Manifolds with complete metrics of positive scalar curvature

Shmuel Weinberger
Joint work with Stanley Chang and Guoliang Yu

May 5–14, 2008

Classical background.

Fact

If S_p is the **scalar curvature** *at a point p in a manifold M^n, then*

$$\mathrm{Vol}_M\left(B_\epsilon\left(p\right)\right) = \mathrm{Vol}_{\mathbb{R}^n}(B_\epsilon) - S_p \cdot C\epsilon^{n+2} + \cdots$$

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- *If there is a metric g with positive scalar curvature, then any function $f : M \to \mathbb{R}$ is the scalar curvature of some metric.*

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Remark (Gauss-Bonnet)

If a surface Σ^2 has a complete metric of positive scalar curvature, then $\Sigma^2 \cong S^2$ or $\Sigma^2 \cong \mathbb{R}P^2$.

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Theorem (Atiyah-Singer, due to Lichnerowicz)

If M^n is a compact spin manifold admitting a metric of positive scalar curvature, then

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Example ($K3$ Surface)

- $K3$ is spin.
- $\operatorname{sign} K3 = 16$, so $\langle \hat{A}(K3), [K3] \rangle \neq 0$
- $K3$ has no metric of positive scalar curvature.

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Example ($\mathbb{C}P^2$)

- $\mathbb{C}P^2$ has a metric with positive scalar curvature,
- $\operatorname{sign} \mathbb{C}P^2 = 1$,
- $\mathbb{C}P^2$ is not spin.

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- *M is not spin, or*
- *M is spin, and* $\mathrm{ind}\, \not{D} = 0 \in \mathrm{KO}_n(\mathrm{pt})$.

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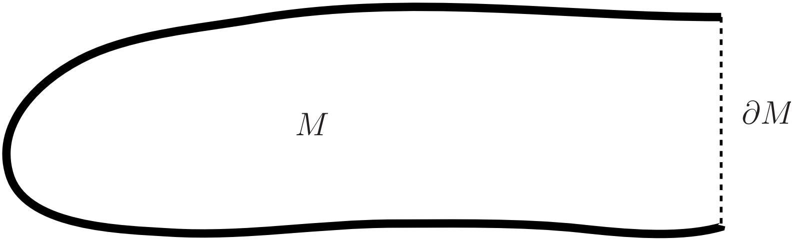
Proof.

Following Rosenberg, same as before but using $K(C^\star \pi)$. $\square$

What happens for interiors of manifolds with boundary?



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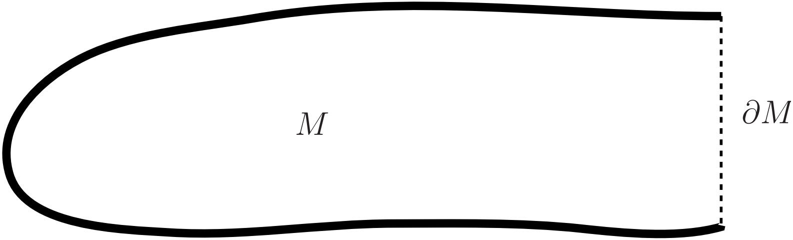

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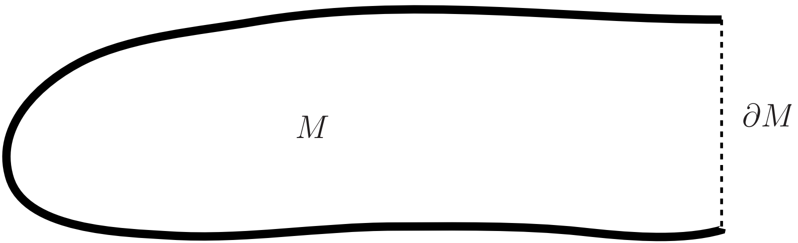

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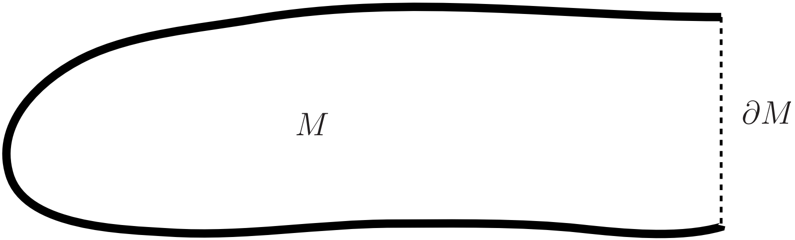

