for k large enough, there must be a very large $(Z/p)^l$ acting freely on some mod p homology manifold with bounded betti numbers. However, if l is large enough, this action will automatically have a large kernel for its action on homology ($GL_r$ doesn't have that much p-torsion). Consequently, one can compute the homology of the quotient space by means of the usual Serre spectral sequence. Now if the rank of this kernel is greater than the homology of the space, one sees that the quotient is infinite dimensional. Putting this all together, we get elements of order p that are acting trivially on M. Since SL(**Z**) is the normal closure of such elements, we see that the action is actually trivial.

**Proposition 2**. There are finitely presented groups that do not act nontrivially on any compact manifold.

Proof: The idea is to use produce finitely presented groups that contain high rank finite subgroups, and that are normally generated by enough elements of order p. Many examples are constructed in [ABJLMS]; it might be worth noting that old ideas related to the unsolvability of the triviality problem (a la Rabin) can be used for this. We follow the exposition in [We1]. Suppose that π is an effectively group with arbitrarily large elementary p-groups, generated by specific words we know. We can make infinitely many HNN extensions, each one conjugating one of these elements to another, so that they are now all conjugate in a larger group, called π'. We denote by g one of these elements of order p. Now, Higman embed π' into a group π". By taking an amalgamated free product $(\pi''*\mathbf{Z}) *_{\mathbf{Z}*\mathbf{Z}} A$ where A is a knot group (say the trefoil group for concreteness), Z*Z is a free subgroup of A that contains a meridian as its first generator, and Z*Z is embedded into $(\pi''*\mathbf{Z})$ so that the first generator goes to [g,t] in the obvious notation. We thus have arranged that any homomorphism that sends any of the elements of order p in our list to e, kills the whole group.

Whether one prefers proposition 1 or 2 is a matter of taste. Of course these both beg the question of whether there is a torsion free group that does not act continuously on any manifold. We note:

**Proposition 3**. Any countable group of finite homological dimension does have a continuous action on some sphere.

Proof: If $B\pi$ is a finite dimensional complex, then it can be thickened to a manifold. The universal cover of this manifold is contractible, and if not a Euclidean space, its product with **R** is (by a well known theorem of Stallings). This manifold has a free $\pi$ action. Its one point compactification is a sphere, and it has a $\pi$ action with a single fixed point.

This action is rather different than the usual ones we think of coming from symmetric spaces and lattices. It has no dynamics!

A smooth version of proposition 3 seems unlikely (but I have no ideas about how to prove such a thing). On the other hand, the hypothesis is far from necessary. Every manifold (of dimension >0) has an action of a free abelian group of infinite rank. On the other hand, presumably the obvious versions of congruence subgroups of SL(Z) should not act, and we know nothing about this.

**Problem:** What groups are discrete subgroups of Homeo($D^n$, rel $\partial$)? Needless to say, they are torsion free (and they are subgroups of Homeo(M) for any m-manifold for $m \geq n$).

## 2. Actions of Lattices

Of course, Zimmer actually suggests that $SL_n(Z)$ shouldn't act on any manifold of dimension < n-1 (except via finite quotients). In dimension n, the action on $T^n$ should be rigid. Indeed beautiful work of Hurder shows that with smoothness that action is infinitesimally rigid.

It seems worth pointing out that in $C^0$ there are many deformations possible. One deformed action is explained in [FS1], but the process is one that can be repeated infinitely often producing uncountably many new and different actions.

The technique here is "insertion" a la [KL]. Whenever one has a $C^1$ action with a fixed point (or indeed finite orbits, or, indeed, an invariant submanifold) one can glue in a tubular nighborhood acted on via the differential, and extend the action to the complement. Essentially one uses the diffeomorphism of the complement of a disk with the complement of a point inside M-D = M-p. In the neighborhood one can alternatively glue in the cone of the action on the sphere at infinity. These actions are quite different from each other or the original action, as one can see by examining orbits of points.

Now, if one wishes, one can insert many times, producing many rings around the fixed point. Also, one can make some of these rings "thick" by coning for a little while before doing a more standard linear insertion. Clearly, this builds uncountably many actions that only differ around a fixed point. (They are distinguished by one another by looking at the limit points of orbits. Note that for compact groups, the action given by insertion is topologically equivalent to the original action.)

But these "bull's eye" structures can be inserted at will at smaller and smaller scales around any of the finite orbits associated to the rational points on the torus. As a result, we have beautiful moduli of "leopard spots", each of which is itself variable according to the bull's eye patterns. Thus we have seen a much more beautiful structure than is asserted in the following:

**Proposition 4**. There are uncountably many non-topologically conjugate $SL_n(Z)$ actions on $T^n$. They can all be deformed to the usual action.

