Topological Classification of Multiaxial $U(n)$-Actions (with an appendix by Jared Bass)

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1 Introduction

Since early 1980s, great progress has been made on the classification of finite group actions on the sphere. Deep but indirect connections to representation theory were discovered. The indirectness is reflected by the existence of non-linear similarities between some linearly inequivalent representations [3], via the equivariant signature operator [MR] (see also [HP??] [4]). Whitehead torsion, which was the cornerstone of the classical theory of lens spaces, plays almost no role at all, especially in the presence of fixed points [12, 13].

On the other hand, the action of positive dimensional groups on topological manifolds has been largely left alone, aside from action by the circle. This paper, inspired by the beautiful results of M. Davis and W. C. Hsiang [8] on concordance classes of smooth multiaxial actions on the homotopy sphere, shows that the classification theory in the topological setting is both completely different and quite comprehensible.

For the purposes of this introduction, we will assume that $G = U(n)$ acts on M locally smoothly. In other words, every orbit has a neighborhood equivariantly homeomorphic to an open subset of an orthogonal representation of G. We will concentrate on multiaxial actions, which means that the representations are of the form $k\rho_n \oplus j\epsilon$, where ρ_n is the defining representation of $U(n)$ on $\mathbb{C}^n$, and ϵ is the trivial representation $\mathbb{R}$. While this may allow different choice of k and j at different locations in the manifold, the results

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presented in the introduction will assume the same k and j everywhere. In such setting, we say the manifold is modeled on $k\rho_n \oplus j\epsilon$.

The isotropy subgroups of multiaxial actions are conjugate to unitary subgroups $U(i)$ of $U(n)$. Then M is stratified by $M_{-i} = U(n)M^{U(i)}$, the set of points fixed by some conjugation of $U(i)$. Correspondingly, the orbit space $X = M/U(n)$ is stratified by $X_{-i} = M^{U(i)}/U(n-i)$.

Our goal is to study the isovariant structure set $S_{U(n)}(M)$. In general, the structure set $S(X)$ of a compact topological manifold X is the homeomorphism classes of topological manifolds simple homotopy equivalent to X. The notion can be extended to $S_G(M)$ for equivariant G-manifolds M and isovariant simple homotopy equivalences. It can also be extended to $S(X)$ for stratified spaces X and stratified simple homotopy equivalences. We have $S_G(M) = S(M/G)$ when the orbit space M/G is homotopically stratified.

Let X_α be the strata of a stratified space X. The pure strata

$$X^\alpha = X_\alpha - X_{<\alpha}, \quad X_{<\alpha} = \cup_{X_\beta \subsetneq X_\alpha} X_\beta$$

are generally noncompact manifolds, and we have natural restriction maps

$$S(X) \to \oplus S^{\text{proper}}(X^\alpha).$$

Here S^{proper} denotes the proper homotopy equivalence version of the structure set. If we further know that all pure strata of links between strata of X are connected and simply connected (or more generally, the fundamental groups of these strata have trivial K-theory, according to Quinn [12] or Steinberger [13]), then the complement $\bar{X}^\alpha$ of (the interior of) a regular neighborhood of $X_{<\alpha}$ is a topological manifold with boundary $\partial\bar{X}^\alpha$ and interior X^α, and the restriction maps natually factor through the structures of $(\bar{X}^\alpha, \partial\bar{X}^\alpha)$

$$S(X) \to \oplus S(\bar{X}^\alpha, \partial\bar{X}^\alpha) \to \oplus S^{\text{proper}}(X^\alpha).$$

The difference between the simple homotopy structure of $(\bar{X}^\alpha, \partial\bar{X}^\alpha)$ and the proper homotopy structure of X^α is captured by the finiteness obstruction at infinity. If, in addition to the simple connectivity of the pure strata of links, all strata of X are also simply connected, then these finiteness obstructions vanish, and we get $S(\bar{X}^\alpha, \partial\bar{X}^\alpha) = S^{\text{proper}}(X^\alpha)$.

The pure strata of links are indeed connected and simply connected for multiaxial $U(n)$-manifolds. Our main result states that the stratified simple homotopy structure on $X = M/U(n)$ is almost determined by the restriction to $S(\bar{X}^{-i}, \partial\bar{X}^{-i})$ for half of strata X_{-i}. More general versions are given by Theorems 5.1, 5.2, 5.3.

Theorem 1.1. *Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$, and $X = M/U(n)$ is the orbit space. If $k \geq n$ and $k-n$ is even, then we have natural splitting*

$$S_{U(n)}(M) = \oplus_{i\geq 0} S(\bar{X}^{-2i}, \partial\bar{X}^{-2i}).$$

If $k \geq n$, $k-n$ is odd and $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold modeled on $k\rho_{n+1} \oplus j\epsilon$, then we have natural splitting

$$S_{U(n)}(M) = S^{\text{alg}}(X) \oplus \left(\oplus_{i\geq 0} S(\bar{X}^{-2i-1}, \partial\bar{X}^{-2i-1})\right).$$

The condition $k \le n$ was always assumed in [7, 8, 9]. In this case, we have $S_{U(n)}(M) = S_{U(k)}(M^{U(n-k)})$. Since $M^{U(n-k)}$ is a multiaxial $U(k)$-manifold modeled on $k\rho_k \oplus j\epsilon$, the first part of the theorem can be applied.

The *algebraic structure set* S^{alg} in the second part of the theorem means the following.

Definition. For any topological space X, let $\mathbb{S}^{\text{alg}}(X)$ be the homotopy fibre of the surgery assembly map $\mathbb{H}_*(X;\mathbb{L}) \to \mathbb{L}(\pi_1 X)$. Then $S^{\text{alg}}(X) = \pi_0 \mathbb{S}^{\text{alg}}(X)$.