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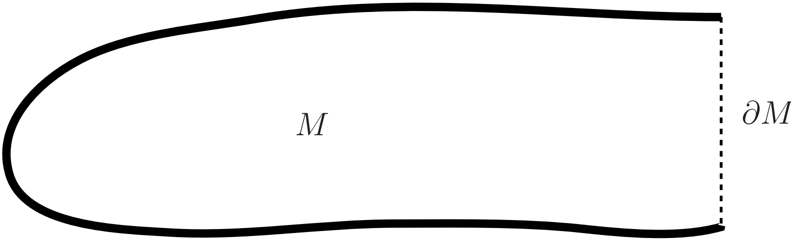

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- This suggests an obstruction in $\mathrm{KO}(B\pi_1 M, B\pi_1^\infty M)$.
- Assembly map for pairs into L-theory. But in the $C^\star$-algebra setting only works really well if the fundamental group injects.
- One mystery of the Baum-Conjecture is the functoriality aspect (even in the torsion free case.) Why should there be functoriality associated to homomorphisms?

Proper homotopy equivalence of non-compact manifolds.

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M is **simply connected at infinity** if every compact $K \subset M$ is contained in a larger compact $C \supset K$, so that $M - C$ is simply connected.

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Theorem (Browder-Livesay-Levine)

M^n, $n > 5$, is the interior of a manifold with simply connected boundary iff

- *M has finitely generated homology and*
- *M is simply connected at infinity.*

The case of lattices.

Theorem (Block-W)

$K\backslash G/\Gamma$ has a complete metric of positive scalar curvature iff $\mathbb{Q}$-rk$(\Gamma) > 2$.

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- We will discuss it in more detail later.

$\square$

Fundamental group at infinity.

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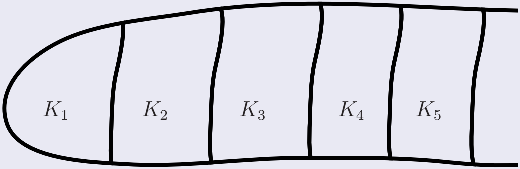

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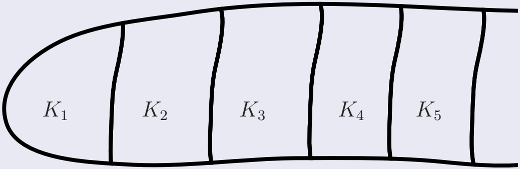

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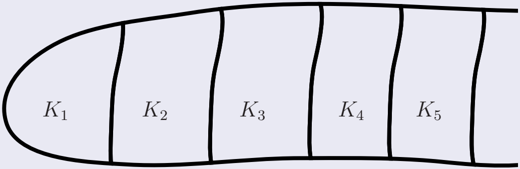

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is pro-equivalent to the constant system $\Gamma \leftarrow \Gamma \leftarrow \cdots$.

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Theorem (Siebenmann's thesis)

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If not, it's somewhat harder to describe the relevant assembly maps that enter the theory, but not impossible.

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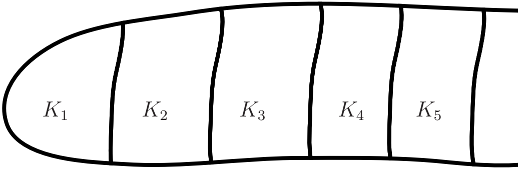

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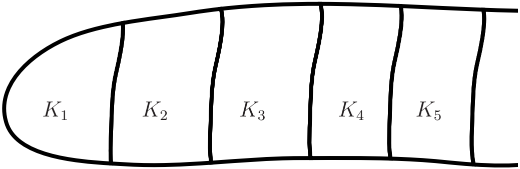

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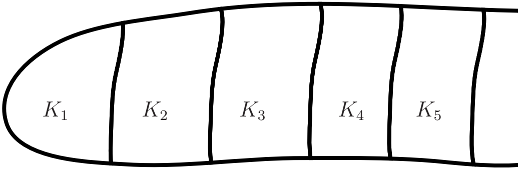

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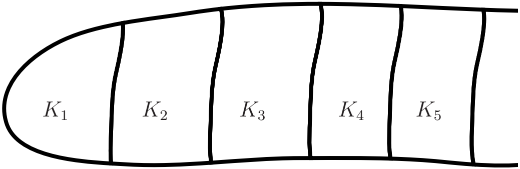

