Equivariant Periodicity for Compact Group Actions

Shmuel Weinberger*
University of Chicago

Min Yan†
Hong Kong University of Science and Technology

October 2, 2003

1 Introduction

Probably the most basic structural phenomenon of high dimensional topology is Siebenmann's periodicity theorem [3] (as amended by Nicas [5]), which asserts that the manifolds homotopy equivalent to M are in a one-to-one correspondence with (a subset of, because of nonresolvable honology manifolds [1]) those homotopy equivalent to $M \times D^4$. The main goal of this paper is to show the following extension of this to the equivariant setting.

Theorem Let G be a compact Lie group. Let M be a homotopically stratified G-manifold with condimension ≥ 3 gap. Let ξ be a complex G-vector bundle over M that has the same isotropy as M everywhere. Then

$$S_G(M, \text{rel } \partial) \cong S_G(D(\xi \oplus \xi), \text{rel } \partial).$$

The same isotropy everywhere condition means that for any open subset U of M, the collection of isotropy groups for U is the same as the collection of isotropy groups for $\xi|_U$. The assumption means that the projection induces a one-to-one correspondences between the isovariant components of ξ and the isovariant components of M.

In fact, one can use topological bundles that have complex structure just over a 1-skeleton on M. We plan to apply this, rather along the lines of Atiyah-Hirzebruch's differentiable Riemann-Roch theorem, in further joint work with Cappell to defining induced maps between structure sets for a wide class of equivariant maps. This should allow application of the tools of algebraic topology, such as homology theory and assembly maps, which do not generalize gracefully to the current situation where functoriality is essentially only known with respect to open inclusions.

*Research was partially supported by an NSF grant
†Research was supported by Hong Kong Research Grant Council Earmark Grant HKUST6071/98P and Direct Allocation Grant DAG01/02.SC20

This paper is a continuation of our earlier one on abelian group actions [7] and the reader can refer to the introduction to that paper for some more discussion of the context, history, and applications of such results, and also for any unexplained notation. In particular the main result of this paper proves a conjecture of that paper. For product bundles associated to representations (and some others), this result extends [8], which dealt with permutation representations of odd order groups, and [7] which dealt with abelian groups, and [9] which produced periodicity representations for some small positive dimensional nonabelian groups.

The proof of the main theorem uses the main technical device of that paper: the products with nonmanifold periodicity spaces. The main difference between this paper and that is the source of the periodicity space. In [7] we used products of modified projective spaces of representations of G. As [9] shows, this can occasionally be done in the nonabelian case, but it rapidly becomes unwieldy. Here we instead, following [2], modify configuration spaces of representation spheres to obtain the periodicity spaces. The superior flexibility of the configuration space construction over projectivization enables one to be able to handle nontrivial bundles, i.e. prove analogs of Thom isomorphism rather than just Bott periodicity. This very quickly leads to our main theorem.

This result was proven during a visit of the second author to University of Chicago; he would like to thank them for their hospitality. The first author would like to thank Kevin Whyte for a valuable conversation. Both authors would also like to thank Sylvain Cappell for useful conversations regarding this work and its continuations.

2 Periodicity Space

Let $V = W \oplus \mathbb{R}^3$ be the direct sum of a unitary complex G-representation W and the trivial G-representation $\mathbb{R}^3$. We also fix two distinct vectors $p, q \in \mathbb{R}^3$ of unit length.

Denote by $S(V)$ and $D(V)$ the induced G-sphere and G-disk. Let

$$C(V) = \frac{S(V) \times S(V)}{(u, v) \sim (v, u)}$$

be the symmetric double of $S(V)$. Contained in $C(V)$ is the diagonal

$$S_\Delta(V) = \{[v, v] \in C(V) \colon v \in S(V)\},$$

which is G-homeomorphic to $S(V)$. The G-space $C(V)$ is stratified with $S_\Delta(V)$ and $C(V) - S_\Delta(V)$ as the lower and upper strata. We use p and q to introduce a special point in the upper stratum

$$b = [(0, p), (0, q)] \in C(\mathbb{R}^3) - S_\Delta(\mathbb{R}^3) \subset C(V^G) - S_\Delta(V^G) \subset C(V) - S_\Delta(V).$$

The tangent space at the point is

$$T_b C(V) = T_{(0,p)} S(W \oplus \mathbb{R}^3) \times T_{(0,q)} S(W \oplus \mathbb{R}^3) = W \oplus \mathbb{R}^2 \oplus W \oplus \mathbb{R}^2. \qquad (2.1)$$

Our intention is to make $C(V)$ into a periodicity space, with the periodicity representation given by the tangent space above.