These deformations are 1-parameter families of actions that start at the "exotic" action at time 0, and end at the standard action at time 1.

And all of these actions map equivariantly to the standard affine action.

These actions are just "modifications" of arithmetic actions. In its most vague form, the Zimmer program asserts that, in some sense[2], all actions of high rank lattices are built up out of such actions. Thus, proposition 4 is just par for the course.

There are other deformations that arise from decomposition space theory (aka "Bing topology"); I will just give one example and then move on.

**Proposition 5.** Let $\pi$ be a countable group with a dense embedding in SO(n). Then SO(n) acts on $S^{n+k}$ with quotient $D^{1+k}$. If one considers the induced action over $D^{1+k}/\sim$ where $\sim$ is a cell-like semicontinuous decomposition, (these exist in profusion if k>1) the total space is still $S^{n+k}$, but the new action of $\pi$ is never equivalent to any of the other ones.

The equivalence of total spaces follows readily from Edwards' characterization of manifolds and the CE approximation theorem (of Siebenmann, which is a consequence of Edwards' theorem, as well), see [Da] and [We2] for such deformations. The actions are thus distinguished by the maximal Hausdorff quotients of their orbit spaces, which will be non-manifold ANR homology manifolds.

Part of the Zimmer program, although it is far from the usual part where the actions are ergodic, seems to me to include considerations of free, and, more generally, proper discontinuous actions. Of course, for these, in the geometric setting, the basic theorem is Mostow rigidity -- and superrigidity easily can be viewed as a description of the infinite covolume geometric actions that follow from the action. The $C^0$ part of this portion of the Zimmer program is then an extension of the well-known Borel conjecture.

On the topological side there is the fabulous and celebrated work of Farrell and Jones [FJ] that tells one that in the cocompact case, there are only the most obvious actions. For uniform lattices, it then follows that if one is interested in actions on Euclidean spaces, the dimension must be strictly larger than the homogeneous space, but there are interesting remarks that should be made about the proper discontinuous actions that exist in (1) the nonuniform case, (2) in dimensions larger than the original action and (3) when there is torsion.

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[2] It is part of the program to figure out what is the correct sense.

Regarding the first of these, it is clear that spirit of the Zimmer program demands that for any aspherical manifold with fundamental group Γ the dimension should be at least that of G/K where G is the Lie group in which Γ sits. It is the **Q-rank** which governs the difference between this dimension and the cohomological dimension of Γ dim(G/K)-cd(Γ) = Q-rk(Γ). When this is 1, it is not hard to give an ad hoc proof that there isn't a geometric realization (the fundamental group of the end is a subgroup, which is a Poincare duality group of dim = dim(G/K)-1 and of infinite index). However, for higher Q-rank, I had been unable to settle it for a number of years, and was very happy when the papers [BKK] and [BF] came out and beautifully confirmed the Zimmer predictions that lower dimensional examples do not exist.

**Problem:** In the minimal dimension (i.e. dim(G/K)), is there any useful way to parametrize the proper homotopy types of aspherical manifolds with fundamental group Γ?

First of all, there is an issue of the fundamental groups at infinity. It is very easy to build in any nonuniform case (of dim>2) uncountably many modifications of the standard action that are distinguished by the fundamental group system of the complements of compact sets (viewed as a pro-system of groups). This is a straightforward modification of constructions of contractible manifolds. However, such actions have proper comparison maps to the standard action, and are concordant to it (i.e. there is a degeneration of each of these to the standard action).

Much more interesting is the observation that Kevin Whyte made (when he was my student) that the manifolds with fundamental group $(F_2)^k$ obtained by taking various products of thrice punctured spheres and punctured tori are all different from each other even up to proper type. It would be quite interesting to get a systematic understanding in some sort of homological fashion. Here the problem should be much more doable when the Q-rank is large. Perhaps the answer is quadratic in part of the group homology when Q-rank > 1/2dim(G/K), cubic when the fractions is around 1/3dim(G/K), etc. by analogy to the work of Goodwillie, Klein, and Weiss [GKW]. Nevertheless, this is quite unclear to me at the moment and the discussion below of case (2) suggests that it could be much simpler than that.