In the definition, $\mathbb{L}(\pi)$ is the (simple) surgery obstruction spectrum for the fundamental group π, and $\mathbb{H}_*(X;\mathbb{L})$ is the homology theory associated to the spectrum $\mathbb{L} = \mathbb{L}(e)$. If X is a topological manifold of dimension ≥ 5 (or dimension 4 in case $\pi_1 X$ is not too bad), then $S^{\text{alg}}(X)$ is the usual structure set that classifies topological (in fact, homological) manifolds simple homotopy equivalent to X. For a general topological space X, however, $S^{\text{alg}}(X)$ no longer carries such geometrical meaning and is only the result of some algebraic computation.

For a taste of what to expect when k and j are not assumed constant, the following is the simplest case of Theorem 5.2. The proof is given at the end of Section 5.

Theorem 1.2. *Suppose the circle S^1 acts semifreely on a topological manifold M^m, such that the fixed points M^{S^1} is a locally flat submanifold. Let $M_0^{S^1}$ and $M_2^{S^1}$ be the unions of those connected components of M^{S^1} that are respectively of dimensions m mod 4 and $m+2$ mod 4. Let N be the complement of (the interior of) an equivariant tube neighborhood of M^{S^1}, with boundaries $\partial_0 N$ and $\partial_2 N$ corresponding to the two parts of the fixed points. Then*

$$S_{S^1}(M) = S(M_0^{S^1}) \oplus S(N/S^1, \partial_2 N/S^1, \operatorname{rel} \partial_0 N/S^1).$$

We note that N/S^1 is a manifold with boundary divided into two parts ∂_0 and ∂_2. The second factor means the homeomorphim classes of manifolds simple homotopy equivalent to N/S^1 that restricts to a simple homotopy equivalence on ∂_2 and a homeomorphism on ∂_0. We also note that it is a special feature of the circle action that the condition of the extendability of M to a multiaxial $U(2)$-manifold is not needed.

The terms in the decomposition in Theorem 1.1 have the following interpretation in terms of the isovariant structure set

$$S(\bar{X}^{-i}, \partial \bar{X}^{-i}) = S_{U(n-i)}(M^{U(i)}, \operatorname{rel} U(n-i) M^{U(i+2)}).$$

Here $M^{U(i)}$ is actually a multiaxial $U(n-i)$-manifold modeled on $k\rho_{n-i} \oplus j\epsilon$, and $U(n-i)M^{U(i+2)}$ is the stratum of the multiaxial $U(n-i)$-manifold two levels down. The right side classifies those $U(n-i)$-manifolds isovariantly simple homotopy equivalent to $M^{U(i)}$, such that the restriction to the stratum two levels down are already equivariantly homeomorphic. The decomposition in Theorem 1.1 is then equivalent to the decomposition

$$S_{U(n)}(M) = S_{U(n)}(M, \operatorname{rel} U(n) M^{U(i)}) \oplus S_{U(n-2i)}(M^{U(i)}),$$

where i and $k-n$ have the same parity. The map to the second factor is the obvious restriction. The fact that this restriction is onto has the following interpretation.

Theorem 1.3. *Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$. Suppose $k \ge n > i$ and one of the following holds.*

1. *$k-n$ and i are even.*
2. *$k-n$ and i are odd, and $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold modeled on $k\rho_{n+1} \oplus j\epsilon$.*

Then for any $U(n-i)$-isovariant simple homotopy equivalence $\phi\colon V \to M^{U(i)}$, there is a $U(n)$-isovariant simple homotopy equivalence $f\colon N \to M$, such that $\phi = f^{U(i)}$ is the restriction of f.

The theorem means that half of the fixed point subsets can be homotopically replaced. The homotopy replacement of the fixed point subset of the whole group has been studied in [5, 6]. Here the replacement is for the fixed point subsets of certain subgroups and is therefore a new kind of replacement.

The algebraic calculation can be explicitly carried out for the special case that M is the unit sphere of the representation $k\rho_n \oplus j\epsilon$. For $k \geq n$, let $A_{n,k}$ be the numbers of Schubert cells of dimensions 0 mod 4 in the complex Grassmannian $G(n,k)$, and let $B_{n,k}$ be the number of cells of dimensions 2 mod 4. Specifically, $A_{n,k}$ is the number of n-tuples $(\mu_1,\dots,\mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k-n, \quad \sum \mu_i \text{ is even},$$

and $B_{n,k}$ is the similar number for the case $\sum \mu_i$ is odd. Then the following computation is carried out in Section 6.

Theorem 1.4. *Suppose $S(k\rho_n \oplus j\epsilon)$ is the unit sphere of the representation $k\rho_n \oplus j\epsilon$, $k \geq n$.*

1. *If $k-n$ is even, then we have*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{0\leq 2i<n} A_{n-2i,k}} \oplus \mathbb{Z}_2^{\sum_{0\leq 2i<n} B_{n-2i,k}},$$

with the only exception that there is one less copy of $\mathbb{Z}$ in case n is odd and $j=0$.

2. *If $k-n$ is odd, then we have*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{A_{n,k-1}+\sum_{0\leq 2i+1<n} A_{n-2i-1,k}} \oplus \mathbb{Z}_2^{B_{n,k-1}+\sum_{0\leq 2i+1<n} B_{n-2i-1,k}},$$

with exceptions that there is one less copy of $\mathbb{Z}$ in case n is even and $j=0$, and there is one more copy of $\mathbb{Z}_2$ in case n is odd and $j>0$.