For a subgroup H of G, we fix a finite generating set. Then $[v,w] \in C(V)^H$ means that for each generator h, either

$$hv = v, \quad hw = w,$$

or

$$hv = w, \quad hw = v.$$

Let $h_1, \dots, h_r$ be the generators that fix v, and let $h'_1, \dots, h'_s$ be the remaining generators that do not fix v but exchange v and w instead. If $s = 0$, then H fixes v and w, so that $[v,w] \in C(V^H)$. If $s > 0$, then v is fixed by the subgroup K generated by $h_i, h'_j h'_k, h'_j h_i h'_j$. Moreover, K is a normal subgroup of H of index 2. Thus if let $h \in H$ be any element generating H/K and denote

$$S_{H,2}(V^K) = \{[v, hv] \colon v \in S(V^K)\}, \tag{2.2}$$

then we have

$$C(V)^H = C(V^H) \cup \left(\bigcup_{K \subset H, |H/K|=2} S_{H,2}(V^K) \right). \tag{2.3}$$

By the way K is constructed, the union is in fact a finite one. Moreover, the WH-action exchanges $S_{H,2}(V^K)$ for various index 2 subgroups K of H, and for $K \neq K'$ we have

$$S_{H,2}(V^K) \cap S_{H,2}(V^{K'}) = S_\Delta(V^H). \tag{2.4}$$

The space $C(V)$ is not yet a periodicity space due to the following problems:

1. The singular part $S_\Delta(V)$ of $C(V)$ needs to be "killed";

2. The extra fixed points $S_{H,2}(V^K)$ needs to be "killed";

3. The top stratum is not equivariantly simply connected: $\pi_1(C(V^H) - S(V^H)) = \mathbb{Z}_2$.

To solve the first and the second problems, we need to find an equivariant "coboundary" for

$$\hat{S}(V) = S_\Delta(V) \cup \left(\bigcup_{K \subset H \subset G, |H/K|=2} S_{H,2}(V^K) \right).$$

It is easy to see that

$$\hat{S}(V) = \{[v, gv] \colon v \in S(V), g \in G, g^2 v = v\}.$$

Moreover, we have the intersections among the pieces that make up $\hat{S}(V)$:

$$\begin{aligned} S_\Delta(V) \cap S_{H,2}(V^K) &= S_\Delta(V^H), \\ S_\Delta(V^H) \cap S_\Delta(V^{H'}) &= S_\Delta(V^{H''}), \\ S_{H,2}(V^K) \cap S_{H',2}(V^{K'}) &= \begin{cases} S_\Delta(V^{H''}) & \text{if } H'' = K'' \\ S_{H'',2}(V^{K''}) & \text{if } H'' \neq K'' \end{cases}, \end{aligned}$$

where H'' is the subgroup generated by H, H', and K'' is the subgroup generated by K, K', hh' (h and h' generate H/K and H'/K', respectively). The intersections imply that $\hat{S}(V)$ has a natural G-stratification with two types of closed strata, both labeled by conjugacy classes of isotropy subgroups. The first type is (0 stands for the lower strata, $[H]$ stands for the conjugacy class of H)

$$\hat{S}(V)_{0,[H]} = GS_\Delta(V^H),$$

which gives the usual stratification of the G-sphere $S_\Delta(V)$ (any G-manifold has such natural G-stratification). The second type is (1 stands for the upper strata)

$$\hat{S}(V)_{1,[H]} = G\left(S_\Delta(V^H) \cup \left(\bigcup_{K\subset H, |H/K|=2} S_{H,2}(V^K)\right)\right).$$

The partial order among the strata is given by the inclusion of subgroups up to conjugation and the inclusion $\hat{S}(V)_{0,[H]} \subset \hat{S}(V)_{1,[H]}$.

Note that $v \leftrightarrow [v, v]$ is a homeomorphism between $S_\Delta(V)$ and $S(V)$, and $v \leftrightarrow [v, hv]$ is a homeomorphism between $S_{H,2}(V^K)$ and $S(V^K)/H$ (which is really a quotient by $H/K \cong \mathbb{Z}_2$). Since $S_\Delta(V)$ and $S_{H,2}(V^K)$ are spheres or quotients of spheres, we may construct the "coboundary" by extending the spheres to the disks. Specifically, inside the G-space

$$\frac{D(V) \times D(V)}{(u, v) \sim (v, u)},$$

we introduce the subsets

$$D_\Delta(V) = \{[v, v] : v \in D(V)\},$$
$$D_{H,2}(V^K) = \{[v, hv] : v \in D(V^K)\},$$

where $h \in H$ is any element generating H/K. The "coboundary" is then the subset

$$\hat{D}(V) = D_\Delta(V) \cup \left(\bigcup_{K\subset H\subset G, |H/K|=2} D_{H,2}(V^K)\right).$$

Moreover, similar homeomorphisms give us

$$D_\Delta(V) \cong D(V) = coneS(V) \cong coneS_\Delta(V),$$
$$D_{H,2}(V^K) \cong D(V^K)/H = coneS(V^K)/H \cong coneS_{H,2}(V^K),$$

which combine to form a homeomorphism of G-stratified spaces

$$\hat{D}(V) \cong cone\hat{S}(V). \tag{2.5}$$

Note that all the strata of $\hat{S}(V)$ contain the minimum stratum $S(V^G)$, which is a sphere of dimension ≥ 2 acted trivially by the group. Therefore we may take the cone point in (2.5) not as a seperate stratum, but instead as an interior point of the closed stratum $D(V^G)$.