If we assume that we have the correct proper homotopy type then one might have a hope that the proper homotopy equivalence is properly homotopic to a homeomorphism, but this is rarely the case. As Stanley Chang and I noticed, when Q-rank > 2, there is always a finite cover where this fails [ChW][3]. Our obstruction is of exponent 2 and our examples are all virtually standard. On the other hand whenever $H^{4i}(\Gamma ; \mathbf{R}) \neq 0$ for some i, surgery arguments (e.g. like the ones in that paper that make

[3] There is a good heuristic for proper rigidity when Q-rank =2. It is true when Q-rank<2 by the work of [FJ].

use, of course, of the Borel-Serre picture of arithmetic manifolds [BoS][4]) produce (infinitely many) manifolds in the given proper homotopy type distinguished by $p_i$, the i-th Pontrjagin class. These examples will not go away by passing to a finite index subgroup. (In characteristic 0, cohomology can only get bigger on passing to a finite cover.) And, many examples then follow from known results, e.g. of Borel [Bo2].

The situation of being above the cd in the uniform case (or the nonuniform case) is very similar to the issues involved in the proper nonuniform discussion we just had. However, in the uniform case we can be much more explicit about what occurs if, say, the dimension supercedes the cd by >2 and we assume that the fundamental group system is trivial (i.e. equivalent to the constant system Γ). In that case, it is not hard to show that the manifold is the interior of a manifold with boundary, which by the theorem of Browder-Casson-Haefliger-Sullivan-Wall (see [We3, Wa]) will automatically be a topological block bundle over K\G/ Γ. The proper homotopy types are then parametrized by the homotopy classes of maps [K\G/ Γ : BAut($S^{c-1}$)] where c is the difference of dimensions. Rationally this is isomorphic to a group cohomology. For c even, the invariant is simply the Euler class of this bundle. When c is odd, a relatively straightforward calculation shows that it lies in $H^{2(c-1)}$(K\G/ Γ ; **Q**) and all elements are realized[5].

Note: as here c is the analogue of the **Q**-rank in the previous discussion, I currently hope that the answers might be simpler than one thinks in the nonuniform case. Including the results of classification[6] one gets:

**Summary proposition 6:** The actions of a uniform lattice Γ on K\G×$\mathbf{R}^c$ that have appropriate fundamental group systems at ∞ (in their quotient) ↔ [K\G/Γ:$BTop_c$] if c>2. Rationally[7] this can be calculated as a sum of group cohomology groups $H^t$(K\G/Γ; **Q**) ⊕$H^{4i}$(K\G/Γ; **Q**) where t = c, if c is even, and t = 2(c-1) if c is odd, and the sum is over all positive i.

When there is torsion in Γ the situation is much more complicated in several respects. Even in the situation of **cocompact** proper discontinuous actions there can different failures of rigidity. Some examples appear in [CK] – and are based on failures

4 Borel-Serre theory implies that Q-rank>2 boils down to the fundamental group condition at infinity. (In [ChW] this is interpreted and verified using Margulis arithmeticity for all, even nonarithmetic, lattices.)

5 The interpretation of this, not as well-known as it should be, invariant is as follows: every odd dimensional bundle (rationally) has a section that splits off a lower dimensional bundle. While the Euler class of this complementary bundle is not uniquely determined, its square is.

6 See the discussion after proposition 7 below.

7 Since there are no group structures here at least on the left, it is worth clarifying there here the map from left to right is finite to one and has image that contains a lattice in the target vector space.

of excision in algebraic and hermitian K-theory, i.e. the functors Nil and UNil. When nonzero, these groups tend to be infinitely generated torsion groups.

Rather different examples, based on a connection to embedding theory and somewhat technical arguments involving stratified space theory can be found in [We3,4][8] for actions of crystallographic groups. These were extended in [Shi] to many other lattices. They can be infinitely generated non-torsion, e.g. detected by analogues of the Alexander polynomial. However, all of these examples (e.g. the algebraic and the embedding theoretic) are virtually trivial in the sense that they become standard on passing to a sufficiently large cover that keeps all of the torsion. (When one passes to a cover, only some of the coefficients of the polynomial survive, so any particular example will be killed on passing to a suitable finite cover.) This can be proved by a geometric argument related to [SW] (the equivariant version of α-approximation theorem [CF]) that when there are no fixed sets of subgroups that are included in one another in codimension <3[9].

It is worth noting that the above cohomological discussion about the proper analysis for nonuniform lattices can be married to the discussion of the role of the singularities (and embeddings) that come from torsion. In any case, suffice it to say that the analogue of the $\oplus H^{4i}(K\backslash G/\Gamma; \mathbf{Q})$ part of the previous theorem is a sum of similar terms, one for each stratum that has **G**-rank>2, and the form of the term is an equivariant K-group. As a result nonuniform lattices of Q-rank>2 with torsion almost never have proper rigidity.

In any case, all of these examples certainly can be viewed as "obtained from some procedure applied to the arithmetic examples" – although the exact procedures involved are somewhat involved and might be hard to visualize.

## 3. Leaving the aspherical setting.

One of the audacious aspects of the Zimmer program is that the manifolds studied are general compact manifolds. Here is a theorem of this sort from [FW].