The computation generalizes the classical computation for the fake complex projective space [14, Section 14C].

If N is isovariant simple homotopy equivalent to the representation sphere $S(k\rho_n \oplus j\epsilon)$, then joining with the representation sphere $S(\rho_n)$ give a manifold $N * S(\rho_n)$ isovariant simple homotopy equivalent to the representation sphere $S((k+1)\rho_n \oplus j\epsilon)$. This gives the suspension map

$$*S(\rho_n)\colon S_{U(n)}(S(k\rho_n \oplus j\epsilon)) \to S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon)).$$

A consequence of the calculation in Theorem 1.4 is the following. The detailed about the suspension map is given in Section 7.

Theorem 1.5. *The suspension map is injective.*

Finally, in Section 8, we extend all the results to the similarly defined multiaxial $Sp(n)$-manifolds.

2 Strata of Multixial $U(n)$-Manifold

The concept of multiaxial manifold was introduced and studied in [7, 8, 9]. Our definition is more general in that the actions are not assumed to be locally smooth, and the local model may be different at different parts of the manifold.

Let $U(n)$ be the unitary group of linear transformations of $\mathbb{C}^n$ preserving the Euclidean norm. By a *unitary subgroup*, we mean the specific subgroup $U(i)$ of $U(n)$ that fixes the last $n-i$ coordinates. Although it is more intrinsic to define a unitary subgroup more generally as a subgroup fixing a linear subspace of $\mathbb{C}^n$, such a unitary subgroup is always conjugate to some $U(i)$ in $U(n)$. We fix specific unitary subgroups only to accommodate simpler presentation in this paper.

The *normaliser* of the unitary group is $NU(i) = U(i) \times U(n-i)$, where $U(n-i) = NU(i)/U(i)$ is the *Weyl group* that fixes the first i coordinates. It is usually clear from the context when $U(k)$ is a unitary subgroup (fixing the last $n-k$ coordinates) and when $U(k)$ is a Weyl group (fixing the first $n-k$ coordinates).

Definition. A topological $U(n)$-manifold M is *multiaxial*, if any isotropy group is conjugate to a unitary subgroup $U(i)$, and for any $i > j$, $M^{-i} = M_{-i} - M_{-i-1}$ is a locally flat submanifold in M_{-j}.

In the definition, the multiaxial manifold M is stratified by $M_{-i} = U(n)M^{U(i)}$, the set of points fixed by some conjugations of $U(i)$. Correspondingly, the orbit space $X = M/U(n)$ is stratified by $X_{-i} = M_{-i}/U(n)$.

The locally flat assumption can be relaxed. What we really need is the homotopy consequence of the assumption. Specifically, we need the links between adjacent strata to be spheres, and the fundamental groups of the pure strata of the links of M^{-i} in M to be connected and simply connected (with the exception that the link of M_{-1} in M can be the circle). Quinn [12] showed that such homotopy properties imply that the orbit space is homotopically stratified. Then the pure stratum $M^{-i} = M_{-i} - M_{-i-1}$ is an open manifold that can be completed into a manifold with boundary $U(n) \times_{U(n-i)} (\bar{M}^{U(i)}, \partial \bar{M}^{U(i)})$, by deleting (the interior of) regular neighborhoods of lower strata. The pure stratum $X^{-i} = X_{-i} - X_{-i-1}$ is a homological manifold [2], and can also be completed into a homological manifold with boundary $(\bar{X}^{-i}, \partial \bar{X}^{-i})$.

For a multiaxial $U(n)$-manifold M, the fixed set $M^{U(i)}$ is a multiaxial $U(n-i)$-manifold, where $U(n-i)$ is the Weyl group. The following is some sort of "hereditary property" for multiaxial manifolds.

Lemma 2.1. *If M is a multiaxial $U(n)$-manifold, then $M_{-i}/U(n) = M^{U(i)}/U(n-i)$.*

The lemma shows that, as far as the orbit space is concerned, the study of a stratum of a multiaxial manifold is the same as the study of a "smaller" multiaxial manifold. In particular, if a multiaxial $U(n)$-manifold M does not have free points, then the minimal isotropy groups are conjugate to $U(m)$ for some $m > 0$, and the study of the $U(n)$-manifold M is the same as the study of the $U(n-m)$-manifold $M^{U(m)}$. Since the $U(n-m)$-action on $M^{U(m)}$ has free points, we may always assume the existence of free points without loss of generality. In the setting of multiaxial manifolds modeled on $k\rho_n \oplus j\epsilon$ studied in [7, 8, 9], this means that we may always assume $k \geq n$. We remark that $k \leq n$ was assumed in these earlier works.

Lemma 2.1 is a consequence of the two subsequent propositions.

Proposition 2.2. *If $H \subset K \subset G = U(n)$ are conjugate to unitary subgroups, then the NH-action on $(G/K)^H$ is transitive. In other words, if $H \subset K$ and $g^{-1}Hg \subset K$, then $g = \nu k$ for some $\nu \in NH$ and $k \in K$.*

Proof. The subgroups K and H consist of the unitary transformations of $\mathbb{C}^n$ that respectively fix some subspaces V_K and V_H. Then $H \subset K$ means $V_K \subset V_H$ and $g^{-1}Hg \subset K$ means $gV_K \subset V_H$. Therefore there is a unitary transformation ν that preserves V_H and restricts to g on V_K. Then $\nu^{-1}g$ preserves V_K, so that $\nu^{-1}g \in K$. Moreover, the fact that ν preserves V_H means that $\nu \in NH$.