The cone relation (2.5) implies that the strata of $\hat{D}(V)$ and $\hat{S}(V)$ satisfy

$$\hat{D}(V)^\alpha = \hat{S}(V)^\alpha \times (0,1], \quad \alpha = (0,[H]) \text{ or } (1,[H]),$$

with the pair $(D(V^G), S(V^G))$ of smallest strata as the only exception. Consequently (together with $\dim V^G \geq 3$), the pair $(\hat{D}(V), \hat{S}(V))$ has the isovariant π-π property.

Now we solve the problem that $\pi_1(C(V^H) - S(V^H))$ is nontrivial. Take an embedding

$$\rho\colon S^1 \to C(\mathbb{R}^3) - S_\Delta(\mathbb{R}^3) - \{b\} \tag{2.6}$$

representing the generator of the fundamental group $\pi_1(C(\mathbb{R}^3) - S_\Delta(\mathbb{R}^3)) = \mathbb{Z}_2$. Since the space $C(\mathbb{R}^3)$ (which is homeomorphic to $\mathbb{CP}^2$) is orientable, the normal bundle of ρ in $C(\mathbb{R}^3)$ is trivial. We fix one trivialization of the normal bundle.

The assumption $V = W \oplus \mathbb{R}^3$ allows us to naturally consider $S_\Delta(\mathbb{R}^3)$ and $C(\mathbf{R}^3)$ as subsets of $S_\Delta(V)$ and $C(V)$. The assumption also gives a natural trivialization of the normal bundle of $S(\mathbb{R}^3)$ in $S(V)$, with W as the fibre. Since ρ is contained in the top stratum $C(V) - \hat{S}(V)$, the normal bundle of ρ in $C(V)$ can be naturally identified

$$\nu = \frac{[0,1] \times \mathbb{R}^5 \times (W \oplus W)}{(0,u,w_1,w_2) \sim (1,u,w_2,w_1)}. \tag{2.7}$$

Using the assumption that W is a complex representation, a path

$$\sigma_t = \begin{pmatrix} \cos\dfrac{\pi t}{2} & -e^{i\pi t}\sin\dfrac{\pi t}{2} \\ \sin\dfrac{\pi t}{2} & e^{i\pi t}\cos\dfrac{\pi t}{2} \end{pmatrix} : W \oplus W \to W \oplus W$$

may be constructed to connect the transformation $(w_1, w_2) \to (w_2, w_1)$ to the identity transformation in the space of equivariant automorphisms of $W \oplus W$. The path σ_t can be interpreted as an explicit trivialization of the normal bundle (2.7), which can be used to identify a tubular neighborhood of ρ in $C(V)$ with $S^1 \times D^5 \times D(W \oplus W) \cong S^1 \times D(\mathbb{R}^5 \oplus W \oplus W)$. An equivariant surgery on $C(V)$ that replaces $S^1 \times D(\mathbb{R}^5 \oplus W \oplus W)$ by $D^2 \times S(\mathbb{R}^5 \oplus W \oplus W)$ kills the fundamental groups $\pi_1(C(V^H) - S_\Delta(V^H))$ for all H. We denote the result of the surgery by $C_1(V)$.

Since the surgery is performed inside $C(V) - \hat{S}(V)$, it can also be applied to (the top stratum of) $C(V) \cup_{\hat{S}(V)} \hat{D}(V)$ and gives rise to

$$P(V) = C_1(V) \cup_{\hat{S}(V)} \hat{D}(V).$$

To use $P(V)$ as a periodicity space, we need to verify that $C_1(V)^H - \hat{S}(V)^H = C_1(V^H) - \hat{S}(V)^H$ (see (2.3)) is connected and simply connected for any subgroup H. Since the surgery already makes $C_1(V^H) - S_\Delta(V^H)$ connected and simply connected, it suffices to show that

$$\dim(\hat{S}(V)^H - S_\Delta(V^H)) + 3 \leq \dim(C(V)^H - S_\Delta(V^H)).$$

For the special case that H is trivial, this means

$$\dim\{(v, gv)\colon v \in V, g \in G, gv \neq v, g^2 v = v\} + 4 \leq 2 \dim V.$$

For a general subgroup H, we need to substitute V and G by V^H and WH in the inequality. Since V already contains $\mathbb{R}^3$, the inequality is almost true (with $+4$ replaced by $+3$). The classical 4-fold periodicity allows us to add $\mathbb{R}^4$ to V, so that $V \oplus \mathbb{R}^4$ satisfies the inequality. Therefore as far as proving periodicity theorem is concerned, $P(V)$ can be taken as a periodicity space.

3 Periodicity Bundle

In this section, a fibrewise construction of the periodicity space will be carried out. Attention is paid to the fact that the isotropy groups acting on fibres depend on the points of the base manifold.

