Finiteness Conditions for CW-Complexes

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A CW-complex [12] is, roughly speaking, a space obtained by repeatedly attaching cells (perhaps infinitely many) by maps of spheres into finite subcomplexes, and assigning the whole the weak topology. This definition may be viewed as a convenient restriction to place on a space, or homotopy type, under which many results in homotopy theory may be obtained [5]. We go further, and regard it as a potentially effective description of the space, of interest in its own right. It is, then, important for us to be able to replace a CW-complex, when possible, by a simpler one—ideally, we would like a canonically simple one in each homotopy type. This is rather an unreasonable request, but in the simply-connected case, one can get close to it (Proposition 4.1).

Questions of this kind were first considered by Whitehead [13], who posed in particular the question, when is a CW-complex homotopy equivalent to a finite dimensional one? This is answered by Theorem E below. Another interesting problem of this kind comes from the fact that any ANR is dominated by a CW-complex (by a result of Whitehead [13], it must then have the homotopy type of one), and a compact ANR by a finite complex. The question arises (Milnor [5]) whether every complex which is dominated by a finite complex is homotopy equivalent to a finite complex. This is answered negatively by Theorem F below; however, the question whether a compact ANR is always equivalent to a finite CW-complex remains open; in particular, for compact topological manifolds (a negative answer here would, of course, settle the triangulation problem).

In this paper, we shall often use *complex* for CW-complex, and *equivalent* for homotopy equivalent. X will always have the homotopy type of a connected complex, with fundamental group π, of integral group ring Λ. Tilde will denote universal coverings; the homology and homotopy, etc., of $\widetilde{X}$ will be regarded as Λ-modules. We shall find it convenient to refer to relative homology and homotopy groups as groups associated to inclusion maps. More generally, let $\varphi : K \to X$ be a map, with mapping cylinder $M = X \cup_\varphi (K \times I)$; then we may define $\pi_n(\varphi)$, $H_n(\varphi)$ as $\pi_n(M, K \times 1)$ and $H_n(M, K \times 1)$. Alternatively, we may define $\pi_n(\varphi)$ by homotopy classes of commutative diagrams

$$\begin{array}{ccc} S^{n-1} & \subset & D^n \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ K & \xrightarrow{\varphi} & X \end{array}$$

(Hilton and Eckmann). We call φ n-connected if K, X are connected and $\pi_i(\varphi) = 0$ for $1 \leqq i \leqq n$. Observe that the usual homology and homotopy exact sequences (of pairs, triples etc.) can be reformulated in this notation.

In § 1 we give conditions which are equivalent to the condition that X has the homotopy type of a complex with finite (resp. countable) skeleta. It turns out that if X is dominated by a complex satisfying such a condition, X also satisfies it. In § 2 we do the same for finite dimension. The conditions can all be taken together, except (§ 3) when they require X to be equivalent to a finite complex; there is an obstruction to this which lies in the projective class group of Λ. In § 4, we reformulate some of our arguments in an attempt to discover when a finite complex can be replaced by one with a smaller number of cells; our result is fairly general, but not of much practical value.

A number of problems are left unsolved in this paper; the most vital is that we fail, except in one simple case, to give sufficient conditions for X to be equivalent to a 2-dimensional complex. It seems that a closer examination of possible systems of generators and relations for π is necessary for this.

Our results overlap somewhat with those of Eilenberg and Ganea [17] and Swan [18] in the cases when $\tilde{X}$ has the homotopy type of a point, resp. a sphere.

1. Complexes of finite type

Our object in this section is to characterise in algebraic terms complexes equivalent to ones of finite type (i.e., having a finite number of cells of each dimension). The characterisation we obtain is not wholly satisfactory, in that extraneous elements are introduced, but it is sufficient to give some results of interest.

We shall describe various conditions on X.

F1: The group π is finitely generated.

F2: The group π is finitely presented, and for any finite complex K^2 and map $\varphi : K^2 \to X$ inducing an isomorphism of fundamental groups, $\pi_2(\varphi)$ is a finitely generated module over Λ.

Fn ($n \geqq 3$): Condition F$(n-1)$ holds, and for any finite complex K^{n-1} and $(n-1)$-connected map $\varphi : K^{n-1} \to X$, $\pi_n(\varphi)$ is a finitely generated Λ-module.

F: All the Fn hold.

The unsatisfactory feature of condition Fn ($n \geqq 2$) is that it gives one condition for each of a whole class of admissible maps φ, about which nothing is *a priori* evident. We shall in fact show (if $n \geqq 3$) that, if F$(n-1)$ holds, admissible maps φ exist, and (if $n \geqq 2$) that, if any one $\pi_n(\varphi)$ is finitely generated, then so are all the others. Thus in this case, Fn reduces to a single condition.

We can now state the main result of this section.

THEOREM A. *The following conditions on X are equivalent*:

(i) *X is equivalent to a complex with finite n-skeleton* (*resp. to a complex of finite type*),

(ii) *X is dominated by a complex with finite n-skeleton* (*resp. by one of finite type*),

(iii) *X satisfies* Fn (*resp.* F).

We shall prove (i) equivalent to each of (ii) and (iii); it is trivial that (i) $\Rightarrow$ (ii). To prove that (i) $\Rightarrow$ (iii), we introduce a lemma of J. H. C. Whitehead [11, Lemma 15, p. 46] which will be needed several times in the sequel.