The transitivity of the NH-action on $(G/K)^H$ means that if $gK \in (G/K)^H$, then $gK = \nu K$ for some $\nu \in NH$. Since $gK \in (G/K)^H$ means $g^{-1}Hg \subset K$, and $gK = \nu K$ means $g = \nu k$ for some $k \in K$, we see that the transitivity is the same as the group theoretical property above. □

Proposition 2.3. *If G acts on a set M, such that every pair of isotropy groups satisfy the property in Proposition 2.2, then $GM^H/G = M^H/NH$ for any isotropy group H.*

Proof. We always have the natural surjective map $M^H/NH \to GM^H/G$. Over a point in GM^H/G represented by $x \in M^H$, the fibre of the map is $(Gx)^H/NH$. Therefore the map is one-to-one if and only if the action of NH on $(Gx)^H = (G/G_x)^H$ is transitive. □

3 Homotopy Property of Multixial $U(n)$-Manifold

Although our definition of multiaxial $U(n)$-manifold is more general than those in [7, 8, 9] that are modeled on linear representations, many homotopy properties of the linear model are still preserved.

First we consider the link between adjacent strata of the orbit space $X = M/U(n)$ of a multiaxial $U(n)$-manifold M. By the link of X_{-j} in X_{-j+1}, we really mean the link of the pure stratum $X^{-j} = X_{-j} - X_{-j-1}$ in X_{-j+1} (same for the strata of M), and this link may be different along different connected component of X^{-j}. So for any $x \in X_{-j}$, we denote by X^x_{-j} the connected component of X_{-j} containing x. By the link of X^x_{-j} in X_{-j+1}, we really mean the link of $X^x_{-j} - X_{-j-1}$ in X_{-j+1}. We also denote by $M^{U(j),x}$ the corresponding connected component of $M^{U(j)}$, so that $X^x_{-j} = M^{U(j),x}/U(n-j)$.

Lemma 3.1. *For any $x \in X_{-i}$ and $1 \le j \le i$, the link of X^x_{-j} in X_{-j+1} is homotopic to $\mathbb{C}P^{r^x_j}$, and $r^x_j = r^x_{j-1} + 1$.*

The lemma paints the following picture for the strata in a (connected) multiaxial $U(n)$-manifold. For any $x \in X^{-i}$, the stratification near x is given by

$$X = X^x_0 \supset X^x_{-1} \supset \cdots \supset X^x_{-i}.$$

The *first gap* r^x_1 of x depends only on the connected component X^x_{-1} and determines the homotopy type $\mathbb{C}P^{r_1^x+j-1}$ of the link of X^x_{-j} in X^x_{-j+1}. Moreover, we have

$$\begin{aligned}
&\dim M^{U(j-1),x} - \dim M^{U(j),x} \\
&= \dim X^x_{-j+1} + \dim U(n-j+1) - \dim X^x_{-j} - \dim U(n-j) \\
&= \dim \mathbb{C}P^{r_1^x+j-1} + 1 + (n-j+1)^2 - (n-j)^2 \\
&= 2(r_1^x + n).
\end{aligned}$$

The picture also shows that, near a point of M with isotropy group $gU(i)g^{-1}$, $gU(j)g^{-1}$ is the isotropy group of some nearby point for any $1 \le j \le i$.

If the multiaxial manifold is modeled on $k\rho_n \oplus j\epsilon$, then the first gap is independent of the connected component, and $r_1 = k - n$ in case $k \ge n$. On the other hand, multiaxial $U(1)$-manifolds are nothing but semi-free S^1-manifolds, for which any fixed point component has even codimension $2c$, and the first (and the only) gap of the component is $c - 1$.

Proof. The link of X^x_{-j} in X_{-j+1} is the quotient of the link of M_{-j} in M_{-j+1} by the free action of the Weyl group $N_{U(j)}U(j-1)/U(j-1) = S^1$. Since M^{-j} is a locally flat submanifold of M_{-j+1}, the link is a sphere. The quotient of the sphere by a free S^1-action must be homotopic to a complex projective space $\mathbb{C}P^{r_j}$.

Let $m_j = \dim M^{U(j),x}$ and $x_j = \dim X^x_{-j}$. By $X^x_{-j} = M^{U(j),x}/U(n-j)$, we have

$$x_j = \dim M^{U(j),x} - \dim U(n-j) = m_j - (n-j)^2.$$

Since the link of X^x_{-j} in X^x_{-j+1} is homotopic to $\mathbb{C}P^{r_j}$, we also have

$$x_{j-1} - x_j = 2r_j + 1.$$

Since the isotropy group in a multiaxial manifold is always conjugate to some unitary subgroup, we know $M^{U(j)} = M^{T^j}$ for the maximal torus T^j of $U(j)$. Here T^j is the specific torus group acting by scalar multiplication on the first j coordinates of $\mathbb{C}^n$. Now we fix j and consider M as a T^j-manifold. By the multiaxial assumption, the isotropy groups of the T^j-manifold M are the tori that are in one-to-one correspondence with the choices of some coordinates from the first j coordinates of $\mathbb{C}^n$. The number k of chosen coordinates is the rank of the isotropy torus. Since all the tori of the same rank k are conjugate to the specific torus group T^k, their fixed point components containing $\tilde{x} \in M^{U(j)}$ (whose image in X_{-j} is x) have the same dimension, which is $\dim M^{U(k),x} = m_k$.