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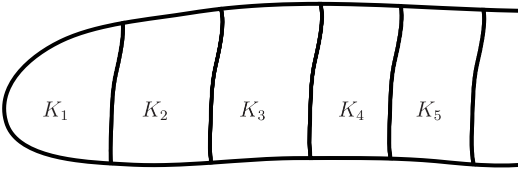

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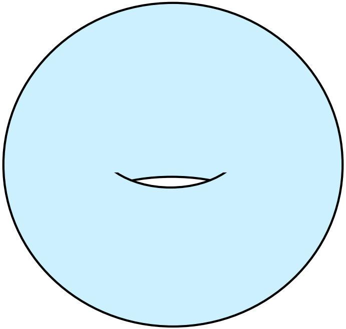
and other terms measured off group homology's limits.

Key example: Whitehead manifold.

Example

$h^1(S^1 \times D^2)$,



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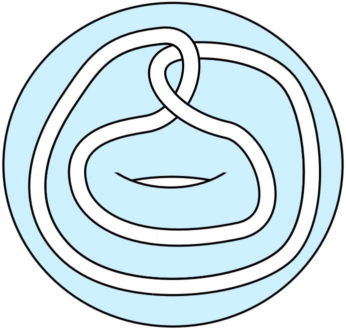

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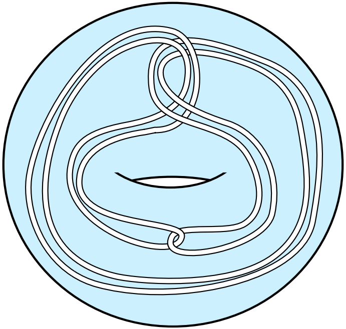

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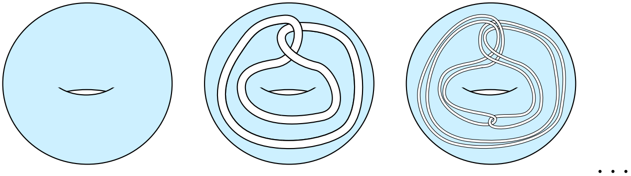

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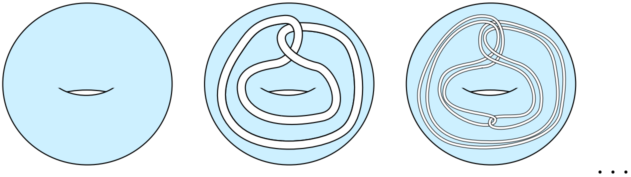

$$\text{Whitehead} = S^3 - \bigcap_i h^i(S^1 \times D^2).$$

Remark

Whitehead $\not\cong \mathbb{R}^3$, *but* $\mathbb{R} \times$ *Whitehead* $\cong \mathbb{R}^4$.

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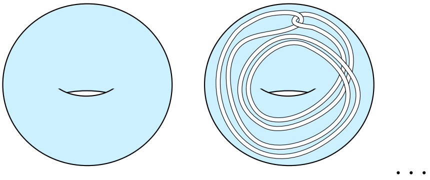

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Remark

There are uncountably many variants of this construction.

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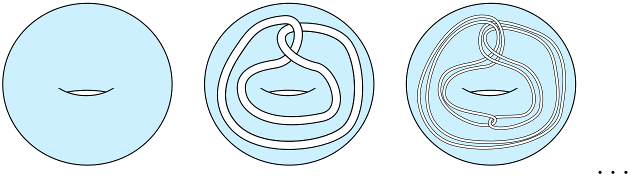

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What does the moduli space of these manifolds look like?

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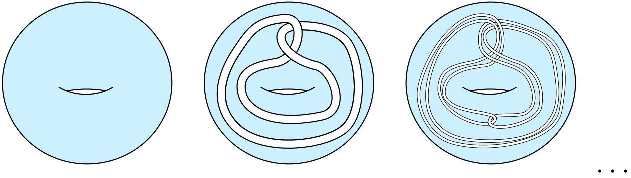

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What does the moduli space of these manifolds look like? A little bit like the space of Penrose tilings.