Let M be a compact G-manifold. Let η be a complex G-vector bundle over M. Let $\epsilon^3 = M \times \mathbb{R}^3$ be the three dimensional trivial bundle. Denote $\xi = \eta \oplus \epsilon^3$ and the projection $\pi\colon \xi \to M$. For any subgroup H, ξ^H is a WH-bundle over M^H.

We have the sphere bundle $S(\xi)$, the disk bundle $D(\xi)$, and the fibrewise symmetric product bundle $C(\xi)$ (which contains the diagonal bundle $S_\Delta(\xi)$ as lower stratum). All these are G-bundles.

Similar to the case $\xi = M \times V$, for any $K \subset H$ satisfying $|H/K| = 2$, denote

$$S_{H,2}(\xi^K|_{M^H}) = \{[v, hv]\colon v \in S(\xi^K|_{M^H})\}, \quad h \text{ generates } H/K. \tag{3.1}$$

Note that the bundle ξ^K is restricted to M^H because v and hv should lie in the same fibre of $S(\xi)$. Similar to (2.3), the fixed point

$$C(\xi)^H = C(\xi^H) \cup \left(\bigcup_{K \subset H, |H/K|=2} S_{H,2}(\xi^K|_{M^H}) \right). \tag{3.2}$$

The WH-action exchanges $S_{H,2}(\xi^K|_{M^H})$ for various K, and by (2.4),

$$S_{H,2}(\xi^K|_{M^H}) \cap S_{H,2}(\xi^{K'}|_{M^H}) = S_\Delta(\xi^H), \quad \text{for } K \neq K'.$$

To make (3.2) into a periodicity bundle, a "coboundary" for

$$\begin{aligned} \hat{S}(\xi) &= S_\Delta(\xi) \cup \left(\bigcup_{K \subset H \subset G, |H/K|=2} S_{H,2}(\xi^K|_{M^H}) \right) \\ &= \{[v, gv]\colon v \in S(\xi), g \in G_{\pi(v)}, g^2 v = v\} \end{aligned} \tag{3.3}$$

needs to be constructed. The space $\hat{S}(\xi)$ is G-stratified with strata

$$\begin{aligned} \hat{S}(\xi)_{0,[H]} &= G S_\Delta(\xi^H), \\ \hat{S}(\xi)_{1,[H]} &= G \left(S_\Delta(\xi^H) \cup \left(\bigcup_{K \subset H, |H/K|=2} S_{H,2}(\xi^K|_{M^H}) \right) \right), \end{aligned}$$

and the projection $\hat{\pi}\colon \hat{S}(\xi) \to M$ is a stratified system of G-bundles, with $\hat{\pi}^{-1}(x) = \hat{S}(\xi_x)$, where $\xi_x = \eta_x \oplus \mathbb{R}^3$ is a G_x-representation. Similar to (2.5), a fibrewise cone construction

$$\hat{D}(\xi) = cone_M \hat{S}(\xi) = \hat{S}(\xi) \times [0, 1] \cup_{\hat{S}(\xi) \times 0 \nearrow} M$$

may be carried out, with the cone point not introducing an extra stratum. By the same reason as before, the pair $(\hat{D}(\xi), \hat{S}(\xi))$ (and the restriction on any G-invariant subset of M) has the isovariant π-π property.

Next, the fibrewise fundamental group $\pi_1(C(\xi_x^H) - S_\Delta(\xi_x^H))$, $H \subset G_x$, needs to be killed. To do this, the special loop (2.6) may be extended to an embedding

$$\rho_M \colon M \times S^1 \to C(\epsilon^3) - S_\Delta(\epsilon^3) - M \times b \to C(\xi) - S_\Delta(\xi) - M \times b \tag{3.4}$$

The normal bundle of the embedding ρ_M is similar to (2.7)

$$\nu_M = \frac{[0,1] \times \mathbb{R}^5 \times (\eta \oplus \eta)}{(0, u, w_1, w_2) \sim (1, u, w_2, w_1)}, \tag{3.5}$$

Then a fibrewise surgery as before produces a periodicity bundle (more precisely, a stratified system of bundles)

$$\pi_P \colon P(\xi) = C_1(\xi) \cup_{\hat{S}(\xi)} \hat{D}(\xi) \to M.$$

The periodicity representation is the fibrewise tangent bundle of the section $M \times b$ in the top stratum $C_1(\xi) - S_\Delta(\xi)$. The fibrewise tangent bundle is easily identified with $\eta \oplus \eta \oplus \epsilon^4$, so that the corresponding disk bundle

$$E(\xi) = D(\eta \oplus \eta \oplus \epsilon^4)$$

is embedded in the top stratum $C_1(\xi) - S_\Delta(\xi)$. The triviality of the fundamental groups $\pi_1(C(\xi_x^H) - S_\Delta(\xi_x^H))$ then implies that the embedding $E(\xi) \to C_1(\xi) - S_\Delta(\xi) = P(\xi) - \hat{D}(\xi)$ induces isomorphisms on the fundamental groups of the isovariant components.