LEMMA 1.1. *Let P be a finite connected complex, K a connected subcomplex with $\pi_r(P, K) = 0$ for $1 \leqq r < n$. Then there is a formal deformation $D: P \to Q$ rel K such that, for $r < n$, Q has no r-cells outside K, and for $r \geqq n+2$, Q has the same number of r-cells outside K as P does.*

The only property of the formal deformation D which interests us is that it is a homotopy equivalence. We note that, instead of supposing K a subcomplex of P, we can take an $(n-1)$-connected map $i: K \to P$; for then replace P by the mapping cylinder of i (which collapses onto P).

Proof that (i) $\Rightarrow$ (iii). We may suppose X a complex with finite n-skeleton. Then we see by the standard algorithm ('edge path group') for computing π that, if $n \geqq 1$, π is finitely generated (so F1 holds), and if $n \geqq 2$, π is finitely presented. Suppose $n \geqq 3$, and $\varphi: K^{n-1} \to X$ an $(n-1)$-connected map. We may suppose that the image of φ is contained in X^{n-1}; then it follows from the exact sequence

$$\pi_n(X, X^n) \to \pi_{n-1}(X^n, K^{n-1}) \to \pi_{n-1}(X, K^{n-1}) \to \cdots$$

that the induced map from K^{n-1} to X^n is $(n-1)$-connected. By Lemma 1.1, we may replace X^n by an equivalent finite complex Y^{n+1} with K^{n-1} as $(n-1)$-skeleton. But

$$\pi_n(\varphi) = \pi_n(X, K) \cong \pi_n(\widetilde{X}, \widetilde{K}) \cong H_n(\widetilde{X}, \widetilde{K}) ,$$

using the Hurewicz theorem. (This will frequently be used in this form; we will usually write $\pi_n(\varphi) \cong H_n(\widetilde{\varphi})$, assuming φ $(n-1)$-connected). This last is a quotient module of $H_n(\widetilde{X}^n, \widetilde{K}) \cong H_n(\widetilde{Y}^{n+1}, \widetilde{K})$, which in turn is a quotient of $H_n(\widetilde{Y}^n, \widetilde{K}) \cong C_n(\widetilde{Y})$, a free Λ-module of finite rank. So $\pi_n(\varphi)$ is finitely generated. The argument in the case $n = 2$ is similar; if X^2 is finite, and $\varphi: K^2 \to X$ induces an isomorphism of fundamental groups, we replace X^2 by an equivalent finite Y^3, obtained from K^2 by attaching some 2-cells (necessarily by trivial maps) and 3-cells. Then again, $\pi_2(\varphi)$ is a quotient of $H_2(\widetilde{Y}^3, \widetilde{K}^2)$, and this is evidently finitely generated.

Proof that (iii) $\Rightarrow$ (i). We now need a general construction, which was

drawn to my attention by J. Milnor, who used it to establish our main results in the simply-connected case (unpublished). Suppose given an $(n-1)$-connected map $\varphi: K \to X$. If $n \geqq 3$, φ induces an isomorphism of fundamental groups, so we can regard $\pi_n(\varphi)$ as a Λ-module. Select Λ-generators $\{\alpha_i\}$ (or, if $n = 2$, generators) for $\pi_n(\varphi)$. Then the $\partial\alpha_i$ belong to $\pi_{n-1}(K)$: use them to attach n-cells to K. Now use the α_i themselves to extend φ over these cells (recall that an element of $\pi_n(\varphi)$ is represented by a pair of maps $\beta: S^{n-1} \to K$, $\gamma: D^n \to X$ with $\gamma \mid S^{n-1} = \varphi\beta$). If the resulting space is L and map $\psi: L \to X$, then since the map α in the exact sequence

$$\pi_n(L, K) \xrightarrow{\alpha} \pi_n(\varphi) \longrightarrow \pi_n(\psi) \longrightarrow \pi_{n-1}(L, K) = 0$$

is onto (for, if $n \geqq 3$, $\pi_n(L, K) \cong H_n(\tilde{L}, \tilde{K}) = C_n(\tilde{L})$, and the α_i were chosen as generators of $\pi_n(\varphi)$), $\pi_n(\psi)$ vanishes, and so ψ is n-connected. We describe this construction as *attaching cells to K to make φ n-connected.*

LEMMA 1.2. *Let K be a CW-complex, X have the homotopy type of one, and suppose $\psi: K \to X$ to be $(n-1)$-connected. Then we can attach cells to K, of dimensions $\geqq n$, to make φ a homotopy equivalence.*

PROOF. The above construction allows us to attach n-cells to K to make φ n-connected, and give a complex L^n. Repeat the process indefinitely, giving $\varphi_{n+r}: L^{n+r} \to X$. Let $L = \bigcup L^{n+r}$ with the weak topology; then the φ_{n+r} induce a map $\psi: L \to X$. Evidently ψ is N-connected for all N, by a theorem of Whitehead [12, Theorem 1], ψ is a homotopy equivalence.