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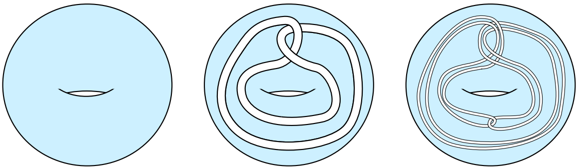

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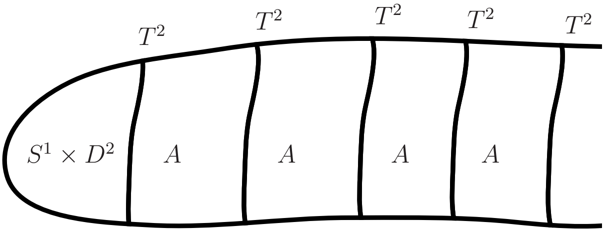
$$\text{Whitehead} = S^3 - \bigcap_i h^i(S^1 \times D^2).$$

Remark

No nice metric on these manifolds.

Observations about the Whitehead manifold.



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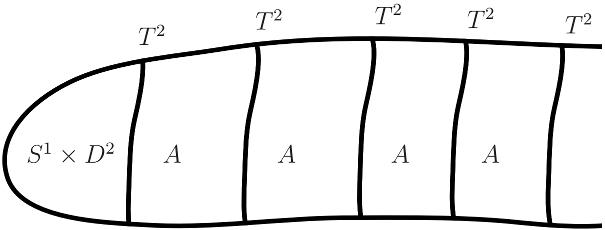

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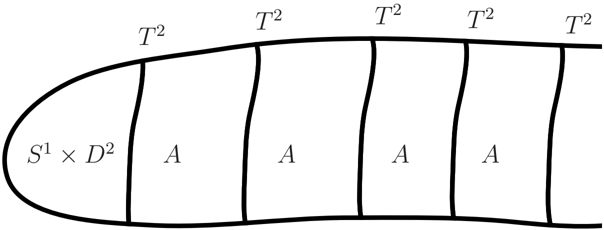

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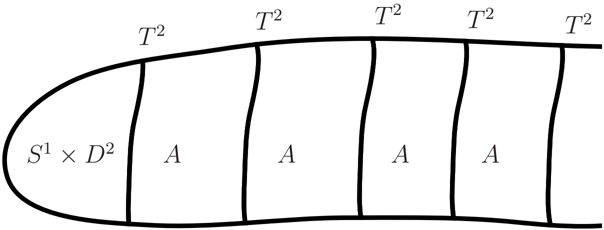

- A is aspherical (by Papakyriakopoulos' sphere theorem, because A is the complement of a non-split link)
- Naively construed, "π_1^∞" is trivial.

The Whitehead manifold and positive scalar curvature.

Theorem

The Whitehead manifold has no complete metric of positive scalar curvature.

Proof:

$\square$

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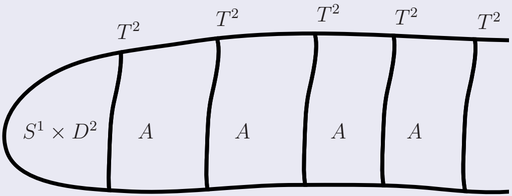

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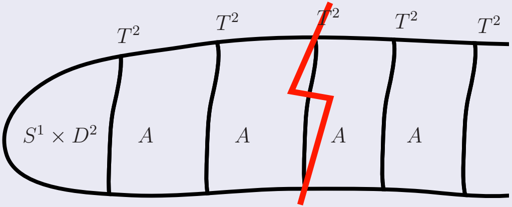
The Whitehead manifold and positive scalar curvature.

Proof:



$\square$

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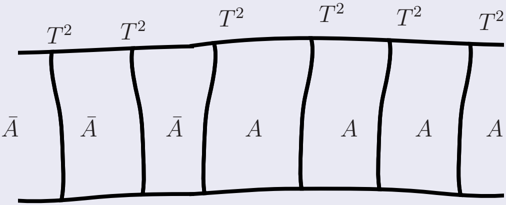

∎

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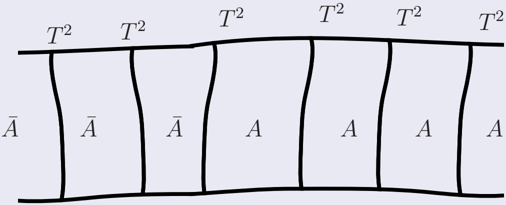

- DW = double of Whitehead manifold along T^2

□

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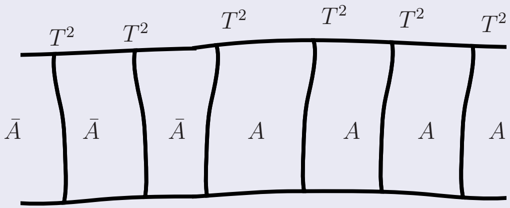