**Theorem**: If $M = X/\pi$ is a compact manifold so that $\pi = \pi_1(M)$ is torsion free and has no normal abelian subgroups, and the isometry group Iso(X) is not a compact extension of π, then π is a lattice in a semisimple Lie group and there is a Riemannian fiber bundle $M \to K\backslash G/\pi$.

If π has torsion, one has to allow the possibility of "orbifibering" over an orbifold.

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8 We are just now getting to the point where we can give complete classifications of **some** proper cocompact actions that are not rigid. However, it is my feeling that the picture of this subject is currently too complicated to be able to give a survey here.
9 When there are codimension 2 situations, one can use counterexamples to the "smith conjecture" on unknottedness of fixed sets of cyclic group actions on the sphere, e.g. [Gi], to build knot theoretic failures of rigidity that do not die in any finite cover.

Note that all the fibers are isometric here, and that therefore the structural group of the bundle is compact. Moreover, the representations $\pi \to \mathrm{Iso}(F)$ control the possible M's. (Taking a representation with infinite image gives an example where no intermediate covers are Riemannian products. Of course many of them are differentiable products.)

The following result, sketched in chapter 12 of [We3], frames our discussion.

**Proposition 7**. Suppose that $M = X/\pi$ is a compact manifold whose fundamental group is a torsion free lattice in a Lie group and that $H_*(X)$ is finitely generated. Then, unless $\dim M = 4$, there is a map $M \to K\backslash G/\pi$ so that the inverse image of every open ball is homeomorphic to X.

So from a slightly blurred perspective this map is like a fiber bundle map: rather than controlling the inverse image of each point, we control the inverse image of each little open ball. The relevant M's that occur here, though, are much richer than the ones that occur in the previous theorem. Rather than homomorphisms from $\pi$ to a Lie group, the objects that occur in this proposition are much more algebraic topological. For example, when $X = S^k \times \mathbf{R}^n$, if one restricts attention just to the homotopically trivial fibrations, then the relevant classification is (independent of $k>1$[10]) the homotopy classes of maps $[K\backslash G/\pi : G/Top]$, where the space G/Top was analyzed completely by Sullivan-Kirby-Siebenmann; rationally it is $\oplus H^{4i}(K\backslash G/\pi\ ;\ \mathbf{Q})$. Incidentally, even when $\pi$ has nontrivial homomorphisms to $O(k+1)$ they never give rise to nontrivial elements in this cohomology[11]. These actions of $\pi$ on $S^k \times \mathbf{R}^n$ are never "Lie theoretic".

**Conjecture 7'**: If M is a manifold with torsion free fundamental group $M = X/\pi$ ($\pi = \pi_1 M$) so that $H_*(X)$ is finitely generated, then there is an aspherical homology manifold N with fundamental group $\pi$, and for any such N, there is a map $f: M \to N$ so that for any small open set $O \subset N$, $f^{-1}(O) \sim O \times X$.

There are several things that need to be explained about this conjecture. The first is that the maps allowed here, when N is a manifold, are exactly the maps produced in the previous theorem; moreover, despite allowing more general N's than $K\backslash G/\pi$ in the conjecture, by Quinn's work on resolution of homology manifolds [Q], the conjecture is a consequence of the theorem.

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[10] This is due to $G_c/Top_c$ stability as explained in [We3]

[11] These cohomology classes are essentially Pontrjagin classes (that can be very high dimensional in Top despite the low dimensionality of the "fiber"); the vanishing for representations is immediate from the Chern-Weil description of characteristic classes.

**Proposition 8**. The conjecture is true for $\pi \subset GL_n(\mathbf{R})$ discretely embedded and of finite type[12] (but not necessarily a lattice) and for hyperbolic groups, assuming that $cd(\pi)>4$.

This follows the same lines as the previous proposition, aside from a few points. The first is that one must exclude, say, nonuniform lattices. More precisely, if $\pi$ is the fundamental group of an aspherical manifold, it must satisfy Poincare duality. That Poincare duality follows, assuming finiteness of $K(\pi,1)$, from a manifold's having a finite universal cover was shown in [BW] (as is the coarseness of satisfying Poincare duality among such groups)[13]. Now, the following is an immediate consequence of [BFMW]:

**Proposition 9.** If $\pi$ is a Poincare duality group of dimension > 4 and satisfies the Borel conjecture (see e.g. [FRR][14]) then there is an aspherical homology manifold with fundamental group $\pi$.

Thus the existence of the aspherical homology manifold follows from the results of [FJ] and [BL]. I will soon outline some evidence for not believing that it is possible to improve the existence result to be a manifold. In any case:

**Proposition 10**. Among torsion free hyperbolic groups, being the fundamental group of a closed aspherical manifold is a coarse condition; it follows from the Gromov boundary being a sphere.

