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RELATIVE HYPERBOLIZATION AND ASPHERICAL BORDISMS

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1. Introduction

In [2, p. 116], Gromov introduced the notion of hyperbolization: it is a procedure for associating to a finite dimensional simplicial complex X a certain nonpositively curved polyhedron $H(X)$. A few pages later [2, pp. 117–118], he discusses the idea of relative hyperbolization: given a subcomplex Y of X, it should produce a new space $H(X, Y)$ which contains Y as a subspace. One of the key properties of such a procedure should be the following:

($*$) If (each component of) Y is aspherical, then so is the relative hyperbolization $H(X, Y)$

Gromov points out that it follows from the existence of such a relative hyperbolization procedure that:

- Any (triangulable) closed manifold M is bordant to an aspherical manifold.
- If a closed aspherical manifold M bounds a (triangulable) manifold, then it bounds an aspherical manifold.

The proof of the second claim uses proerty ($*$), but the proof of the first does not. Unfortunately, Gromov's definition of a relative version of hyperbolization does not quite work. In [1, Section 1g], the first two authors gave a different version of relative hyperbolization (here denoted by $K(X, Y)$) and used it to demonstrate Gromov's first claim, cf. [1, Example 1g.1]. However, they did not know how to prove that the version of relative hyperbolization in [1] satisfies the crucial property ($*$). In fact, it does (as does the simpler version of relative hyperbolization, $J(X, Y)$, defined in Section 2). Our purpose here is to prove that both

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these relative hyperbolization procedures satisfy $(*)$ (Theorems 2.5 and 3.2) and to prove Gromov's second claim (Theorem 2.7).

Actually, Gromov defined several hyperbolization procedures. The specific one in which we are interested is discussed in [1, Section 4c]. Given a finite dimensional simplicial complex X, there is a new polyhedron $H(X)$, called a *hyperbolization* of X, together with a map $c : H(X) \to X$. Some important properties of the construction are listed below. (Proofs of these properties can be found in [1].)

1. $H(X)$ is a nonpositively curved cubical cell complex (and hence, is aspherical).
2. The construction is functorial in the sense that if $i : Y \to X$ is a simplicial embedding, then there is an induced isometric embedding $H(i) : H(Y) \to H(X)$.
3. The link of a vertex in $H(X)$ is isomorphic to a subdivision of the link of the corresponding vertex in X.
4. The map $c : H(X) \to X$ induces surjections on integral homology groups and on fundamental groups.
5. If X is an n-manifold, then so is $H(X)$. If X is a smooth triangulation of a smooth manifold, then $H(X)$ is a smooth manifold. Moreover, $c : H(X) \to X$ pulls back the stable tangent bundle of X to that of $H(X)$.

2. Relative hyperbolization

Suppose Y is a subcomplex of X and that $\{Y_i\}$ is the set of path components of Y. Let $X \cup CY$ denote the simplicial complex formed by attaching to X the cone on each Y_i. Let y_i denote the cone point corresponding to Y_i in the hyperbolization $H(X \cup CY)$ of $X \cup CY$ and let L_i denote the link of y_i in $H(X \cup CY)$. Then L_i is identified with a subdivision of Y_i. The *relative hyperbolization of X with respect to Y* is defined to be the space $J(X, Y)$ formed by removing a small open conical neighborhood of each y_i from $H(X \cup CY)$. Since the boundary of such a neighborhood is L_i $(= Y_i)$, each Y_i is identified with a subspace of $J(X, Y)$.

Remark 2.1. If X is a manifold with boundary and Y is a union of boundary components, then $J(X, Y)$ is also a manifold with boundary and Y is identified with a union of its boundary components. This gives the proof of Gromov's first claim: for any closed manifold M, $J(M \times [0, 1], M \times 1)$ is a bordism between M and $H(M)$.

Let $\overline{H}(X \cup CY)$ denote the universal cover of $H(X \cup CY)$ and let $\overline{J}(X, Y)$ denote the inverse image of $J(X, Y)$ in $\overline{H}(X \cup CY)$.