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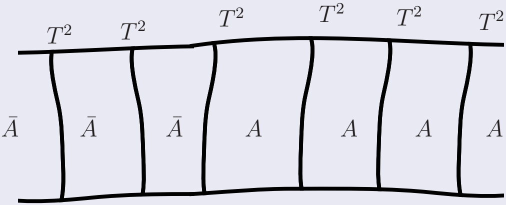

- DW = double of Whitehead manifold along T^2
- DW has a positive scalar curvature metric except at $A \cup \bar{A}$
- DW has positive scalar curvature metric at infinity

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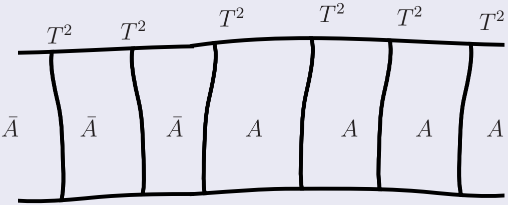


- DW = double of Whitehead manifold along T^2
- DW has a positive scalar curvature metric except at $A \cup \bar{A}$
- DW has positive scalar curvature metric at infinity—a contradiction.

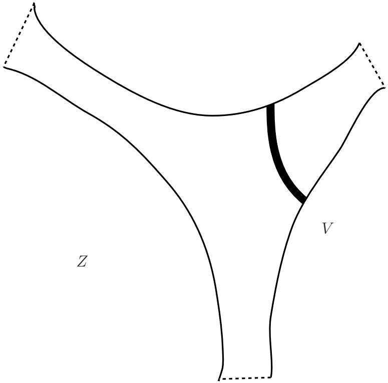
$\square$

Digression: Roe's Partition Manifold Theorem.

Theorem (Roe)

If V is spin and Z positive scalar curvature at infinity, then $\mathrm{ind} = 0$.

Modern Philosophy



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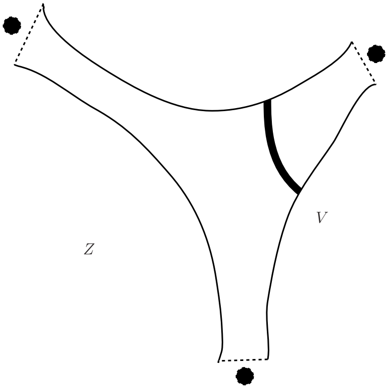
The partition defines a virtual vector bundle on the space at infinity.



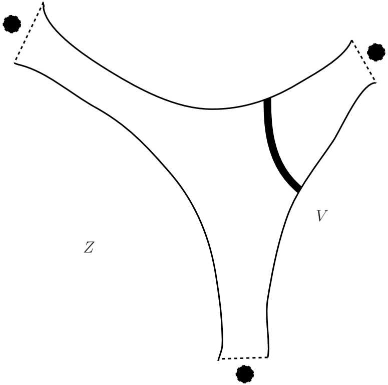
Digression: Roe's Partition Manifold Theorem.

Theorem (Roe)

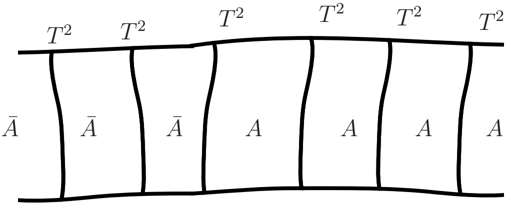
If V is spin and Z positive scalar curvature at infinity, then $\mathrm{ind} = 0$.

Modern Philosophy

The partition defines a virtual vector bundle on the space at infinity. Only the ends of the space at infinity are independent of the quasi-isometry class of the metric.



The non-simply connected case.



If V is not simply connected, attractive to couple Roe's theorem to $C^\star(\pi_1 V)$.