If now X satisfies F1, we can find a finite bouquet K^1 of circles, and a map $\varphi: K^1 \to X$ which induces an epimorphism of fundamental groups, and so is 1-connected. By the lemma, we can attach to K^1 cells of dimension $\geqq 2$ to obtain a complex L (with finite 1-skeleton K) equivalent to X. If X satisfies F2, since π is finitely presented, we can construct a finite complex K'^2 and map $\varphi: K'^2 \to X$ inducing an isomorphism of fundamental groups. By F2, $\pi_2(\varphi)$ is finitely generated, so by the construction, we can attach a finite collection of 2-cells to K'^2 (forming K^2), and extend φ to a 2-connected map $K^2 \to X$; now, again by the lemma, we can attach cells of higher dimension to K^2 to make this a homotopy equivalence. Now let X satisfy Fn, with $n \geqq 3$. Assume, by induction (we have just verified it if $n = 3$), that we can construct an $(n-1)$-connected map $\varphi: K^{n-1} \to X$ with K^{n-1} finite. By Fn, $\pi_n(\varphi)$ is finitely generated; by the construction, we can attach finitely many n-cells to K^{n-1} to make φ n-connected (which performs the induction step), and now, by Lemma 1.2, we can attach cells of higher dimension to make the map a homotopy equivalence. Finally, if X satisfies F, we construct all the K^n, by induction, and let their union (with the weak topology) be K. As in Lemma 1.2, K is equivalent to X.

Before embarking on the proof that (ii) $\Rightarrow$ (i), we give a lemma. I am indebted to J. R. Stallings for pointing out the following result.

LEMMA 1.3. *Let G be a finitely presented group, H a retract of G. Then H is finitely presented.*

PROOF. By *retract*, we mean that homomorphisms $j : H \rightarrow G$, $r : G \rightarrow H$ are given with $rj = 1$. Let $\{g_i : r_j\}$ be a finite presentation for G, F the free group on the g_i, $p : F \rightarrow G$ the corresponding epimorphism. Then $jrp(g_i)$ is in G, so can be expressed as $p(w_i)$. We assert that $\{g_i : r_j, g_i^{-1}w_i\}$ is a presentation for H. For let it define a group L. Let $u : G \rightarrow L$ be the natural projection (which exists by van Dyck's theorem). Note that $rp(r_j) = r(1) = 1$ and

$$rp(g_i^{-1}w_i) = rp(g_i^{-1})rp(w_i) = rp(g_i^{-1})rjrp(g_i) = 1 ,$$

so the relations hold in H, and r factorises through L, $r = vu$. Then $vuj = rj = 1$, and uj is onto since $ujrp(g_i) = up(w_i) = up(g_i)$ give the generators of L. Hence v and uj are inverse isomorphisms of L and H.

Proof that (ii) $\Rightarrow$ (i). Let $Y = Y_0$ dominate X. If the 1-skeleton of Y is finite, $\pi_1(Y)$ is finitely generated, so is its quotient group π, and F1 holds for X. If the 2-skeleton of Y is finite, $\pi_1(Y)$ and hence (by the lemma) π are finitely presented. We attach 2-cells to Y_0 corresponding to the relators $g_i^{-1}w_i$ above, and use the fact that these relations hold in X to extend the retraction over the resulting complex Y_1. Then Y_1 still dominates X, and the fundamental group maps isomorphically. The map from Y_1 to X is then 2-connected, and we can attach cells of dimension $\geqq 3$ to Y_1 to make it a homotopy equivalence, by Lemma 1.2.

Now suppose the n-skeleton of Y finite. By induction, we may suppose the retraction $r : Y \rightarrow X$ $(n-1)$-connected (replacing perhaps Y by some Y_{n-1}). By Lemma 1.2 we may suppose that X is obtained from Y by adding cells of dimensions $\geqq n$. Let $j : X \rightarrow Y$, $r : Y \rightarrow X$ be the given maps with $rj \simeq 1$. Take universal covers; then $\tilde{j}$ and $\tilde{r}$ split each other's homology exact sequences, and $H_i(\tilde{j}) \cong H_{i+1}(\tilde{r})$ for all i. Then $\pi_n(r) \cong H_n(\tilde{r}) \cong H_{n-1}(\tilde{j})$. If $k = j \mid X^{n-2}$, this is a quotient of $H_{n-1}(\tilde{k})$. But r is the inclusion, and $rj \simeq 1$ in X; since the cells of $X - Y$ have dimension $\geqq n$, this homotopy, restricted to X^{n-2}, can be taken to lie in Y, so we can assume that k is just the inclusion of $X^{n-2} = Y^{n-2}$. But now $H_{n-1}(\tilde{k})$ is clearly finitely generated; hence so is $\pi_n(r)$, and we can add a finite number of n-cells to Y to make r n-connected. This completes the induction step, and the proof follows as in the other cases.

COROLLARY 1.4. *If $n = 2$, π is finitely presented or if $n \geqq 3$ and* F$(n-1)$ *holds, there exist finite complexes K^2 or K^{n-1} and $(n-1)$-connected maps*

$\varphi : K \to X$ inducing an isomorphism of fundamental groups. If $\pi_n(\varphi)$ is finitely generated for one such φ, then it is for every such.

If π is finitely presented, such a K^2 exists evidently, and under F$(n-1)$, X is equivalent to a complex with finite $(n-1)$-skeleton. If one such $\pi_n(\varphi)$ is finitely generated, add a finite number of n-cells to make φ n-connected and then (Lemma 1.2) cells of higher dimension to make it a homotopy equivalence. Then X satisfies (i), and so also (iii).