For the case $k = j - 1$ (corank 1 in T^j), there are j such isotropy tori. By a formula of Borel [1, Theorem XIII.4.3], we have

$$m_0 - m_j = j(m_{j-1} - m_j).$$

Written in terms of x_j, we have

$$x_0 + n^2 = j(x_{j-1} + (n-j+1)^2) - (j-1)(x_j + (n-j)^2),$$

or

$$(j-1)^{-1}(x_{j-1} - x_0) - j^{-1}(x_j - x_0) = 1.$$

This gives $x_j - x_0 = j(a - j)$ and

$$x_{j-1} - x_j = 2j - 1 - a.$$

Combined with $x_{j-1} - x_j = 2r_j + 1$, we get $r_j = r_{j-1} + 1$. □

What about the links between any two (not necessarily adjacent) strata of a multiaxial manifold? For multiaxial manifolds modeled on $k\rho_n \oplus j\epsilon$, the pure strata of the links are actually homotopy equivalent to the Grassmannians. We expect that, under our more general assumption, the homotopy type of the pure strata of the links remain the Grassmannians. However, we only need the following weaker statement in the present paper.

Lemma 3.2. *All strata and pure strata of the links in the orbit space X are connected and simply connected, and*

$$\pi_1(X_{-i} - X_{-j}) = \pi_1(X_{-i}), \quad j > i.$$

In particular, we have $\pi_1 X^{-i} = \pi_1 X_{-i}$.

The lemma is an immediate consequence of Lemma 3.1 and Proposition 3.5. The proof of Proposition 3.5 is based on some well known general observations on the fundamental groups about homotopically stratified spaces.

In a homotopically stratified space, the neighborhoods of strata are stratified systems of fibrations over the strata. The fundamental groups are related as follows.

Proposition 3.3. *Suppose $E \to X$ is a stratified system of fibrations over a homotopically stratified space X. If the fibres are nonempty and connected, then $\pi_1 E \to \pi_1 X$ is surjective. If the fibres are (nonempty and) connected and simply connected, then $\pi_1 E \to \pi_1 X$ is an isomorphism.*

Proof. If $E \to X$ is a genuine fibration, then the two claims follow from the exact sequence of homotopy groups associated to the fibration.

Inductively, we only need to consider $X = Z \cup_{\partial Z} Y$, where Y is the union of lower strata, Z is the complement of a regular neighborhood of Y, and ∂Z is the boundary of a regular neighborhood of Y as well as the boundary of Z. Correspondingly, we have $E = E_Z \cup_{E_{\partial Z}} E_Y$, such that $E_Z \to Z$ is a fibration that restricts to the fibration $E_{\partial Z} \to \partial Z$, and $E_Y \to Y$ is a stratified system of fibrations. Then we consider the map

$$\pi_1 E = \pi_1 E_Z *_{\pi_1 E_{\partial Z}} \pi_1 E_Y \to \pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y.$$

If the fibres of $E \to X$ are connected, then $\pi_1 E_Z \to \pi_1 Z$ and $\pi_1 E_{\partial Z} \to \pi_1 \partial Z$ are surjective by the genuine fibration case, and $\pi_1 E_Y \to \pi_1 Y$ is surjective by induction. Therefore the map $\pi_1 E \to \pi_1 X$ is surjective. If the fibres of $E \to Z$ are connected and simply connected, then all the maps are isomorphisms, so that $\pi_1 E \to \pi_1 X$ is an isomorphism.

The proof makes use of the Van-Kampen theorem, which requires Y to be connected (which further implies that ∂Z is connected). In general, the argument can be carried out by successively adding connected components of Y to Z. □

Proposition 3.4. *If X is a homotopically stratified space, such that all pure strata are connected, and all links are not empty, then X is connected. Moreover, if all pure strata are connected and simply connected, and all links are connected, then X is simply connected.*

We remark that a link L of a stratum X_β in another stratum X_α is stratified, with strata L_γ corresponding to the strata X_γ satisfying $X_\beta \subsetneq X_\gamma \subset X_\alpha$. Moreover, the link of L_γ in $L_{\gamma'}$ is the same as the link of X_γ in $X_{\gamma'}$. The proposition implies that, if the pure strata of the link between any two strata sandwiched between X_β and X_α are (nonempty and) connected and simply connected, then the link of X_β in X_α is simply connected.

Proof. If the links are not empty, then any pure stratum is glued to higher pure strata. Therefore the connectivity of all pure strata implies the connectivity of the union, which is the whole X.

Now assume that all pure strata are connected and simply connected, and all links are connected. Let Y be a minimum stratum. Then we have decomposition $X = Z \cup_{\partial Z} Y$ similar to the proof of Proposition 3.3. The complement Z of a regular neighborhood of Y is a stratified space, with the pure strata the same as the pure strata of X, except the stratum Y. Moreover, the links in Z are the same as the links in X. By induction, we may assume that Z (which has one less stratum than X) is simply connected. Moreover, Y is a pure stratum and is already assumed to be simply connected. If we know that ∂Z is connected, then we can apply Van-Kampen theorem and conclude that $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y$ is trivial.

To see that ∂Z is connected, we note that the base of the fibration $\partial Z \to Y$ is connected. So it is sufficient to show that the fibre L of the fibration is also connected. The fibre is the link L of Y in X, and is a stratified space with one less stratum than X. Moreover, L has the same link as X. Since all pure strata of X are connected, by the first part of the proposition, L is connected. □

Proposition 3.5. *Suppose X is a homotopically stratified space, and Y is a closed union of strata of X. If for any link between strata of X, those pure strata of the link that are not contained in Y are connected and simply connected, then $\pi_1(X - Y) = \pi_1 X$.*

Proof. We have decomposition $X = Z \cup_{\partial Z} Y$ similar to the proof of Proposition 3.3. The fibre of the stratified system of fibrations $\partial Z \to Y$ is a stratified space L_y depending on the location of the point $y \in Y$. If Y^y is the pure stratum containing y, then the pure strata of L_y are the pure strata of the link of Y^y in X that are not contained in Y. By Proposition 3.4 and the remark afterwards, the assumption of the proposition implies that L_y is connected and simply connected. Then we may apply Proposition 3.3 to get $\pi_1 \partial Z = \pi_1 Y$. Further application of the Van-Kampen theorem gives us $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y = \pi_1 Z = \pi_1(X - Y)$. □

4 General Splitting Theorem

The homotopy properties in the last section fit into a general scheme for splitting the structure set of certain stratified spaces. We will use the spectra version of the surgery obstruction, homology and structure set. The equality of spectra really means homotopy equivalence.