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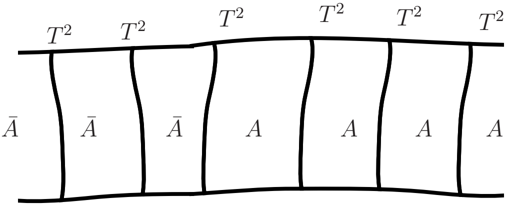

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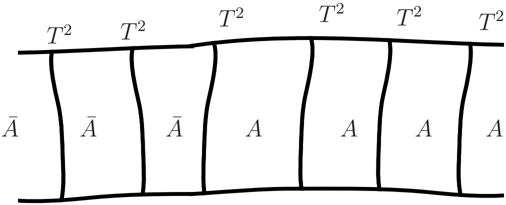

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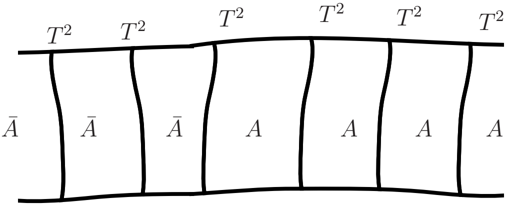

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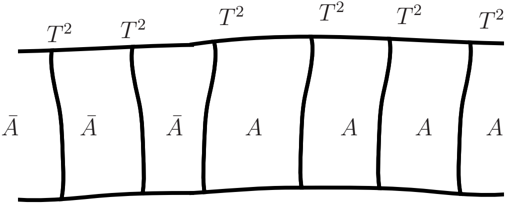

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But $[\not{D}] \in K_2(T^2)$ dies on pushing forward by $K_2(T^2) \to K_2(\mathrm{pt})$, so $\not{D}$ on T^2 does not obstruct positive scalar curvature.

On the other hand, $K_2(T^2) \to K_2(C^\star(\mathbb{Z}^2))$ is injective.

Tilt horizontal direction to the vertical.

Idea

Use $\pi_1(DW)$ rather than $\pi_1(T^2) = \mathbb{Z}^2$.

Question

Do we know strong Novikov conjecture for $\pi_1(DW)$?

DW aspherical.

$$
\begin{tikzcd}
H_2(T^2) \arrow[r, hook] & H_2(DW) & \\
K_2(T^2) \arrow[u] \arrow[r] & K_2(DW) \arrow[u] \arrow[r] & K_2(\pi_1 DW)
\end{tikzcd}
$$

Theorem (Connes-Gromov-Mascovicci)

Novikov conjecture holds for all 2-dimensional cohomology classes.

Taming 3-manifolds.

Theorem

If M^3 is of finite type and has positive scalar curvature at infinity, then M is the interior of a manifold with boundary.

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$S^3 - \text{cantor set} = \widetilde{L^3 \# L^3}$.

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S^3 − cantor set $= \widetilde{L^3 \# L^3}$.

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Whitehead case applies.
Perelman is used—though Hamilton is probably enough. $\square$

Contractible manifolds and positive scalar curvature.

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For all n, there is a contractible manifold M^n having no complete metric of positive scalar curvature.

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Proof:

$\mathbf{n = 1}$ $\mathbb{R}$.

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Proof:

$\mathbf{n = 1}$ $\mathbb{R}$.

$\mathbf{n = 2}$ $\mathbb{R}^2$.

$\square$

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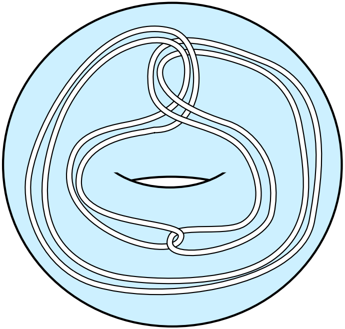
Proof:

$n = 3$ The Whitehead manifold.

$\square$



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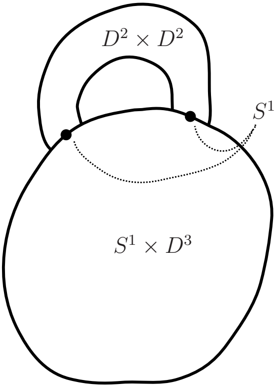
$\mathbf{n = 4}$ The Mazur manifold.

□



Contractible manifolds and positive scalar curvature.

Theorem

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Proof:

- $\mathbf{n = 3}$ The Whitehead manifold.
- $\mathbf{n = 4}$ The Mazur manifold.
- $\mathbf{n > 4}$ Variations on the Mazur manifold.

$\square$

Dimension four?

Question

Are there are interesting 4-manifolds that have complete metrics of positive scalar curvature?

Dimension four?

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Are there are interesting 4-manifolds that have complete metrics of positive scalar curvature?

Question

Is $\mathbb{R}^4$ *the only contractible 4-manifold with positive scalar curvature?*