When the group ring Λ is noetherian, we can give simpler forms for the conditions Fn, viz.,

NF2: π is finitely presented and $H_2(\widetilde{X})$ is finitely generated (over Λ).

NFn ($n \geqq 3$): NF$(n-1)$ holds and $H_n(\widetilde{X})$ is finitely generated.

LEMMA 1.5. *If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence of modules over a noetherian ring, and if A and C are finitely generated, then so is B.*

PROOF. C is finitely generated, hence so is the submodule $\beta(B)$, with generators $\beta(f_j)$ say. Then if the e_i generate A, the $\alpha(e_i)$ and the f_j generate B; for let $x \in B$. Then $\beta(x)$ is dependent on the $\beta(f_j)$, so some $x - \sum \lambda_j f_j$ is in $\operatorname{Ker} \beta = \operatorname{Im} \alpha$, and is a linear combination of the $\alpha(e_i)$.

THEOREM B. *Assume Λ noetherian. Then π is finitely generated, and for $n \geqq 2$, Fn is equivalent to* NFn.

PROOF. The proof that π is finitely generated can be found in [4]. Suppose K a finite complex with fundamental group π. Then $C_i(\widetilde{K})$ is free of finite rank, hence $Z_i(\widetilde{K})$, and also $H_i(\widetilde{K})$ is finitely generated. Now if $\varphi : K \to X$ is $(n-1)$-connected, $\pi_n(\varphi) \cong H_n(\widetilde{\varphi})$, and we have the exact sequence

$$H_n(\widetilde{K}) \to H_n(\widetilde{X}) \to H_n(\widetilde{\varphi}) \to H_{n-1}(\widetilde{K})$$

with extreme terms finitely generated. By Lemma 1.5, $H_n(\widetilde{X})$ is then finitely generated if and only if $\pi_n(\varphi)$ is.

We conjecture that Λ noetherian implies also π finitely presented; however, this does not affect our results. The class of groups with noetherian group rings is certainly quite restricted; they satisfy the maximal condition for subgroups, and the only known examples [4] are finite extensions of polycyclic groups. This certainly includes finite groups and finitely generated abelian groups. Outside this class our proof of Theorem B breaks down; the result breaks down also, and an example of a finite complex K (of dimension 2) with $\pi_2(K)$ a non-finitely generated Λ-module is given by Stallings in [15].

The arguments of this section all apply (with a good deal less trouble) to the countable case. Here, the noetherian condition is irrelevant.

C1: The group π is countable.

Cn: ($n \geqq 2$): C($n-1$) holds and $H_n(\tilde{X})$ is countable.

C: All the Cn hold.

THEOREM C. *The following conditions on X are equivalent:*

(i) *X is equivalent to a complex with countable n-skeleton (resp. to a countable complex),*

(ii) *X is dominated by a complex with countable n-skeleton (resp. by a countable complex),*

(iii) *X satisfies* Cn *(resp.* C).

We leave to the reader the reformulation of our arguments to cover this case. The equivalence of (i) and (ii) (in the case $n = \infty$) is due to Whitehead [13]. There is also a conflation of Theorems A and C (cf. Theorem E).

2. Complexes of finite dimension

The algebraic conditions imposed by finite dimension are much clearer than in the case of finite type.

Dn: $H_i(\tilde{X}) = 0$ for $i > n$, and $H^{n+1}(X; \mathcal{B}) = 0$ for all coefficient bundles $\mathcal{B}$ (possibly non-abelian if $n = 1$).

Here a coefficient bundle is interpreted in the sense of Steenrod [8], with the natural extension to the non-abelian case (cf. Dedecker [3]).

The case $n = 1$ is entirely different from the rest; we shall treat it first.

THEOREM D. *If X satisfies* D1, *it has the homotopy type of a bouquet of circles.*

PROOF. All the homology groups of $\tilde{X}$ vanish, so by Whitehead's theorem, $\tilde{X}$ is contractible. Thus X is a space of type $(\pi, 1)$. So if $\mathcal{B}$ has fibre G, $H^2(X; \mathcal{B})$ may be identified with the set of extensions of G by π, corresponding to the given *operation* of π on G. Now let $\alpha : F \rightarrow \pi$ be a homomorphism of a free group onto π, with kernel, say, G. By hypothesis, the corresponding extension splits, so π is isomorphic to a subgroup of F, hence is free. The result follows.

It is now trivial to add finiteness and countability restrictions (F1 and C1), and to observe that if Y, dominating X, satisfies D1, then X also does.

For $n \geqq 3$, we construct a map φ as in § 1; let $\varphi : K^{n-1} \rightarrow X$ be $(n-1)$-connected. (Here, K need not be finite).

LEMMA 2.1. *Suppose φ as above, and that X satisfies* Dn. *Then $\pi_n(\varphi)$ is a projective Λ-module.*

PROOF. By Lemma 1.2, we can attach to K cells of dimension $\geqq n$ to make φ a homotopy equivalence. So we can suppose X a CW-complex with K as $(n-1)$-skeleton. Now $\pi_n(\varphi) \cong H_n(\tilde{\varphi}) = H_n(\tilde{X}, \tilde{K}) = C_n(\tilde{X})/B_n(\tilde{X}) = C_n/B_n$, say. We shall prove that the short exact sequence

(1) $$B_n \xrightarrow{j} C_n \longrightarrow \pi_n(\varphi)$$

splits; since C_n is free, it will follow that $\pi_n(\varphi)$ is projective.

