Manifolds with complete metrics of positive scalar curvature

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Joint work with Stanley Chang and Guoliang Yu

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Classical background.

**Fact**

If $S_p$ is the **scalar curvature** at a point $p$ in a manifold $M^n$, then

$$\text{Vol}_M(B_\epsilon(p)) = \text{Vol}_{\mathbb{R}^n}(B_\epsilon) - S_p \cdot C \epsilon^{n+2} + \cdots$$
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*If $S_p$ is the scalar curvature at a point $p$ in a manifold $M^n$, then*

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**Remark (Kazhdan-Warner)**

*Suppose $M$ is a compact $n$-manifold with $n > 2$.**
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Fact

If $S_p$ is the \textbf{scalar curvature} at a point $p$ in a manifold $M^n$, then

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Remark (Kazhdan-Warner)

Suppose $M$ is a compact $n$-manifold with $n > 2$.

- If $f : M \rightarrow \mathbb{R}$ is a function with $f(p) < 0$ for some $p \in M$, then there is a metric $g$ so that $f(p)$ is the scalar curvature of $(M, g)$ at $p$. 
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Fact

If $S_p$ is the scalar curvature at a point $p$ in a manifold $M^n$, then

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Suppose $M$ is a compact $n$-manifold with $n > 2$.

- If $f : M \rightarrow \mathbb{R}$ is a function with $f(p) < 0$ for some $p \in M$, then there is a metric $g$ so that $f(p)$ is the scalar curvature of $(M, g)$ at $p$.
- If there is a metric $g$ with positive scalar curvature, then any function $f : M \rightarrow \mathbb{R}$ is the scalar curvature of some metric.
Fact

If $S_p$ is the scalar curvature at a point $p$ in a manifold $M^n$, then

$$\text{Vol}_M(B_\varepsilon(p)) = \text{Vol}_{\mathbb{R}^n}(B_\varepsilon) - S_p \cdot C \varepsilon^{n+2} + \cdots$$

Remark (Gauss-Bonnet)

If a surface $\Sigma^2$ has a complete metric of positive scalar curvature, then $\Sigma^2 \cong S^2$ or $\Sigma^2 \cong \mathbb{R}P^2$. 
Theorem (Atiyah-Singer, due to Lichnerowicz)

If $M^n$ is a compact spin manifold admitting a metric of positive scalar curvature, then

$$\langle \hat{A}(M), [M] \rangle = 0.$$
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**Example (**$K3$ Surface**)**

- $K3$ is spin.
- $\text{sign } K3 = 16$, so $\langle \hat{A}(K3), [K3] \rangle \neq 0$
- $K3$ has no metric of positive scalar curvature.
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Example (\(\mathbb{C}P^2\))

- \(\mathbb{C}P^2\) has a metric with positive scalar curvature,
- \(\text{sign } \mathbb{C}P^2 = 1,\)
- \(\mathbb{C}P^2\) is not spin.
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**Idea of Proof:**

Since \( M \) is spin, \( M \) has a Dirac operator \( \mathcal{D} \).
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Since $M$ is spin, $M$ has a Dirac operator $\mathcal{D}$.

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\mathcal{D}^* \mathcal{D} = \Delta + \text{Scal}.
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If $\text{Scal} > 0$, then $\Delta + \text{Scal} > 0$, 

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$\square$
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### Theorem (Atiyah-Singer, due to Lichnerowicz)

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$$\mathcal{D}^* \mathcal{D} = \Delta + \text{Scal}.$$ 

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Simply connected case.

Theorem (Gromov-Lawson, Hitchin, Stolz)

If \( M^n \) with \( n > 4 \) and \( M \) simply connected, then \( M \) has a metric of positive scalar curvature iff

\[ \text{ind} / D = 0 \in KO_n(\text{pt}). \]
If $M^n$ with $n > 4$ and $M$ simply connected, then $M$ has a metric of positive scalar curvature iff

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- $M$ is not spin, or
- $M$ is spin, and $\text{ind} \, \mathcal{D} = 0 \in \text{KO}_n(pt)$. 


Question

Does $T^n$ have a metric of positive scalar curvature?
Non-simply connected case.

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**Proof.**

Following Rosenberg, same as before but using $K(C^*\pi)$. 

What happens for interiors of manifolds with boundary?

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- Assembly map for pairs into L-theory. But in the $C^*$-algebra setting only works really well if the fundamental group injects.
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- Assembly map for pairs into L-theory. But in the $C^*$-algebra setting only works really well if the fundamental group injects.
- One mystery of the Baum-Conjecture is the functoriality aspect (even in the torsion free case.) Why should there be functoriality associated to homomorphisms?
Proper homotopy equivalence of non-compact manifolds.