4 Periodicity for Surgery Obstruction

Denote by L_G the surgery obstruction space, with the homotopy groups being the surgery obstruction groups of Browder and Quinn. The periodicity is based on the fact that the following maps

$$L_G(M) \xrightarrow{\text{trf}} L_G(P(\xi)) \xrightarrow{\text{incl}} L_G(E(\xi)) \tag{4.1}$$

are homotopy equivalences.

The inclusion in (4.1) is the composition of the following inclusions

$$L_G(E(\xi)) \xrightarrow{\text{incl}_\alpha} L_G(P(\xi) - \hat{D}(\xi)) \xrightarrow{\text{incl}_\beta} L_G(P(\xi)). \tag{4.2}$$

Since the embedding $E(\xi) \to P(\xi) - \hat{D}(\xi)$ induces isomorphisms on the fundamental groups of the isovariant components, incl_α is an equivalence. The second inclusion fits into a fibration

$$L_G(P(\xi) - \hat{D}(\xi)) \xrightarrow{\text{incl}_\beta} L_G(P(\xi)) \longrightarrow L_G(\hat{D}(\xi)).$$

The base is homotopically trivial because the pair $(\hat{D}(\xi), \hat{S}(\xi))$ has the isovariant π-π property. Thus incl_β is also an equivalence. This completes the proof that the inclusion in (4.1) is an equivalence.

To prove the transfer in (4.1) is an equivalence, consider a maximal isotropy subgroup H of M. There is a commutative diagram

$$\begin{tikzcd}
L_G(M - GM^H) \arrow[r] \arrow[d, "\tau"'] & L_G(M) \arrow[r] \arrow[d, "\text{trf}"'] & L_{WH}(M^H) \arrow[d, "\text{trf}'"'] \\
L_G(P(\xi) - GP(\xi)^H) \arrow[r] & L_G(P(\xi)) \arrow[r] & L_{WH}(P(\xi)^H)
\end{tikzcd} \tag{4.3}$$

in which the rows are fibrations and the transfer map trf_1 is taken with respect to the WH-bundle $P(\xi)^H = C_1(\xi)^H \cup \hat{D}(\xi)^H \to M^H$. We will prove the transfer map in the middle to be an equivalence by showing that the maps τ and trf' are equivalences.

The transfer trf' may be compared with a related non-equivariant transfer. Since H is a maximal isotropy subgroup, WH acts freely on the equivariant WH-bundle $\xi^H \to M^H$. Then we have the disk bundle $D(\xi^H/WH)$, the sphere bundle $S(\xi^H/WH)$, and the symmetric product $C(\xi^H/WH)$ of the sphere bundle. Moreover, a surgery can be applied to $C(\xi^H/WH)$ to get $C_1(\xi^H/WH)$ and a stratified bundle

$$P(\xi^H/WH) = C_1(\xi^H/WH) \cup_{S_\Delta(\xi^H/WH)} D(\xi^H/WH) \to M^H/WH.$$

Then there is a transfer map

$$\mathrm{trf}''\colon L(M^H/WH) \to L(P(\xi^H/WH)) \tag{4.4}$$

associated to the bundle. Moreover, there is an inclusion

$$\mathrm{incl}''\colon L(C_1(\xi^H/WH) - S_\Delta(\xi^H/WH)) \to L(P(\xi^H/WH)), \tag{4.5}$$

which is again an equivalence because $(D(\xi^H/WH), S_\Delta(\xi^H/WH))$ has the isovariant π-π property. Since the dimension of the fibre of $C(\xi^H/WH)$ is $2(\dim\xi^H - 1)$, there is also the projection

$$\mathrm{proj}\colon L(C_1(\xi^H/WH) - S_\Delta(\xi^H/WH)) \to L_{+2(\dim\xi^H-1)}(M^H/WH),$$

which is also an equivalence because the projection induces an isomorphism on fundamental groups. Moreover, since $\xi^H = \eta^H \oplus \epsilon^3$ and η is a complex vector bundle, $\dim\xi^H - 1$ is an even number, and we have the classical periodicity isomorphism

$$\mathrm{per}\colon L_{+2(\dim\xi^H-1)}(M^H/WH) \cong L(M^H/WH).$$

Combining the four maps associated to the bundle $P(\xi^H/WH)$ together, we get

$$\mathrm{per}\circ\mathrm{proj}\circ\mathrm{incl}''^{-1}\circ\mathrm{trf}''\colon L(M^H/WH) \to L_{+2\dim\xi^H-2}(M^H/WH). \tag{4.6}$$

Since the complex vector bundle ξ^H/WH is always orientable and even dimensional, the proof of Theorem 4.3 in [2] basically says that the composition (4.6) is an equivalence. Since the periodicity, the projection, and the inclusion are all equivalences, the transfer (4.4) is also an equivalence.