Now B_n is a Λ-module, and so defines a coefficient bundle $\mathcal{B}$ over X. By hypothesis, $H^{n+1}(X;\mathcal{B}) = 0$. But $H^*(X;\mathcal{B})$ is the homology of the complex given by π-homomorphisms of $C_*(\tilde{X})$ into B_n. Write C_i for $C_i(\tilde{X})$ and consider

$$C_{n+2} \xrightarrow{d} C_{n+1} \xrightarrow{c} B_n \xrightarrow{j} C_n \ .$$

The π-homomorphism c is an $(n+1)$-cochain of $(X;\mathcal{B})$, its coboundary is $cd = 0$ (since $d^2 = 0$). Thus we have a cocycle; since $H^{n+1}(X;\mathcal{B}) = 0$, it is a coboundary, so we can write $c = sd = sjc$. But c is onto, so $1 = sj$, and s splits the exact sequence (1). This proves the lemma.

Complement. Let X satisfy D2, $\varphi : K^2 \to X$ be 2-connected. Then $\pi_3(\varphi)$ is projective.

The proof is exactly as above, except that $\pi_3(\varphi)$ is now isomorphic to B_2.

Suppose $\pi_n(\varphi)$ is free. Perform the construction of § 1 with the α_i as free generators of $\pi_n(\varphi)$. Considering (as usual by Lemma 1.2) the resulting Y as a subcomplex of X, we note that $H_i(\tilde{Y}, \tilde{K})$ vanishes for $i \neq n$, $H_i(\tilde{X}, \tilde{K})$ is zero for $i < n$ by hypothesis and for $i > n$ since K is $(n-1)$ dimensional, and X satisfies Dn; also, by construction, $H_n(\tilde{Y}, \tilde{K}) = C_n(\tilde{Y}) \to H_n(\tilde{X}, \tilde{K}) \cong \pi_n(\varphi)$ is an isomorphism. It follows from the exact sequence of the triple that all $H_i(\tilde{X}, \tilde{Y})$ vanish, so the inclusion of Y in X is a homotopy equivalence [12, Theorem 1].

THEOREM E. *X satisfies* Dn $(n \geqq 3)$ *if and only if it is equivalent to an n-dimensional complex K. Moreover, if X also satisfies* Fr *and* Cs *with* $0 \leqq r \leqq s \leqq n-2$, *we may suppose that K has finite r-skeleton and countable s-skeleton; also if* $0 \leqq r \leqq n-2$, $s = n$.

PROOF. We use the inductive construction of § 1 to obtain in turn an r-connected $\varphi_1 : K^r \to X$ with K^r finite, an s-connected $\varphi_2 : K^s \to X$ with K^s countable, and having K^r as r-skeleton; and an $(n-1)$-connected $\varphi_3 : K^{n-1} \to X$, where K^{n-1} has K^s as s-skeleton (here we suppose $s \leqq n-2$). By Lemma 2.1, $\pi_n(\varphi_3)$ is a projective module B, and we can find modules A and F, with F free, and $F \cong A \oplus B$. Now the module

$$B \oplus A \oplus B \oplus A \oplus B \oplus A \cdots$$

is isomorphic to $F' = F \oplus F \oplus F \cdots$ (bracketing after the even terms) and to $B \oplus F'$ (bracketing after the odd terms), and F' is free. We attach to K^{n-1} a bunch of $(n-1)$-spheres, corresponding to generators of F', and extend φ_3 to a map φ_4 of the resulting K' by taking the new spheres to the base point.

Since K' dominates K, the exact sequence of the triple splits, and $H_n(\tilde{\varphi}_4) \cong H_n(\tilde{\varphi}_3) \oplus H_n(\tilde{K}', \tilde{K})$, i.e., $\pi_n(\varphi_4) \cong \pi_n(\varphi_3) \oplus F' = B \oplus F' \cong F'$. Hence $\pi_n(\varphi_4)$ is free, and by the remark preceding the theorem, we can attach n-cells to K' to make φ_4 a homotopy equivalence.

If X satisfies Cn, we take K^{n-1} countable. Then B is countable, F may be supposed countable, and then so is F'. Hence so are K'^{n-1} and K^n. If, however, X satisfies C$(n-1)$ but not Cn, we can take K^{n-1} countable, but B is not then countably generated. The result would of course follow if we knew that B was then free; this is known (cf. [1]) if Λ contains no proper ideals J with $J^2 = J$ (H. Bass). A similar situation arises in case X satisfies F$(n-1)$ but not Fn. We should like to know that infinitely generated projective modules are free; this holds if π is a finite solvable extension of a finitely generated abelian group (H. Bass). In case X satisfies Fn, we shall analyse the problem in § 3; here we note merely that B is then finitely generated as well as projective.

Complement. If X satisfies D2, it is equivalent to a 3-dimensional complex K.