The idea for this appears in a discussion of the case of homotopy tori in [BFMW]. One glues the boundary sphere onto the universal cover to obtain a homology manifold with boundary, and then relates the local index of the interior to that of the boundary. Details will appear in a forthcoming paper with Barthels and Lueck.

It is also worth noting also that the conjecture (affirmed in proposition 8) includes the Borel conjecture in its usual formulation. If M and N are both aspherical manifolds with fundamental group $\pi$, there is then a CE map (see [Da]) $M \to N$, which according to

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[12] I believe that if $B\pi$ has some infinite skeleton then there cannot be a manifold such as our M, but I haven't excluded this yet.

[13] Nonuniform lattices never satisfy Poincare duality. They satisfy Bieri-Eckmann duality with an infinitely generated dualizing module.

[14] The Borel conjecture required here is the following standard generalization of the most commonly stated one: (ignoring orientation issues for simplicity) suppose that M is an aspherical manifold and $f: W \to M$ is a proper homotopy equivalence that is a homeomorphism outside of a compact set, then f is homotopic through such maps to a homeomorphism. For a group, we demand that this hold for all aspherical M with that fundamental group. This extends the usual Borel conjecture from fundamental groups of compact aspherical manifolds to all countable groups of finite cohomological dimensional, and in the case of fundamental groups of compact aspherical manifolds is equivalent to it if one considers M and $M \times$ tori simultaneously. (In other words a conterexample to the extended Borel for some bundle over M will translate into a counterexample to ordinary Borel for $M \times$ some torus.)

[Si] is automatically a uniform limit of homeomorphisms. The conjecture thus accomplishes the liberation of the Borel conjecture from the setting of aspherical manifolds.

Finally, the map comes directly from the argument in [BFMW2]. The underlying homotopy theoretic assumption enables one to set up a controlled surgery problem, that that technology solves. More precisely, it produces a DDP[15] homology manifold controlled homotopy equivalent to the original manifold that "approximately fibers over N". However, Edwards' characterization theorem [Da], Quinn's resolution theory [Q], and the α-approximation theorem [ChF] then apply to show that the homology manifold is homeomorphic to the original manifold M.

The situation for groups with torsion is not nearly as pleasant: there are many more sources of obstruction -- however it is consideration of these that leads me to believe that the homology manifolds arising in the conjecture cannot be replaced by manifolds.

The naïve suggestion would be that π should act proper discontinuously on a contractible (homology) manifold C and that there should be an equivariant map $X \rightarrow C$ that has the properties of the conjecture. This is completely deflated by the failure of the Nielsen conjecture [BW2].

In any case, one would suspect that C actually should have more properties, e.g. that the fixed set of every finite subgroup is also contractible, like K\G in the lattice case. However, even then there are some additional obstructions related to Nil and UNil that would have to vanish. If one is willing to cross M with a manifold with zero Euler characteristic, then one can get rid of the Nil obstructions and, then, if π is a lattice with only odd torsion in rank 1, then one can affirm this conjecture, using -- and presumably it is only a matter of time till the general case will follow.

However, for general groups π, even when C exists, nothing implies that the action is locally smoothable. It is very possible for action near the fixed set to be modeled on non-linearizable (say homotopy linear) actions of a finite group on the sphere. (These are constructed in [BW1].) This seems to be entirely parallel to having homology manifolds which are locally homotopy spheres (=DDP) but not actually being "locally linear" i.e. resolvable.

**Remark**: In [BFMW1] there were a number of conjectures made about the geometric topology of DDP homology manifolds. Despite the years that have passed almost no progress has been made. The reason for this is probably because they have never been "seen": they are constructed as Gromov-Hausdorff limits of polyhedra that are themselves constructed using fairly high power machinery. If one of these arose as a

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[15] DDP stands for "Disjoint Disks Property"; it demands of a space Z that any pair of maps of $D^2 \rightarrow Z$ can be approximated by a disjoint pair. According to Edwards [Da] a resolvable homology manifold of dimension > 4 with this property is a manifold.

boundary of a hyperbolic group, it would naturally have many self homeomorphisms, and other additional structures that could lead an optimist to hope that that would be an importat step towards understanding the non-resolvable homology manifolds.

**Appendix**: **Actions on $M\times R^n$**

This brief appendix gives a bit more information about the possible monodromies possible for the manifolds just discussed (in the torsion free case). But we shall phrase the result in terms of the Zimmer program. Here Cat is any of the geometric categories Top, PL or Diff and Aut(?) is the space of self-homotopy equivalences of the space ?.