Theorem 4.1. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space, satisfying the following properties:*

1. *The link of X_{-1} in X is homotopic to $\mathbb{C}P^r$ with even r.*

2. *The link fibration of X^{-1} in X is orientable.*

3. *For any i, the top two pure strata of the link of X_{-i} in X are connected and simply connected.*

Then there is a natural homotopy equivalence of surgery obstructions

$$\mathbb{L}(X) = \mathbb{L}(X, \mathrm{rel}\, X_{-2}) \oplus \mathbb{L}(X_{-2}).$$

Moreover, we have

$$\mathbb{L}(X, \mathrm{rel}\, X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}),$$

and

$$\pi_1 X = \pi_1(X - X_{-1}) = \pi_1 \bar{X}^0, \quad \pi_1 X_{-1} = \pi_1 X^{-1} = \pi_1 \partial \bar{X}^0.$$

It will be clear from the subsequent argument that $\mathbb{C}P^r$ is only used to get the periodicity for the classical surgery obstructions. Therefore it can be replaced by any orientable manifold of signature one.

To prove the theorem, we first establish the following result, which is essentially a reformulation of the periodicity for the classical surgery obstruction [14, Theorem 9.9].

Proposition 4.2. *Suppose X is a two-strata space, such that the link fibration of X_{-1} in X is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$ with even r. Then*

$$\mathbb{L}(X) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}), \quad \pi_1 X = \pi_1(X - X_{-1}).$$

Proof. Let Z be the complement of a regular neighborhood of X_{-1} in X. Let E be the boundary of Z as well as boundary of the regular neighborhood. Then $X = Z \cup E \times [0,1] \cup X_{-1}$, and $E \to X_{-1}$ is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$.

The surgery obstruction $\mathbb{L}(X)$ of the two-strata space X fits into a fibration

$$\mathbb{L}(E \times [0,1] \cup X_{-1}) \to \mathbb{L}(X) \to \mathbb{L}(Z, E),$$

where the mapping cylinder $E \times [0,1] \cup X_{-1}$ is a regular neighborhood of X_{-1} in X and is a two-strata space with X_{-1} as the lower stratum. The surgery obstruction of the mapping cylinder further fits into a fibration

$$\mathbb{L}(E \times [0,1] \cup X_{-1}) \xrightarrow{\mathrm{res}} \mathbb{L}(X_{-1}) \xrightarrow{\mathrm{trf}} \mathbb{L}(E),$$

given by the restriction and the transfer. Since $E \to X_{-1}$ is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$ with even r, by the fibration version of the classical periodicity for the surgery obstruction [10, 11], the transfer map is a homotopy equivalence. Therefore the second fibration tells us that $\mathbb{L}(E \times [0,1] \cup X_{-1})$ is contractible, and then the first fibration tells us that the spectra $\mathbb{L}(X)$ and $\mathbb{L}(Z, E)$ are homotopy equivalent.

It remains to compute $\mathbb{L}(Z, E)$. The fibration $\mathbb{C}P^r \to E \to X_{-1}$ implies $\pi_1 E = \pi_1 X_{-1}$. By Van-Kampen theorem, we have $\pi_1 X = \pi_1 Z *_{\pi_1 E} \pi_1 X_{-1} = \pi_1 Z = \pi_1(X - X_{-1})$. □

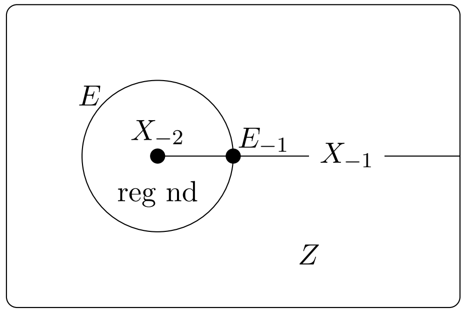

Figure 1: Regular neighborhood of X_{-2} in X

Proof of Theorem 4.1. Let Z be the complement of a regular neighborhood of X_{-2} in X. Let E be the boundary of the regular neighborhood. Then Z and E are two-strata spaces with lower strata $Z_{-1} = Z \cap X_{-1}$ and $E_{-1} = E \cap X_{-1}$. Moreover, E is the boundary of Z in the sense that E has a collar neighborhood in Z. We will use Z and E to denote the two-strata spaces, and use (Z, E) to denote the space Z considered as a four-strata space, in which the two-strata of E are also counted.