PROOF. We construct as usual a 2-connected $\varphi: K^2 \to X$. By the complement to Lemma 2.1, $\pi_3(\varphi)$ is projective. As in the theorem, attach 2-spheres to K to make it free; then if we attach 3-cells to K corresponding to the generators, the same proof as before shows that φ becomes a homotopy equivalence.

COROLLARY. *For $i \leqq j$, Di implies Dj.*

This is trivial if $i = j$ and an i-dimensional complex certainly satisfies Dj for $j \geqq i$.

3. The obstruction to finiteness

We shall need a few elementary properties of finitely generated projective Λ-modules. Let $\mathcal{P}$ be the set of isomorphism classes $[X]$ of such modules X, $F(\mathcal{P})$ the free abelian group with basis $\mathcal{P}$. Let $R(\mathcal{P})$ be the subgroup generated by elements of the form $[X \oplus Y] - [X] - [Y]$, and denote the quotient group $F(\mathcal{P})/R(\mathcal{P})$ by $K^0(\Lambda)$, and the class of X in it by $\{X\}$.

Let $\mathbf{Z}$ be the Λ-module on which π operates trivially; for any X we can form $X \otimes_\Lambda \mathbf{Z}$, and define its rank as $r(X)$. The rank is clearly additive for direct sums, and induces a homomorphism $r : K^0(\Lambda) \to \mathbf{Z}$. This is split by the map $u : \mathbf{Z} \to K^0(\Lambda)$ defined by $u(1) = \{\Lambda\}$; we write $\tilde{K}^0(\Lambda)$ for the cokernel of u. This may be described as the Grothendieck group of finitely generated projective Λ-modules, modulo free modules; it is known [6] as the (reduced) projective class group.

Suppose $\{P\} = \{Q\}$. Then there are positive integers λ_i, μ_j and modules X_i, Y_i, U_j, V_j with

$$[P] - [Q] = \sum_i \lambda_i([X_i \oplus Y_i] - [X_i] - [Y_i]) - \sum_j \mu_j([U_j \oplus V_j] - [U_j] - [V_j]) ,$$

i.e.,

$$[P] + \sum \lambda_i[X_i] + \sum \lambda_i[Y_i] + \sum \mu_j[U_j \oplus V_j] = [Q] + \sum \lambda_i[X_i \oplus Y_i] + \sum \mu_j[U_j] + \sum \mu_j[V_j] .$$

The collections of modules whose classes figure on the two sides of this last equation must be (up to isomorphism) permutations of each other; hence the direct sums of P resp. Q with λ_i copies each of X_i and Y_i, and μ_j copies each of U_j and V_j must be isomorphic; say $P \oplus R \cong Q \oplus R$. Choose a finite set of generators for R; this gives a free F of finite rank and epimorphism $F \to R$ which splits (as R is projective) $F \cong R \oplus S$. Hence $P \oplus F \cong Q \oplus F$. Conversely, if this holds, evidently $\{P\} = \{Q\}$. We deduce also that P and Q have the same class in $\tilde{K}^0(\Lambda)$ if and only if finitely generated free modules F, G exist with $P \oplus F \cong Q \oplus G$.

Now if X satisfies (Fn) and (Dn), we can find an $(n-1)$-connected map $\varphi : K^{n-1} \to X$ with K^{n-1} finite, and then $\pi_n(\varphi)$ is finitely generated and projective. Hence it determines an element of $\tilde{K}^0(\Lambda)$. Our object is to show that this depends only on X. Write P for $\pi_n(\varphi)$, let F be free of finite rank, $F \cong P \oplus Q$, and attach cells to K^{n-1} corresponding to F. This gives a finite complex L^n, and an n-connected map $\psi : L \to X$, with $\pi_{n+1}(\psi) \cong Q$. We observe that the class of Q in $\tilde{K}^0(\Lambda)$ is minus that of P.

LEMMA 3.1. *Let X satisfy* Dn *and $\psi : L \to X$ be n-connected. Then ψ has a homotopy right inverse, so L dominates X.*

PROOF. We replace ψ by an equivalent fiber map, still denoted ψ; note that $\pi_{i-1}(F) \cong \pi_i(\psi)$. The obstructions to finding a cross-section lie in the groups $H^i(X; \pi_i(\psi))$. But for $i \leqq n$, $\pi_i(\psi)$ vanishes, and for $i > n$, the cohomology group vanishes by D$(i-1)$, which holds by Dn and the corollary to Theorem E. Thus there are no obstructions, and a section exists.

LEMMA 3.2. *Let X satisfy* Fn *and* Dn; *let L_1^n and L_2^n be finite, and $\psi_i : L_i \to X$ n-connected, with $Q_i = \pi_n(\psi_i)$ $(i = 1, 2)$. Then Q_1 and Q_2 have the same class in $\tilde{K}^0(\Lambda)$.*

PROOF. The homology structure of ψ_i is very simple: for $j < n$, $\psi_{i*} : H_j(\tilde{L}_i) \cong H_j(\tilde{X})$; for $j > n$ both groups are zero; for $j = n$, $H_n(\tilde{L}_i) = H_n(\tilde{X}) \oplus Q_i$: here ψ_{i*} projects on the first summand, and is split by s_{i*}, where s_i is a right inverse to ψ_i.