Next the two transfers trf' and trf'' are compared. By (3.2) and (3.3),

$$\begin{aligned}
C_1(\xi)^H &= C_1(\xi^H) \cup S_H,\\
\hat{S}(\xi)^H &= S_\Delta(\xi^H) \cup S_H,\\
P(\xi)^H &= \left(C_1(\xi^H) \cup S_H\right) \cup_{(S_\Delta(\xi^H)\cup S_H)} \left(D_\Delta(\xi^H) \cup D_H\right),
\end{aligned}$$

where

$$\begin{aligned}
S_H &= \bigcup_{K\subset H, |H/K|=2} S_{H,2}(\xi^K|_{M^H}),\\
D_H &= \bigcup_{K\subset H, |H/K|=2} D_{H,2}(\xi^K|_{M^H}).
\end{aligned}$$

Moreover, by the way $P(\xi^H/WH)$ is constructed,

$$\begin{aligned} C_1(\xi^H/WH) &= C_1(\xi^H)/WH, \\ S_\Delta(\xi^H/WH) &= S_\Delta(\xi^H)/WH, \\ P(\xi^H/WH) &= \left(C_1(\xi^H) \cup_{S_\Delta(\xi^H)} D_\Delta(\xi^H)\right)/WH. \end{aligned}$$

Thus the only difference between $P(\xi^H/WH)$ and the quotient $P(\xi)^H/WH$ is the extra strata D_H/WH and S_H/WH. Forgetting the extra strata gives a commutative diagram

$$\begin{tikzcd} L_{WH}(M^H) \arrow[r, "\mathrm{trf}'"] \arrow[d, equal] & L_{WH}(P(\xi)^H) \arrow[d, "\mathrm{forget}"'] & L_{WH}(C_1(\xi)^H - \hat{S}(\xi)^H) \arrow[l, "\mathrm{incl}'"'] \arrow[d, equal] \\ L(M^H/WH) \arrow[r, "\mathrm{trf}''"] & L(P(\xi^H/WH)) & L(C_1(\xi^H/WH) - S_\Delta(\xi^H/WH)) \arrow[l, "\mathrm{incl}''"'] \end{tikzcd}$$

It has been argued before that incl'' in (4.5) is an equivalence. By the similar π-π reason, the inclusion incl' is also an equivalence. Consequently, the transfer trf' is an equivalence.

Now consider the induced map τ in (4.3). The map fits into a commutative diagram

$$\begin{tikzcd} L_G(M - GM^H) \arrow[r, equal] \arrow[d, "\mathrm{trf}"'] & L_G(M - GM^H) \arrow[d, "\tau"'] \\ L_G(P(\xi|_{M-GM^H})) \arrow[r, "\mathrm{incl}"] & L_G(P(\xi) - GP(\xi)^H) \\ L_G(P(\xi|_{M-GM^H}) - \hat{D}(\xi|_{M-GM^H})) \arrow[r, "\mathrm{incl}"] \arrow[u, "\mathrm{incl}_1"] & L_G(P(\xi) - \hat{D}(\xi) - GP(\xi)^H) \arrow[u, "\mathrm{incl}_5"] \\ L_G(E(\xi|_{M-GM^H})) \arrow[r, "\mathrm{incl}_3"] \arrow[u, "\mathrm{incl}_2"] & L_G(E(\xi) - GE(\xi)^H) \arrow[u, "\mathrm{incl}_4"] \end{tikzcd} \tag{4.7}$$

Since $M - GM^H$ has fewer isotropy subgroups than M, it can be assumed, by induction, that the transfer on the top left is already an equivalence. Then τ can be proved to be an equivalence if the indexed inclusions $\mathrm{incl}_1, \dots, \mathrm{incl}_5$ can be shown to be equivalences.

The composition of incl_1 and incl_2 is the same as the composition in (4.2), except M is replaced by $M - GM^H$. Moreover, because $GE(\xi)^H = E(\xi) \cap GP(\xi)^H$, the composition of incl_4 and incl_5 is the same as (4.2), except the isovariant components of (the maximal isotropy subgroup) H and its conjugates are deleted. The modifications do not affect the arguments for the inclusions in (4.2) to be equivalences. Therefore by the same arguments, the inclusion maps $\mathrm{incl}_4, \mathrm{incl}_5$ are equivalences.

Finally, consider the spaces $E(\xi|_{M-GM^H})$ and $E(\xi) - GE(\xi)^H$ on the two sides of incl_3. Denote $\zeta = \eta \oplus \eta \oplus \epsilon^4$. Then

$$\begin{aligned} E(\xi|_{M-GM^H}) &= D(\zeta|_{M-GM^H}), \\ E(\xi) - GE(\xi)^H &= D(\zeta) - GD(\zeta)^H \\ &= D(\zeta|_{M-GM^H}) \cup G \times_H (D(\zeta|_{M^H}) - D(\zeta^H)). \end{aligned}$$

For any proper subgroup K of H, the following is a one-to-one correspondence:

- the K-isovariant components of M and of $M - GM^H$, via inclusion.