**Proposition A1:** Suppose that $\Gamma$ is a countable group of finite cohomological dimension and we have a homomorphism $\rho:\Gamma \to \pi_0\mathrm{Aut}(M)$. Then there is a Cat action of $\Gamma$ on on $M\times R^n$ for some n iff there is a lift of $B\rho$ to BAut(M,t) where (M,t) is the component of the space of maps $M\to$ BCat that contains the stable tangent bundle of M, and Aut denotes the automorphism group. Moreover, the action can be taken free.

**Note**: Note that if the original $\rho:\Gamma \to \pi_0\mathrm{Aut}(M)$ does not preserve the stable tangent bundle, then the automorphism space is empty and the proposition is vacuous. If M is stably parallelizable, then the condition is equivalent to just solving the lifting problem:

$$\begin{array}{r}\mathrm{BAut}(M)\\ \downarrow\quad\\ B\rho: B\Gamma \to B\pi_0\mathrm{Aut}(M).\end{array}$$

This propositon is the (un)natural marriage of Cooke's obstruction theory for lifting homotopy actions to actions [Co] combined with Mazur's theory of stable differential topology [Ma] where one uses finite dimensionality of $B\Gamma$ is get manifold structure when "assembling". The proof is quite straightforward and is left to the reader. (In the parallelizable case, one simply argues that the action produced by Cooke gives rise to a finite dimensional CW complex with a map $\pi_1 \to \Gamma$ whose induced cover is homotopy equivalent to M. "Thickening" this complex to be a parallelizable manifold gives one whose $\Gamma$ cover is $M\times R^n$ according to Mazur.)

**Remark:** On the other hand, for finite groups this approach seems doomed to failure. The theorem *does* give rise to obstructions: for example, the Nielson realization problem was disproved for nilmanifolds via this obstruction, see [RS]. However, for trivial $\rho$ clearly it is impossible to ever accomplish the task achieved by the proposition, that is, constructing a free action. There is some literature on this problem for finite groups, but it is hemmed in by natural and strong hypotheses.

Note that according to Sullivan [Su] and Wilkerson [Wi] when M is a compact (or even finite type) simply connected manifold $\pi_0\mathrm{Aut}(M)$ is an arithmetic group. In practice the lifting problem has its subtle aspects, and I hope to devote a future paper to some examples of it.

The actions produced by the method are always free and proper discontinuous. As a result, the n here can be rather larger than the cd. (Think about the case of a product of free groups [BKK] and M = a point.)

Much more in the spirit of the Zimmer program would be to produce actions with small values of n. With some trepidation, I would like to suggest:

**Conjecture**: If M is a compact manifold of dimension less than that of the smallest representation of an irreducible lattice $\Gamma$ in a Lie group G of rank >1, and n << dim(G/K), then all actions of $\Gamma$ on $M \times R^n$ factor through a finite group.

The condition on n should force a certain amount of recurrence. If n is (rather) less than dim(G/K), one can suspect that the amount is enough to impose the features of rigidity.

## 4. Conjugacy of translations

Suppose G is a connected Lie group and g is an element. Left translation by g defines a dynamical system on G. In this section, I would like to discuss in some simple case when these dynamical systems are topologically conjugate in what is just an experimental exploration. My hope is that this will serve as a first setting for which surgical ideas have dynamical applications. In any case, it gives some more examples in the style of [LZ] of lattices that act ergodically[16] on a compact manifold in infinitely many distinct ways.

**Proposition 11.** Suppose g is an element of a torus, then translation by g is topologically conjugate to tranlation by h iff there is an element of $GL_n(\mathbf{Z})$ taking g to h. If <g> is dense in the torus, then this element is unique and is the unique continuous conjugating map.

Note that g can recovered by its eigenvalues on $L^2(T)$.

**Corollary.** If G is a compact Lie group then any g can be conjugated to an element of a maximal torus. Translation by g is conjugate measure theoretically to h iff there is an element of $GL_n(\mathbf{Z})$ taking g to h (for the usual action of $GL_n(\mathbf{Z})$ as automorphisms of the torus).

Much more interesting is the situation of continuous conjugacy.

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[16] Of course, many nonergodic actions – even with continuous deformation – appear in earlier sections.

**Corollary.** **(**G,g) is topologically conjugate to (G, h) iff G/cl(<g>) ≈ G/cl(<h>) by an isomorphism that pulls back the principle cl<g> = cl<h> bundles.