The composite $s_2\psi_1$ induces isomorphisms of fundamental groups, and of the homology of the universal cover, up to and including dimension $(n-1)$, so is $(n-1)$-connected. By Lemma 1.1, L_2 is equivalent to a finite complex L_3^{n+1} with L_1 as subcomplex and extra cells only in dimensions n, $n+1$. So the chain complex of $(\widetilde{L}_3, \widetilde{L}_1)$ is

$$0 \longrightarrow C_{n+1} \xrightarrow{d} C_n \longrightarrow 0 ,$$

say, and has homology modules Q_1 and Q_2. If R is the image of d, we have short exact sequences

$$Q_1 \longrightarrow C_{n+1} \longrightarrow R , \quad R \longrightarrow C_n \longrightarrow Q_2 .$$

The second splits since Q_2 is projective, so $C_n \cong Q_2 \oplus R$, R is projective, the first sequence splits, and $C_{n+1} \cong Q_1 \oplus R$. So $Q_1 \oplus C_n \cong Q_1 \oplus Q_2 \oplus R \cong Q_2 \oplus C_{n+1}$; since C_n and C_{n+1} are free, it follows that Q_1 and Q_2 have the same class in $\widetilde{K}^0(\Lambda)$.

THEOREM F. *X is dominated by a finite complex of dimension n if and only if X satisfies* Fn *and* Dn. *When this holds, and $n \geqq 2$, there is an obstruction in $\widetilde{K}^0(\Lambda)$, depending only on the homotopy type of X, which vanishes if X is finite, and whose vanishing is sufficient for X to be equivalent to a finite complex of dimension* max (3, n). *Any finite $(n-1)$-type $(n \geqq 2)$ contains complexes X, satisfying* Fn *and* Dn, *and corresponding to all elements of $\widetilde{K}^0(\Lambda)$.*

PROOF. The sufficiency in the first sentence follows from the construction of § 1 and from Lemma 3.1; the necessity is evident. We define the obstruction as $(-1)^n$ times the projective class of the modules Q of Lemma 3.2; by that lemma, this is independent of L, and by an argument immediately preceding it, also of the integer n. If X is finite, we may take ψ as the identity; Q is then zero. Conversely, let the obstruction vanish; let $\varphi : K^{n-1} \to X$ be $(n-1)$-connected with $A = \pi_n(\varphi)$, then A determines zero in $\widetilde{K}^0(\Lambda)$ (if $n = 2$, read 2 for $n-1$ and 3 for n here and below). Then there is a finitely generated free module F with $A \oplus F$ free. Form the bouquet of K with $(n-1)$-spheres corresponding to generators of F, and extend φ by mapping them to the base point. This makes $\pi_n(\varphi)$ free; we may therefore add a finite number of n-cells to make φ a homotopy equivalence.

It remains to construct the examples. Let $K = K^n$ be any finite complex; we shall give examples with the same $(n-1)$-type. Let A, B be finitely generated projective modules with $A \oplus B = F$ free. Attach an n-sphere to K for each generator of F, forming X^n. Since X^n retracts onto K (we can map these spheres to the base-point), we have $\pi_n(X^n) \cong \pi_n(K) \oplus \pi_n(X^n, K)$, and $\pi_n(X^n, K) \cong H_n(\widetilde{X}^n, \widetilde{K}) \cong F$. Note that $\widetilde{X}^n$ has the homotopy type of a bouquet of $\widetilde{K}$ with

one sphere for each $\mathbf{Z}$-generator of F. Choose these as the union of $\mathbf{Z}$-bases for A and B, and write $\tilde{X}^n \simeq K \vee T^n_A \vee T^n_B = Y \vee T^n_B$, say. Now suppose inductively X^{n+2k} constructed, and that $\tilde{X}^{n+2k} \simeq Y \vee T^{n+2k}_B$ has the homotopy type of the bouquet of Y with $(n+2k)$-spheres corresponding to a decomposition $\pi_{n+2k}(X^{n+2k}) \cong \pi_{n+2k}(Y) \oplus B$. We form X^{n+2k+1} by attaching cells by the images of free Λ-generators of F under the projection $F \to B \subset \pi_{n+2k}(X^{n+2k})$. This has the effect of attaching $(n+2k+1)$-cells to $\tilde{X}^{n+2k}$ by the images of free $\mathbf{Z}$-generators of F. But the generators of B will make the spheres of T^{n+2k}_B all trivial; those of A are attached trivially, hence $\tilde{X}^{n+2k+1} \simeq Y \vee T^{n+2k+1}_A$. The other half of the induction step is similar. Thus we attach cells of all dimensions, with the ultimate effect of cancelling out T^n_B and obtaining $\tilde{X} \simeq Y$. The resulting X has obstruction $(-1)^n$ times the class of A, and so arbitrary (we use the inclusion of K in X to calculate it). The conditions Fn and Dn hold since X^n dominates X (there is again no obstruction to constructing a 'cross-section' $X \to X^n$).