**Definition**

$M$ is **simply connected at infinity** if every compact $K \subset M$ is contained in a larger compact $C \supset K$, so that $M - C$ is simply connected.
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Theorem (Browder-Livesay-Levine)

$M^n$, $n > 5$, is the interior of a manifold with simply connected boundary iff

- $M$ has finitely generated homology and
- $M$ is simply connected at infinity.
The case of lattices.

**Theorem (Block-W)**

$K \backslash G / \Gamma$ has a complete metric of positive scalar curvature iff $\mathbb{Q}$-rk$(\Gamma) > 2$. 

**Theorem (Chang)**

$K \backslash G / \Gamma$ never has a complete metric of positive scalar curvature in the obvious QI class.

**Idea of Proof:** Marry Novikov idea to Roe's partitioned manifold index theorem. We will discuss it in more detail later.
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Definition (Fundamental group at infinity)

$K_1 \subset K_2 \subset K_3 \subset \cdots \subset M$
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\( \pi_1^\infty(M) = \Gamma \) means that the pro-system

\[ \pi_1(M - K_1) \leftarrow \pi_1(M - K_2) \leftarrow \pi_1(M - K_3) \leftarrow \cdots \]
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means that the pro-system

\[ \pi_1(M - K_1) \leftarrow \pi_1(M - K_2) \leftarrow \pi_1(M - K_3) \leftarrow \cdots \]

is pro-equivalent to the constant system \( \Gamma \leftarrow \Gamma \leftarrow \cdots \).
Proper homotopy equivalence of non-compact manifolds.

**Theorem (Siebenmann’s thesis)**

The obstruction to putting a boundary on a tame manifold lies in $\tilde{K}_0(\mathbb{Z}\pi_1^\infty)$. 

Takes the fear out of non-compactness when you are tame. If tame at infinity, then there is a relative assembly theory, relative Novikov conjecture for $L$-classes and so on. If not, it’s somewhat harder to describe the relevant assembly maps that enter the theory, but not impossible.
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To what extent does the theory of pairs capture the issues?
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\[ K_1 \quad K_2 \quad K_3 \quad K_4 \quad K_5 \]

Answer

In the fundamental group tame case, pretty well
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In the fundamental group tame case, pretty well—but not in general. There are $\lim^1$ terms,

$$0 \to \lim^1 H_{*-1}^\text{lf}(K_i) \to H_*^\text{lf}(M) \to \lim H_*^\text{lf}(K_i) \to 0,$$
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and other terms measured off group homology’s limits.
Key example: Whitehead manifold.

Example

$h^1(S^1 \times D^2)$,
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$h^1(S^1 \times D^2)$, $h^2(S^1 \times D^2)$, $h^3(S^1 \times D^2)$, ...
Key example: Whitehead manifold.

\[ \text{Whitehead} = S^3 - \bigcap_i h^i(S^1 \times D^2). \]
Key example: Whitehead manifold.

Whitehead = $S^3 - \bigcap_i h^i(S^1 \times D^2)$.

Remark

$Whitehead \not\cong \mathbb{R}^3$, but $\mathbb{R} \times Whitehead \cong \mathbb{R}^4$. 
Key example: Whitehead manifold.

\[ \text{Whitehead} = S^3 - \bigcap h^i(S^1 \times D^2). \]

Remark

There are uncountably many variants of this construction.
Key example: Whitehead manifold.

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\text{Whitehead} = S^3 - \bigcap_i h^i(S^1 \times D^2).
\]

Question

What does the moduli space of these manifolds look like?
Key example: Whitehead manifold.

Whitehead = $S^3 - \bigcap_i h^i(S^1 \times D^2)$.

Question

*What does the moduli space of these manifolds look like? A little bit like the space of Penrose tilings.*
Key example: Whitehead manifold.

Whitehead manifold:

\[ \text{Whitehead} = S^3 - \bigcap_i h^i(S^1 \times D^2). \]

Remark

*No nice metric on these manifolds.*
Observations about the Whitehead manifold.

$S^1 \times D^2$ $T^2$ $T^2$ $T^2$ $T^2$ $T^2$

$A$ $A$ $A$ $A$ $A$

A is aspherical (by Papakyriakopoulos' sphere theorem, because A is the complement of a non-split link).

Naively construed, $\pi_1$ is trivial.
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- Naively construed, “$\pi_1^\infty$” is trivial.
The Whitehead manifold and positive scalar curvature.

**Theorem**

*The Whitehead manifold has no complete metric of positive scalar curvature.*

**Proof:**

$DW$ = double of Whitehead manifold along $T^2$

$DW$ has a positive scalar curvature metric except at $A \cup \bar{A}$

$DW$ has positive scalar curvature metric at infinity — a contradiction.
Proof:

The Whitehead manifold and positive scalar curvature.

\[ S^1 \times D^2 \]

\[ T^2 \quad T^2 \quad T^2 \quad T^2 \quad T^2 \]

\[ D^2 \]

\[ T^2 \]

\[ A \quad A \quad A \quad A \]

Double of Whitehead manifold along \( T^2 \) has a positive scalar curvature metric except at \( A \cup \overline{A} \). \( T^2 \) has positive scalar curvature metric at infinity—contradiction.
The Whitehead manifold and positive scalar curvature.