As a special case, consider G = SU(n) and g generic, so that <g> is dense in T the maximal torus. Then the maps associated to g and h, SU(n)/T → BT classifying the principle bundles differ by the automorphism of $GL_n(\mathbf{Z})$ in other words, these are the same bundles, but their structure as principle bundles is different. So we want to know what kinds of automorphisms of G/T are possible. The automorphisms of the cohomology algebra of this space was much studied and for G = SU(n) the only possibility is the Weyl group combined with complex conjugation (see e.g. [EL]) Thus we obtain, generically there is rigidity:

**Corollary:** For generic g ∈ SU(n), translation by g is only topologically conjugate to translation by elements conguate in SU(n) to g or its complex conjugate.

Much more interesting is the situation for nongeneric elements. We will see that these are frequently topologically conjugate. We shall focus on the extreme cases: where cl<g> is finite or a circle[17].

First of all I want to show some conjugacies that are "soft", i.e. that follow from general principles with no calculations[18]. We will consider $Z_k$ where k is a product of a large number of primes. For each prime separately, you have the action of the Weyl group moving the generator to another element of the maximal torus. However, if there are r prime factors, one has something like $(\#\text{Weyl})^{r-1}$ associated representations of $Z_k$ modulo action of the Weyl element on the whole maximal torus.

**Proposition 12[19].** These "Weyl mixed" translations are all topologically conjugate for G = SU(n), n>2.

Proof: For simplicity of notation let us consider the case of 2 factors. Thus we are considering G/(P×Q) versus G/(P'×Q') where P and Q have relatively orders, and P and P' are conjugate in G, as are Q and Q'. First of the quotients are homotopy equivalent[20]

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[17] In general is a product $\mathbf{Z}_r$×T for some torus.

[18] On the other hand, they have the usual shocking feeling that follows from applying obviously discontinuous p-adic constructions on standardly manifold theoretic objects: in other words, they are reminiscent of the use of Frobenius to get actions on etale homotopy types that arose in the proof of the Adams conjecture.

[19] This proposition is true for all simply connected compact Lie groups other than SU(2); Indeed, I suspect it is true smoothly and for arbitrary compact Lie groups of rank>1 and perhaps even more general abelian subgroups of G.

[20] All homotopy equivalences are assumed to preserve identification of fundamental group.

by localization theory [HMR]. To build an equivalence between (simple) spaces[21], it suffices to build rationally compatible equivalences between their localizations at each prime. At P, G/P → G/(P×Q) is an equivalence, and G/P ≈ G/P' by a conjugacy that it homotopic to the identity as a map of G (it is induced by an element of G, which is connected). At Q, the argument is the same. At other primes, it's even easier, both quotients are equivalent to G. These are all visibly rationally compatible.

In fact they are simple homotopy equivalent. As n>2, the Lie group we are considering has rank >1. Therefore one can use an extra circle from the maximal torus to compute that the Reidemeister torsion of these spaces vanish. As the Whitehead groups of cyclic groups are torsion free [BMS], the Whitehead torsion is detected by Reidemeister torsion and we get its vanishing as well.

Thus we can consider G/(P'×Q') ∈ S(G/(P×Q)), where S denotes the structure set of surgery theory. We shall analyze this in terms of the "old fashioned surgery exact sequence" of Sullivan and Wall [Wa]. The first obstruction we have is the normal invariant of this homotopy equivalence ∈ [G/(P×Q) : F/Cat]. However, by passing to P and Q covers, this map clearly vanishes (on those covers, it's homotopic to a diffeomorphism), so the map is nullhomootpic. As a result G/(P×Q) and G/(P'×Q') are normally cobordant. To complete the proof, we have to analyze a final surgery obstruction. If dim(G) is even the relevant surgery group vanishes[22].

However if dim(G) is odd, then there are additional obstructions, the most prominent being the ρ –invariant. This is the G-singature of any free G manifold bounding our manifold. If S is an $S^1$ in T disjoint from P×Q and P'×Q' we can use it compute the ρ –invariants of G/(P×Q) and G/(P'×Q') simultaneously. Note that G bounds the mapping cylinder of G → G/S, and that both finite groups act homologically trivially on this manifold with boundary. Thus for both, the ρ –invariants is the signature of this manifold with boundary (almost always 0) × the trivial representation. In all cases, these manifolds can't be distinguished by their ρ –invariants, completing the argument.

**Remark:** It is clear that one can apply the same method to some positive dimensional topologically cyclic subgroups of the maximal torus -- although one will be compelled to only mix using subgroups of the Weyl group. This, though, would give an amusing example of the use of "soft" topological methods from homotopy theory together with

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[21] A space is simple is its fundamental group is abelian and it acts trivially on higher homotopy. Localization theory works well for nilpotent, and hence, simple, spaces (see [HMR]).