Lemma 2.2. *Let $\overline{L}_i$ be the link of any cone point $\overline{y}_i$ in $\overline{H}(X \cup CY)$. Then $\overline{J}(X,Y)$ retracts onto $\overline{L}_i$. Hence, $\pi_1(\overline{L}_i) \to \pi_1(\overline{J}(X,Y))$ is an injection.*

Proof. Since $\overline{H}(X \cup CY)$ is $CAT(0)$, geodesic contraction provides a deformation retraction of $\overline{H}(X \cup CY) \setminus \overline{y}_i$ onto $\overline{L}_i$. The restriction of this to $\overline{J}(X,Y)$ gives the desired retraction. □

Corollary 2.3. *For each Y_i, $\pi_1(Y_i) \to \pi_1(J(X,Y))$ is injective.*

Remark 2.4. This corollary provides a proof of the following theorem of Hausmann [3]. Suppose that a (not necessarily connected) closed manifold M is a boundary. Then M bounds a manifold N such that for each path component M_i of M, the homomorphism $\pi_1(M_i) \to \pi_1(N)$ is injective. Moreover, $M_i \to N$ is a "pseudo covering projection" in the sense that each M_i is a retraction of some covering space of N.

Theorem 2.5. *$J(X,Y)$ is aspherical if and only if each component of Y is aspherical.*

In order to prove this, we need to introduce a space $\widetilde{H}(X \cup CY)$, the "universal branched cover of $\overline{H}(X \cup CY)$ along the cone points". Let S denote the union of the set of cone points in $\overline{H}(X \cup CY)$. Then $\overline{H}(X \cup CY) \setminus S$ is connected. Let Z be its universal cover. Then $\widetilde{H}(X \cup CY)$ is defined to be the metric completion of Z. It is clear that $\widetilde{H}(X \cup CY)$ is formed by adjoining to Z a new cone point for each end of Z which corresponds to a copy of the inverse image of a $\overline{L}_i$ in Z. Thus, $\widetilde{H}(X \cup CY)$ is homeomorphic to the universal cover of $\overline{J}(X,Y)$ with each copy of the universal cover of $\overline{L}_i$ coned off. In other words, the universal cover $\widetilde{J}(X,Y)$ of $J(X,Y)$ can be idenitified with inverse image of $\overline{J}(X,Y)$ in $\widetilde{H}(X \cup CY)$.

Lemma 2.6. *$\widetilde{H}(X \cup CY)$ is $CAT(0)$.*

Proof. Since $\overline{H}(X \cup CY)$ is a piecewise Euclidean cubical cell complex, this same type of structure is induced on $\widetilde{H}(X \cup CY)$. Moreover, $\widetilde{H}(X \cup CY)$ is connected and simply connected. So, it suffices to show that $\widetilde{H}(X \cup CY)$ is locally $CAT(0)$. This is clear except possibly in neighborhoods of the cone points. Here we need to show that the link of each cone point in $\widetilde{H}(X \cup CY)$ is $CAT(1)$ (cf. [2, p. 120]). The link of such a cone point is the universal cover of the link of its image in $\overline{H}(X \cup CY)$. Since $\overline{H}(X \cup CY)$ is $CAT(0)$, the link of each of its cone points is $CAT(1)$. Since any covering space of a $CAT(1)$ piecewise spherical complex is also $CAT(1)$, the cone points in $\widetilde{H}(X \cup CY)$ also have $CAT(1)$ links. The lemma follows. □

Proof of Theorem 2.5. The "only if" part of this theorem follows immediately from Lemma 2.2. So, suppose each component of Y_i is aspherical. The link $\widetilde{L}_i$ of a cone point in $\widetilde{H}(X \cup CY)$ is the universal cover of Y_i; hence, it is contractible. By Lemma 2.6, $\widetilde{H}(X \cup CY)$ is contractible. Since $\widetilde{H}(X \cup CY)$ is formed from $\widetilde{J}(X, Y)$ by attaching cones on the $\widetilde{L}_i$, it follows that $\widetilde{J}(X, Y)$ is also contractible. Hence, $J(X, Y)$ is aspherical (since $\widetilde{J}(X, Y)$ is a covering space of it). □

We are now in position to prove Gromov's second claim.