The same isotropy everywhere assumption implies the following are one-to-one correspondences:

- the K-isovariant components of $D(\zeta)$ and of M, via projection;
- the K-isovariant components of $D(\zeta|_{M-GM^H})$ and of $M - GM^H$, via projection.

Combining the three one-to-one correspondences together, we conclude that inclusion gives a one-to-one correspondence between the K-isovariant components of $D(\zeta)$ and of $D(\zeta|_{M-GM^H})$. Then by the codimension ≥ 3 gap condition, the corresponding K-isovariant components of $D(\zeta)$ and of $D(\zeta|_{M-GM^H})$ have the same fundamental groups (same as the fundamental groups of the corresponding isovariant components of M and $M - GM^H$). Since K-isovariant components of $D(\zeta)$ are the same as the K-isovariant components of $D(\zeta) - GD(\zeta)^H$, we conclude that incl_3 induces isomorphisms on the fundamental groups of corresponding isovariant components. Consequently, incl_3 induces an equivalence on the surgery obstruction.

5 Periodicity for K-theory

The discussion of the third section of [9] is still valid here. There are two periodicity problems concerning the K-theory. The first is the stablization of the surgery obstruction. The second is the destablization of the structure set. By mostly formal arguments, both problems can be settled by considering the special case G acts freely on M and by considering the diagram

$$\begin{array}{ccc} K_G^{\leq 1}(C_1(\xi)) & \overset{\overline{\mathrm{incl}}}{\longleftarrow} & K_G^{\leq 1}(C_1(\xi) - \hat{S}(\xi)) \\ \overline{\mathrm{trf}} \uparrow & \nwarrow \phi & \mathrm{incl} \downarrow \\ K_G^{\leq 1}(M) & \xrightarrow{\mathrm{trf}} & K_G^{\leq 1}(P(\xi)) \end{array} \tag{5.1}$$

in which $C_1(\xi)$ is no longer considered as stratified, but only as a G-bundle over M with polyhedron $C_1(V)$ as the fibre. The map ϕ is the restriction to the closed stratum $C_1(\xi)$ followed by forgeting the stratification in $C_1(\xi)$. The geometrical meanings of the maps in (5.1) imply that the two triangles (and hence the whole diagram) are commutative. Since $\overline{\mathrm{incl}}$ induces isomorphisms on the fundamental groups of isovariant components, it is an equivalence on the K-theory. As argued in [9], what needs to be done is to show that $\overline{\mathrm{trf}}$ is an equivalence after localizing at 2. In fact, we are going to see that the transfer is the multiplication by 3.

Since G acts freely, $\overline{\mathrm{trf}}$ is the same as the transfer $K^{\leq 1}(C_1(\xi)/G) \to K^{\leq 1}(M/G)$ for the nonequivariant bundle $C_1(\xi)/G \to M/G$, which was studied in [4]. Fix a base point of $C_1(\xi)/G$ and use its projection $b = Gx$ ($x \in M$) as the base point of M/G. Denote by $V = \xi_x$ the fibre of ξ over x. Then $C_1(V)$ is the fibre of the bundle $C_1(\xi)/G \to M/G$, and the universal cover $\tilde{C}_1(\xi)$ of the total space $C_1(\xi)/G$ is given by the following pullbacks

$$\begin{array}{ccccc} \tilde{C}_1(\xi) & \longrightarrow & C_1(\xi) & \longrightarrow & C_1(\xi)/G \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & M & \longrightarrow & M/G \end{array}, \tag{5.2}$$

where $\tilde{M}$ is the universal cover of M. Since $C_1(V)$ is connected and simply connected, $\pi_1(C_1(\xi)/G)$ may be identified with $\pi_1(M/G)$ via the projection. Then by fixing a base point $\tilde{b} \in \tilde{M}$ above b, the fibre of $\tilde{M} \to M/G$ is exactly $\pi_1(M/G)\tilde{b}$. From the pullback (5.2), the fibre $\tilde{C}_1(V)$ of the bundle $\tilde{C}_1(\xi) \to M/G$ is then exactly $\pi_1(M/G)\tilde{b} \times C_1(V)$ as a $\pi_1(M/G)$-space. The monodromy of the $\pi_1(M/G)$-equivariant bundle $\tilde{M} \to M/G$ then gives rise to a homomorphism

$$\pi_1(M/G) \to [\tilde{C}_1(V), \tilde{C}_1(V)]_{\pi_1(M/G)}, \tag{5.3}$$

where $[\tilde{F}, \tilde{F}]_\pi$ is the equivalent homotopy classes of self-homotopy equivalences of a π-space $\tilde{F}$. Note that in case $\tilde{F} = \pi \times F$, with the π-action being left multiplication on the first and the trivial action on the second, we have a natural map

$$[F, F] \to [\tilde{F}, \tilde{F}]_\pi, \quad \tilde{\phi}(g, x) = (g, \phi(x)). \tag{5.4}$$

Now we are in exactly the same situation. The pullback (5.2) implies that the homomorphism (5.3) is a composition

$$\pi_1(M/G) \to [C_1(V), C_1(V)] \to [\tilde{C}_1(V), \tilde{C}_1(V)]_{\pi_1(M/G)}, \tag{5.5}$$

where the first map is the monodromy of the bundle $C_1(\xi)/G \to M/G$, and the second map is (5.4).