[22] If P and Q are odd. Otherwise there can be a $\mathbf{Z}_2$ which is a codimension one arf invariant. This element always acts trivially on topological structure sets as is well known (but can be nontrivial smoothly). In many cases this obstruction can be seen to be trivial even smoothly, e.g. if there is a fixed circle for all of of the elements of the Weyl group used at P and Q, using a variant of the trick used above and the "numerical Levine formula" of [CW, Dv, Pa].

surgery giving rise to topological conjugacies of dynamical systems that do have some recurrence.

In order to make further progress (e.g. to show that the quotient of G under the relation of topological conjugacy has trivial hausdorfification) we have to study what happens for individual primes. We shall facilitate matters greatly by assuming that **p is a large prime** compared to the dimension. Moreover, the calculations are suggestive of what occurs for the case of a single circle.

Consider now $\mathbf{Z}_p \subset T$ as a diagonal matrix in SU(n), denoted by $(a_1, a_2,\dots a_n)$ where the a's are integers mod p. We have $\Sigma a_i = 0$ since our torus is in SU(n). The Chern classes are, of course, the symmetric functions of the $a_i$'s.

**Proposition 13:** For G = SU(n), two $G/\mathbf{Z}_p$'s have same homotopy type (for p large w.r.t. n) iff their first nonzero Chern classes agree. Moreover, they then have the same homeomoprhism (=diffeomorphism) type.

There is a fibration $G/\mathbf{Z}_p \to B\mathbf{Z}_p \to BSU(n)$. The map $B\mathbf{Z}_p \to BSU(n)$ factors through $B\mathbf{Z}_p \to BT$, where T is the maximal torus and it is easy enough to understand. BSU(n) through dimension = dim(G) at a large prime can be thought of as a product of Eilenberg-MacLane spaces determined by the Chern classes. So $BSU(n) \to \prod K(\mathbf{Z}, 2i)$ is an isomorphism. The Chern classes thus determine the Postnikov decomposition of $G/\mathbf{Z}_p$. The r-th k-invariant is obtained by pulling back the r+1st Chern class from $B\mathbf{Z}_p$. However, once a Chern class is nonzero, the cohomology from $B\mathbf{Z}_p$ pulls back trivially in higher dimensions, and $G/\mathbf{Z}_p$ looks like that Postnikov piece (at p) × with a product of the remaining odd dimensional spheres. This proves the proposition by the classification of spaces via Postnikov towers (and the fact that two finite dimensional spaces are homotopy equvalent iff their Postnikov towers agree through their dimensions).

A similar method computes their Pontrjagin classes in terms of the Chern class of the representation ρ pulled back to $B\mathbf{Z}_p$. As a result, only one nonzero Pontrjagin class enters. If our prime is sufficiently large, F/Cat can be caught in terms of Pontrjagin classes and we've computed the normal invariant. The rest of the surgery exact sequence is computed as before. Also, if p is sufficiently large, Top/O has no homotopy groups in the relevant range, so all of these manifolds are diffeomophic because they are homeomorphic and their universal covers are diffeomorphic.

**Corollary**: By choosing p large it is possible to get topological conjugacy classes of elements as dense in G as you would like. So the largest hausdorff quotient of G/~ is a point.

Now let us consider the same analysis applied to $BS^1$ in place of $B\mathbf{Z}_p$. Again we have $\Sigma a_i = 0$, but now as an equation in **Z**. Now consider the 2nd Chern class = $1/2(0^2 -$

$\Sigma a_i^2$). So there are only finitely many circles in the maximal torus that give rise to bundles with the same 2nd Chern class.

I have not checked the following but it seems reasonable in light of the previous discussion to believe the following:

**Problem**: Is it true that if G is irreducible, then the map from G-conjugacy classes to topological conjugacy of the translations is finite to one?

The first case where one sees infinite indeterminacy is SU(2) × SU(2). Here the maximal torus is of dimension 2. The quotient by any circle is topologically $S^2 \times S^3$. This follows very easily from old work of Smale [Sm]. Smale showed that any simply connected spin 5-manifold with $H^2 \cong \mathbf{Z}$ is diffeomorphic to $S^2 \times S^3$. That the quotient has these properties is easy enough to see from the Gysin sequence associated to the cover $G/S \to G/T$. (S is the circle, and T is the maximal 2-torus).

The classifying map for the circle bundle $G \to G/S$ is a generator of $H^2$, and these are equivalent under diffeomorphism. So all of these circle actions are the same, and thus we have an analysis of the topological conjugacy in this way.

**Remark**: As Lashof and Zimmer have observed, whenever the compact group G has lattice subgroup, examples where $G/A \cong G/B$ for nonconjugate subgroups A and B give rise to different ergodic lattice actions on the same manifold. Thus, the observations and methods of this section shed a bit of light on these phenomena.

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