Theorem 2.7. *Suppose that each component of a closed manifold M is aspherical and that M is the boundary of a (triangulable) manifold. Then M bounds an aspherical manifold.*

Proof. Suppose $M = \partial N$. As in Remark 2.1, M is also the boundary of the manifold $J(N, M)$. By Theorem 2.5, $J(N, M)$ is aspherical. □

Remark 2.8. Theorem 2.7 is valid for any bordism theory.

3. Another version

When (X, Y) is a manifold with boundary, the construction of the relative hyperbolization $J(X, Y)$ is perfectly adequate. However, in more general situations it has a serious defect: it changes the local topology near Y. A regular neighborhood of Y in $J(X, Y)$ is homeomorhic to $Y \times [0, 1]$. It would be preferable for this to be homeomorphic to the original regular neighborhood of Y in X. This can be acheived by the procedure of [1]. The details are explained below.

Replace X by its barycentric subdivision. Let R_i denote the first derived neighborhood of Y_i in X, let R_i° be its relative interior and let $\partial R_i = R_i \setminus R_i^\circ$. Also, let R, R° and ∂R denote the union of the R_i, the R_i° and the ∂R_i, respectively. Set $\widehat{X} = X \setminus R^\circ$. Apply the construction of the previous section to the pair $(\widehat{X}, \partial R)$ to obtain $J(\widehat{X}, \partial R)$. Our second version of relative hyperbolization, is the space $K(X, Y)$ formed by gluing each R_i back onto $J(\widehat{X}, \partial R)$ along ∂R_i. Next, we want to establish that Lemma 2.2 and Theorem 2.5 hold for $K(X, Y)$.

For the analog of Lemma 2.2 we need to define a covering space $\overline{K}(X, Y)$ of $K(X, Y)$ which retracts onto each R_i. If ∂R_i is connected, then $\overline{K}(X, Y)$ is defined to be $\overline{H}(\widehat{X} \cup C(\partial R))$ with a neighborhood of each cone point removed and replaced by a copy of the appropriate R_i. If the ∂R_i are not connected, then the definition of $\overline{H}(\widehat{X} \cup C(\partial R))$ needs to be modified. For each path component Y_i, define a graph Ω_i: it is the suspension of $\pi_0(\partial R_i)$. Denote the suspension points by v_i and x_i. Let Ω be the wedge of the Ω_i (i.e., identify the x_i to a common point

x). There is a continuous map $K(X,Y) \to \Omega$ which collapses $J(\widehat{X}, \partial R)$ to x, collapses Y_i to v_i and which takes each component of ∂R_i to the midpoint of the corresponding edge of Ω_i. A map $H(\widehat{X} \cup C(\partial R)) \to \Omega$ is defined in a similar fashion. Define a graph of groups on Ω by putting the group $\pi_1(H(\widehat{X} \cup C(\partial R))$ on the vertex x, the trivial group on each of the other vertices and the trivial group on each edge. Let T be the universal cover of this graph of groups. (T is a tree.) The space $\overline{H}(\widehat{X} \cup C(\partial R))$ is defined by gluing together copies of the the universal cover of $H(\widehat{X} \cup C(\partial R))$ in a pattern given by T. There is one such copy for each vertex lying above x. Two copies are glued together at a common cone point whenever the corresponding vertices of T are each connected by an edge to a vertex lying over some v_i. So, the link of a cone point in $\overline{H}(\widehat{X} \cup C(\partial R))$ is isomorphic to some ∂R_i (which need not be connected). This version of $\overline{H}(\widehat{X} \cup C(\partial R))$ is clearly simply connected and $CAT(0)$. Using the tree T, a covering space $\overline{K}(X,Y)$ of $K(X,Y)$ is defined in a similar fashion. Alternatively, $\overline{K}(X,Y)$ is formed from $\overline{H}(\widehat{X} \cup C(\partial R))$ by removing a neighborhood of each cone point and replacing it with a copy of the appropriate R_i.