By Theorem 2.1 of the second part of [4], the transfer $K^{\leq 1}(C_1(\xi)/G) \to K^{\leq 1}(M/G)$ is algebraically determined by the $\mathbb{Z}\pi_1(M/G)$-$\mathbb{Z}\pi_1(M/G)$-bimodule structure on the homologies $H_i(\tilde{C}_1(V))$ induced by the homomorphism (5.3). The first $\mathbb{Z}\pi_1(M/G)$ refers to the group $\pi_1(M/G)$ on the right (appearing as a subscript) of (5.3). The second $\mathbb{Z}\pi_1(M/G)$ refers to the group $\pi_1(M/G)$ on the left of (5.3). Since both $\pi_1(M/G)$ in (5.3) were considered as acting on the left of $\tilde{C}_1(V)$, the action of the group $\pi_1(M/G)$ on the left of (5.3) needs to be modified by the inverse in the bimodule structure.

In our case, the factorization (5.5) implies that

$$H_i(\tilde{C}_1(V)) = \mathbb{Z}\pi_1(M/G) \otimes H_i(C_1(V)).$$

Moreover, the bimodule structure is the following: The first $\mathbb{Z}\pi_1(M/G)$ acts by left multiplication on the first factor only. The second $\mathbb{Z}\pi_1(M/G)$ acts on the second factor only, and is induced by the monodromy of the bundle $C_1(\xi)/G \to M/G$.

Since $H_i(C_1(V))$ is a finitely generated abelian group, $H_i(\tilde{C}_1(V))$ has finitely generated projective resolutions as (the first and the left) $\mathbb{Z}\pi_1(M/G)$-module. Thus according to Theorem 2.1 of the second part of [4], the transfer is determined by the element

$$\sum (-1)^i [H_i(\tilde{C}_1(V))] = [\mathbb{Z}\pi_1(M/G)] \otimes \sum (-1)^i [H_i(C_1(V))] \tag{5.6}$$

in a certain Grothendieck group $K_0(\mathbb{Z}\pi_1(M/G) - \mathbb{Z}\pi_1(M/G))$.

Let $2k-2$ be the real dimension of the complex bundle η. Then ξ is a real orientable bundle of real dimension $2k+1$. Based on the computation in [2], it is easy to see that

$$H_i(C_1(V)) = \begin{cases} H_i S^{2k} & i = 0, 2k \\ 0 & i = 1 \\ \tilde{H}_{i-2k-1}(RP^{2k-1}) & \text{other } i \end{cases} = \begin{cases} \mathbb{Z} & i = 0, 2k, 4k \\ \mathbb{Z}_2 & i = 3, 5, ..., 2k-1 \\ 0 & \text{other } i \end{cases}$$

as abelian groups. Since ξ is orientable, the monodromy preserves the orientation of the fibre. Then by tracing the computation in [2], we find the action of $\mathbb{Z}\pi_1(M/G)$ on $H_i(C_1(V))$ to be trivial. Thus the element (5.6) is the same as the element $[\mathbb{Z}\pi_1(M/G)] \otimes [\mathbb{Z}^3] = 3[\mathbb{Z}\pi_1(M/G)]$ in the Grothendieck group $K_0(\mathbb{Z}\pi_1(M/G) - \mathbb{Z}\pi_1(M/G))$. This implies that $\overline{\mathrm{trf}}$ is multiplication by 3.

References

[1] J. Bryant, S. Ferry, W. Mio, S. Weinberger: *Topology of homology manifolds*, Ann. of Math., **143**(1996)435-467

[2] F. T. Farrell, L. E. Jones, *A topological analogue of Mostow's rigidity theorem*, Jour. AMS, **2**(1989)257-370

[3] R. Kirby, L. Siebenmann: Foundational essays on topological manifolds, smoothings, and triangulations. Princeton: Princeton University Press 1977

[4] W. Lück, *The transfer maps induced in the algebraic K_0- and K_1-groups by a fibration I & II*, Math. Scand., **59**(1986)93-121, J. Pure and Applied Algebra, **45**(1987)143-169

[5] A. Nicas: Induction theorems for groups of homotopy manifold structure sets. Mem. A.M.S. **267**, (1982)

[6] S. Weinberger, The topological classification of stratified spaces. University of Chicago Press 1993

[7] S. Weinberger, M. Yan, *Equivariant periodicity for abelian group actions*, Adv. in Geom., **1**(2001)49-70

[8] M. Yan, *The periodicity in stable equivariant surgery*, Comm. in Pure and Applied Math., **46**(1993)1013-1040

[9] M. Yan, *Equivariant periodicity in surgery for actions of some nonabelian groups*, In: Proceedings of Georgia Topology Conference, 1993, 478-508. International Press 1997