Consider a commutative diagram of natural maps of surgery obstructions.

$$\begin{array}{ccccc} \mathbb{L}(Z) & \xrightarrow{\simeq} & \mathbb{L}(Z, E) & \longrightarrow & \mathbb{L}(E) \\ \downarrow \simeq & & \uparrow & & \\ \mathbb{L}(X, \mathrm{rel}\, X_{-2}) & \longrightarrow & \mathbb{L}(X) & \longrightarrow & \mathbb{L}(X_{-2}) \end{array}$$

Both horizontal lines are fibrations of spectra. The vertical $\simeq$ is due to the fact that the inclusion $Z \to X - X_{-2}$ of two-strata spaces is a stratified homotopy equivalence. The horizontal $\simeq$ will be a consequence of the fact that $\mathbb{L}(E)$ is homotopically trivial. The two equivalences give natural splitting to the map $\mathbb{L}(X, \mathrm{rel}\, X_{-2}) \to \mathbb{L}(X)$. Then the bottom fibration implies $\mathbb{L}(X)$ is naturally homotopy equivalent to $\mathbb{L}(X, \mathrm{rel}\, X_{-2}) \oplus \mathbb{L}(X_{-2})$.

To see the triviality of $\mathbb{L}(E)$, we note that the link of E_{-1} in E is the same as the link $\mathbb{C}P^r$ of X_{-1} in X. Therefore we may apply Proposition 4.2 to E and get

$$\mathbb{L}(E) = \mathbb{L}(\pi_1(E - E_{-1}), \pi_1 E_{-1}).$$

Let L be the link of X_{-i} in X, then we have stratified systems of fibrations

$$L - L_{-1} \to E - E_{-1} \to X_{-2}, \quad L_{-1} - L_{-2} \to E_{-1} \to X_{-2}.$$

By the third condition, the fibres are always connected and simply connected, and we may apply Proposition 3.3 to get $\pi_1(E - E_{-1}) = \pi_1 E_{-1} = \pi_1 X_{-2}$. By the π-π theorem of the classical surgery theory, we conclude that $\mathbb{L}(E)$ is homotopically trivial.

Like E, the link of Z_{-1} in Z is also the same as the link $\mathbb{C}P^r$ of X_{-1} in X. Then Proposition 4.2 tells us

$$\mathbb{L}(X, \mathrm{rel}\, X_{-2}) = \mathbb{L}(Z) = \mathbb{L}(\pi_1 Z, \pi_1 Z_{-1}).$$

By $Z \simeq X - X_{-2}$, $Z_{-1} \simeq X_{-1} - X_{-2} = X^{-1}$ and Lemma 3.2, we have

$$\pi_1 Z = \pi_1(X - X_{-2}) = \pi_1(X - X_{-1}) = \pi_1 X, \quad \pi_1 Z_{-1} = \pi_1 X^{-1} = \pi_1 X_{-1}.$$

By $X - X_{-1} \simeq \bar{X}^0$ and applying Proposition 3.3 to $\partial \bar{X}^0 \to X_{-1}$, which is a stratified system of fibrations with the top strata of X_{-i} in X as fibres, we get

$$\pi_1 Z = \pi_1 \bar{X}^0, \quad \pi_1 Z_{-1} = \pi_1 \partial \bar{X}^0. \qquad \square$$

The natural splitting for the surgery obstruction in Theorem 4.1 induces similar natural splitting for the structure set.

Theorem 4.3. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space, satisfying the following properties:*

1. *The link of X_{-1} in X is homotopic to $\mathbb{C}P^r$ with even r.*
2. *The link fibration of X^{-1} in X is orientable.*
3. *The pure strata of all links are connected and simply connected..*

Then there is a natural homotopy equivalence of structure sets

$$\mathbb{S}(X) = \mathbb{S}(X, \operatorname{rel} X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Moreover, we have

$$\mathbb{S}(X, \operatorname{rel} X_{-2}) = \mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}^{\text{alg}}(X, X_{-1}).$$

The third condition can be replaced by the (weaker) third condition in Theorem 4.1, plus the requirement that the fundamental groups of the pure strata of all links have trivial K-theory.

Proof. By the topological h-cobordism theory [12, 13], the third condition implies that the neighborhoods of strata have block bundle structure, the stratified space can be considered as being of the "PT category", and the structure set can be computed by the "unstable surgery fibration" [15, Chapter 8]

$$\mathbb{S}(X) \to \mathbb{H}(X; \mathbb{L}(\operatorname{loc} X)) \to \mathbb{L}(X).$$

By Theorem 4.1, we have natural splitting of the surgery spectra

$$\mathbb{L}(X) = \mathbb{L}(X, \operatorname{rel} X_{-2}) \oplus \mathbb{L}(X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}) \oplus \mathbb{L}(X_{-2}).$$

Since the splitting is natural, it can be applied to the coefficient $\mathbb{L}(\operatorname{loc} X)$ in the homology and induces compatible assembly maps

$$\mathbb{H}(X; \mathbb{L}(\operatorname{loc}(X, \operatorname{rel} X_{-2}))) \to \mathbb{L}(X, \operatorname{rel} X_{-2}), \quad \mathbb{H}(X; \mathbb{L}(\operatorname{loc} X_{-2})) \to \mathbb{L}(X_{-2}).$$

The stratified surgery theory tells us that the homotopy fibre of the first assembly map is the structure set $\mathbb{S}(X, \operatorname{rel} X_{-2})$. Moreover, we have $\mathbb{H}(X; \mathbb{L}(\operatorname{loc} X_{-2})) = \mathbb{H}(X_{-2}; \mathbb{L}(\operatorname{loc} X_{-2}))$ because the coefficient spectrum $\mathbb{L}(\operatorname{loc} X_{-2})$ is concentrated on X_{-2}. Therefore the homotopy fibre of the second assembly map is the structure set $\mathbb{S}(X_{-2})$. Then we have the decomposition of $\mathbb{S}(X)$ as stated in the theorem.

It remains to compute $\mathbb{S}(X, \operatorname{rel} X_{-2})$. The coefficient $\mathbb{L}(\operatorname{loc}(X, \operatorname{rel} X_{-2}))$ of the homology depends on the location.