Lemma 3.1. *$\overline{K}(X,Y)$ retracts onto R_i.*

Proof. Fix a cone point $\overline{y}_i$ in $\overline{H}(\widehat{X} \cup C(\partial R))$ and identify ∂R_i with the link of $\overline{y}_i$. Let $\overline{J}(\widehat{X}, \partial R)$ denote the inverse image of $J(\widehat{X}, \partial R)$ in $\overline{H}(\widehat{X} \cup C(\partial R))$. As in the proof of Lemma 2.2, geodesic contraction from $\overline{H}(\widehat{X} \cup C(\partial R))$ onto $\overline{y}_i$, induces a retraction of $\overline{J}(\widehat{X}, \partial R)$ onto ∂R_i. Under this retraction each of the other boundary components is taken to ∂R_i by a map which is null-homotpic. Hence, we can extend it to a retraction $\overline{K}(X,Y) \to \partial R_i$ by mapping the copy of R_i corresponding to $\overline{y}_i$ via the identity map and all other R_j inessentially. ☐

For the analog of Theorem 2.5, we want to relate the universal covering space $\widetilde{K}(X,Y)$ of $K(X,Y)$ to a branched covering space $\widetilde{H}(\widehat{X} \cup C(\partial R))$ of $\overline{H}(\widehat{X} \cup C(\partial R))$. To this end, we define a new graph of group stucture on Ω. The vertex group corresponding to x is $\pi_1(J(\widehat{X}, \partial R)$, the vertex group corresponding to v_i is $\pi_1(R_i)$ and the edge group corresponding to an edge e of Ω_i is the image of $\pi_1(\partial R_{i,e})$ in $\pi_1(R_i)$, where $\partial R_{i,e}$ denotes the component of ∂R_i corresponding to e. The inclusions of edge groups in vertex groups are the obvious ones. (By the previous lemma, the map from an edge group to the vertex group for x is an inclusion.) Let $\widetilde{T}$ be the tree corresponding to this graph of groups. Let $\widetilde{H}(\widehat{X} \cup C(\partial R))$ be the branched covering space of $\overline{H}(\widehat{X} \cup C(\partial R))$

corresponding to $\widetilde{T}$. The covering space $\widetilde{K}(X,Y)$ of $K(X,Y)$ corresponding to $\widetilde{T}$ is obviously simply connected. Moreover, $\widetilde{K}(X,Y)$ can be constructed from $\widetilde{H}(\widehat{X} \cup C(\partial R))$ by removing a neighborhood of each cone point and replacing it with a copy of the universal cover $\widetilde{R}_i$ of the appropriate R_i.

Theorem 3.2. *$K(X,Y)$ is aspherical if and only if each component of Y is aspherical.*

Proof. As before, the "only if" part follows from Lemma 3.1. As in the proof of Theorem 2.5, $\widetilde{H}(\widehat{X} \cup C(\partial R))$ is locally $CAT(0)$. Hence, it is aspherical. Supposing each Y_i to be aspherical, we have that each $\widetilde{R}_i$ is contractible. Since $\widetilde{K}(X,Y)$ is formed from $\widetilde{H}(\widehat{X} \cup C(\partial R))$ by replacing (contractible) neighborhoods of cone points by (contractible) copies of $\widetilde{R}_i$, $\widetilde{K}(X,Y)$ is homotopy equivalent to $\widetilde{H}(\widehat{X} \cup C(\partial R))$. So, $\widetilde{K}(X,Y)$ is aspherical and hence, so is $K(X,Y)$. □

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