Proof:

\[ S^1 \times D^2 \] is also a manifold.

\[ \text{DW} = \text{double of Whitehead manifold along } T^2 \]

\[ \text{DW} \] has a positive scalar curvature metric everywhere.

\[ A \cup \overline{A} \]

\[ \text{DW} \] has positive scalar curvature metric at infinity—a contradiction.
The Whitehead manifold and positive scalar curvature.

Proof:

\[ \overline{A} \quad \overline{A} \quad \overline{A} \quad A \quad A \quad A \quad A \]

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Digression: Roe’s Partition Manifold Theorem.

**Theorem (Roe)**

*If $V$ is spin and $Z$ positive scalar curvature at infinity, then $\text{ind} = 0$.***

Modern Philosophy
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*If* $V$ *is spin and* $Z$ *positive scalar curvature at infinity, then* $\text{ind} = 0$.

Modern Philosophy

The partition defines a virtual vector bundle on the space at infinity.
Digression: Roe’s Partition Manifold Theorem.

**Theorem (Roe)**

*If $V$ is spin and $Z$ positive scalar curvature at infinity, then $\text{ind} = 0$.***

**Modern Philosophy**

The partition defines a virtual vector bundle on the space at infinity. Only the ends of the space at infinity are independent of the quasi-isometry class of the metric.
If $V$ is not simply connected, attractive to couple Roe’s theorem to $C^*(\pi_1 V)$.
The non-simply connected case.

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The non-simply connected case.

If \( V \) is not simply connected, attractive to couple Roe’s theorem to \( C^*(\pi_1 V) \). In this case, \( V = T^2 \).

But \([\mathcal{D}] \in K_2(T^2)\) dies on pushing forward by \( K_2(T^2) \to K_2(\text{pt})\),
The non-simply connected case.

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But $[\mathcal{D}] \in K_2(T^2)$ dies on pushing forward by $K_2(T^2) \to K_2(\text{pt})$, so $\mathcal{D}$ on $T^2$ does not obstruct positive scalar curvature.
The non-simply connected case.

If $V$ is not simply connected, attractive to couple Roe’s theorem to $C^*(\pi_1 V)$. In this case, $V = T^2$.

But $[\mathcal{P}] \in K_2(T^2)$ dies on pushing forward by $K_2(T^2) \to K_2(\text{pt})$, so $\mathcal{P}$ on $T^2$ does not obstruct positive scalar curvature.

On the other hand, $K_2(T^2) \to K_2(C^*(\mathbb{Z}^2))$ is injective.
Idea

Use $\pi_1(DW)$ rather than $\pi_1(T^2) = \mathbb{Z}^2$.

Question

_Do we know strong Novikov conjecture for $\pi_1(DW)$?

$DW$ aspherical.

\[
\begin{array}{ccc}
H_2(T^2) & \hookrightarrow & H_2(DW) \\
\uparrow & & \uparrow \\
K_2(T^2) & \rightarrow & K_2(DW) \\
& & \rightarrow \\
& & K_2(\pi_1 DW)
\end{array}
\]

Theorem (Connes-Gromov-Mascovici)

Novikov conjecture holds for all 2-dimensional cohomology classes.
Theorem

If $M^3$ is of finite type and has positive scalar curvature at infinity, then $M$ is the interior of a manifold with boundary.
Taming 3-manifolds.

Theorem

If $M^3$ is of finite type and has positive scalar curvature at infinity, then $M$ is the interior of a manifold with boundary.

Example

$S^3 - \text{cantor set} = \overline{L^3} \# L^3$. 
Taming 3-manifolds.

**Theorem**

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**Example**

$S^3 - \text{cantor set} = \widetilde{L^3} \# L^3$.

**Proof:**

Whitehead case applies.
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Whitehead case applies. Perelman is used.
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Example

$S^3$ − cantor set $= \overline{L^3 \# L^3}$.

Proof:

Whitehead case applies. Perelman is used—though Hamilton is probably enough.
Contractible manifolds and positive scalar curvature.

**Theorem**

For all $n$, there is a contractible manifold $M^n$ having no complete metric of positive scalar curvature.
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\( n = 1 \quad \mathbb{R}. \)
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\( n = 3 \) The Whitehead manifold.
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- $n = 3$ The Whitehead manifold.
- $n = 4$ The Mazur manifold.
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$n > 4$ Variations on the Mazur manifold.
Question

Are there are interesting 4-manifolds that have complete metrics of positive scalar curvature?
Dimension four?

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**Question**

Is $\mathbb{R}^4$ the only contractible 4-manifold with positive scalar curvature?