In certain cases, the structure of the projective class group is known, and our obstruction theory can be made more precise. A result of [6] is that for p prime, $\tilde{K}^0(\mathbf{Z}_p)$ is isomorphic to the group of classes of ideals modulo principal ideals in the cyclotomic field of p^{th} roots of unity; it is also known [14] that $\tilde{K}^0(\mathbf{D}_{2p})$, where $\mathbf{D}_{2p}$ denotes the dihedral group, is isomorphic to the ideal class group in the maximal real subfield of the above. A number of calculations may be found, for example in H. Hasse, *Über die Klassenzahlen abelscher Zahlkörper*; in particular, both groups are trivial for $p = 2, 3, 5, 7, 11, 13, 17, 19$, and apparently there are no other cases where either is known to be trivial. The class number of the field of p^{th} roots of unity tends to ∞ with p, so the groups are by no means all trivial. If π is finite, the group $\tilde{K}^0(\pi)$ is finite [9]. If π is a free abelian group, the result $\tilde{K}^0(\pi) = 0$ can be deduced easily from theorems of Hilbert and Grothendieck. For this result, I am indebted to H. Bass. If π is free, $\tilde{K}^0(\pi) = 0$; in fact, every finitely generated Λ-module is free (Bass [16]).

PROPOSITION 3.3. *If X satisfies* D2 *and* F2, *and π is free, then X has the homotopy type of a finite bouquet of* 1*-spheres and* 2*-spheres.*

PROOF. We can find a finite bouquet K of 1-spheres, and a map $\varphi : K \to X$, inducing an isomorphism of fundamental groups. By Lemma 2.1 and the above result of Bass, $\pi_2(\varphi)$ is a free Λ-module, so we can attach a finite set of 2-cells to K, necessarily with trivial attaching maps, to make φ a homotopy equivalence.

4. Simplifying finite complexes

Our objective of reducing any CW-complex to a canonically simple one of the same homotopy type is not achieved by reaching a finite complex; it is still very useful to reduce the number of cells of such a complex. For example, we have the following theorem, due to Milnor (unpublished), which can also be deduced in a very roundabout way from results of Smale [7].

PROPOSITION 4.1. *Suppose X a simply-connected complex such that $H_i(X)$ has rank β_i and $\tau_{i+(1/2)}$ torsion coefficients, for each $i \geqq 0$. Then X is equivalent to a complex with $\alpha_i = \tau_{i-(1/2)} + \beta_i + \tau_{i+(1/2)}$ i-cells for each i.*

The proof follows our usual pattern, and makes use of the classical *normal form* for chain complexes. Our best result in this direction is

THEOREM G. *Let A be a chain complex of free Λ-modules of finite rank, and Λ-homomorphisms; $f: A \rightarrow C(\widetilde{X})$ a Λ-morphism inducing isomorphisms of chains in dimensions 1 and 2 and of homology throughout. Then f can be realised geometrically as induced by a cellular $g: Y \rightarrow X$ with $A = C(\widetilde{Y})$, and g a homotopy equivalence.*

PROOF. Suppose Y^r and $g_r: Y^r \rightarrow X$ already constructed, g_r r-connected. We can attach $(r+1)$-cells to Y^r to kill $\pi_{r+1}(g_r)$, and to ensure $C_{r+1}(\widetilde{Y}^{r+1}) = A_{r+1}$, we need an epimorphism $A_{r+1} \rightarrow \pi_{r+1}(g_r) = H_{r+1}(\tilde{g}_r)$. We shall check that this is given naturally, and hence obtain the result by induction; we can take $Y^2 = X^2$ to give a basis for the induction, so need not worry about fundamental groups.

To compute $H_{r+1}(\tilde{g}_r)$ we use the algebraic mapping cone [11, § 8] of f. Let $M_i = C_i(\widetilde{X}) \oplus A_{i-1}$; define the boundary in M in the usual way. Then there is an exact sequence of chain complexes $C(\widetilde{X}) \rightarrow M \rightarrow A$ whose exact homology sequence is that of f. To do the same for g_r, we must truncate A at dimension r, giving M', say. Then

$$H_{r+1}(\tilde{g}_r) = H_{r+1}(M') = \frac{\operatorname{Ker}\{C_{r+1}(\widetilde{X}) \oplus A_r \rightarrow C_r(\widetilde{X}) \oplus A_{r-1}\}}{\operatorname{Im}\{C_{r+2}(\widetilde{X}) \rightarrow C_{r+1}(\widetilde{X}) \oplus A_r\}} .$$

Since, by hypothesis, f induces homology isomorphisms, M is acyclic. $H_{r+1}(M')$ differs from $H_{r+1}(M) = 0$ only in that the denominator lacks the boundaries of elements of $A_{r+1} \subset M_{r+2}$. Hence the boundary in M induces an epimorphism

$$A_{r+1} \rightarrow H_{r+1}(\tilde{g}_r) .$$

We use a basis of A_{r+1} to attach $(r+1)$-cells to Y^r, and the above homomorphism to extend g_r to an $(r+1)$-connected map of the resulting Y^{r+1} to X. This process gives a chain map $C_{r+1}(\widetilde{Y}^{r+1}) \rightarrow C_{r+1}(\widetilde{X})$ which, since the above epimorphism is induced by the boundary in M, and so by f, agrees with f

modulo boundaries in $C(\widetilde{X})$. Thus the extended map can be altered by a homotopy to induce $f: A_{r+1} \to C_{r+1}(\widetilde{X})$. The induction is now complete; as usual, whether A (and so Y) is finite or infinite, we use Whitehead's theorem to show that we end with a homotopy equivalence.

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