1. At $x \in X^0 = X - X_{-1}$, the coefficient is $\mathbb{L}(D^p) = \mathbb{L}(e)$, where D^p is a ball neighborhood of x in the manifold pure stratum X^0.

2. At $x \in X^{-1}$, the coefficient is $\mathbb{L}(c\mathbb{C}P^r \times D^p)$, where $c\mathbb{C}P^r$ is the cone on the link of X_{-1} in X, and D^p is a ball neighborhood of x in the manifold pure stratum X^{-1}. Since r is even, the surgery obstruction $\mathbb{L}(c\mathbb{C}P^r \times D^p)$ is contractible by Proposition 4.2.

3. At $x \in X_{-2}$, we have $x \in X^{-i}$ for some $i \geq 2$. Let L be the link of X_{-i} in X, and let D^p be a ball neighborhood of x in the manifold pure stratum X^{-i}. Then the coefficient is

$$\begin{aligned}\mathbb{L}(cL \times D^p, \operatorname{rel} cL_{-2} \times D^p) &= \mathbb{L}(cL \times D^p - c \times D^p, \operatorname{rel} cL_{-2} \times D^p - c \times D^p) \\ &= \mathbb{L}(L \times [0,1] \times D^p, \operatorname{rel} L_{-2} \times [0,1] \times D^p) \\ &= \Omega^{p+1}\mathbb{L}(L, \operatorname{rel} L_{-2}).\end{aligned}$$

   We may apply Theorem 4.1 to get $\mathbb{L}(L, \operatorname{rel} L_{-2}) = \mathbb{L}(\pi_1 L^0, \pi_1 L^{-1})$. By the third condition, the pure strata L^0 and L^{-1} are connected and simply connected. Therefore the surgery obstruction is contractible.

Thus the coefficient is the surgery obstruction spectrum $\mathbb{L} = \mathbb{L}(e)$ on the top pure stratum $X^0 = X - X_{-1}$ and is trivial on X_{-1}. Therefore the homology is

$$\mathbb{H}(X; \mathbb{L}(\operatorname{loc}(X, \operatorname{rel} X_{-2}))) = \mathbb{H}(X, X_{-1}; \mathbb{L}).$$

Moreover, Theorem 4.1 tells us

$$\mathbb{L}(X, \operatorname{rel} X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

Therefore the homotopy fibre of the assembly map is $\mathbb{S}^{\text{alg}}(X, X_{-1})$.

By excision, we have $\mathbb{H}(X, X_{-1}; \mathbb{L}) = \mathbb{H}(\bar{X}^0, \partial\bar{X}^0; \mathbb{L})$. By Theorem 4.1, we also know $\mathbb{L}(\pi_1 X, \pi_1 X_{-1}) = \mathbb{L}(\pi_1 \bar{X}^0, \pi_1 \partial\bar{X}^0)$. Therefore the homotopy fibre of the assembly map is also the structure set $\mathbb{S}(\bar{X}^0, \partial\bar{X}^0)$ of the manifold $(\bar{X}^0, \partial\bar{X}^0)$. □

We note that, in the setup of Theorem 4.3, the restriction to X_{-2} factors through X_{-1}. Then the fact that the restriction $\mathbb{S}(X) \to \mathbb{S}(X_{-2})$ is naturally split surjective implies that the restriction $\mathbb{S}(X_{-1}) \to \mathbb{S}(X_{-2})$ is also naturally split surjective, and we get

$$\mathbb{S}(X_{-1}) = \mathbb{S}(X_{-1}, \operatorname{rel} X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Another way of looking at this is that, if a stratified space X is the singular part of a stratified space Y satisfying the conditions of Theorem 4.3, i.e., $X = Y_{-1}$, then we have the natural splitting

$$\mathbb{S}(X) = \mathbb{S}(X, \operatorname{rel} X_{-1}) \oplus \mathbb{S}(X_{-1}).$$

The following computes $\mathbb{S}(X, \operatorname{rel} X_{-1})$ for the case relevant to the multiaxial manifolds.

Theorem 4.4. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space, such that for any i, the top pure stratum of the link of X_{-i} in X are connected and simply connected. Then*

$$\mathbb{S}(X, \operatorname{rel} X_{-1}) = \mathbb{S}^{\text{alg}}(X).$$

Proof. Similar to the proof of Theorem 4.3, the simple connectivity assumption implies that the structure set $\mathbb{S}(X, \mathrm{rel}\, X_{-1})$ is the homotopy fibre of the assembly map

$$\mathbb{H}(X; \mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\, X_{-1}))) \to \mathbb{L}(X, \mathrm{rel}\, X_{-1}),$$

and the coefficient $\mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\, X_{-1})) = \mathbb{L}$. We also get $\pi_1(X - X_{-1}) = \pi_1 X$ from Proposition 3.5. Therefore the assembly map is $\mathbb{H}(X; \mathbb{L}) \to \mathbb{L}(\pi_1 X)$, and the homotopy fibre is $\mathbb{S}^{\mathrm{alg}}(X)$. □

5 Structure Set of Multiaxial Action

Let M be a multiaxial $U(n)$-manifold. By Lemma 3.2, the pure strata of links in the orbit space $X = M/U(n)$ are all connected and simply connected. Since $U(n)$ is connected, the link fibrations are always orientable.

Recall the concept of the first gap defined after the statement of Lemma 3.1. The number $r = r_1^x$ depends only on the connected component of the singular part X_{-1}. For any connected component X_{-1}^x, the number is characterized as the link of X_{-1}^x in X being homotopy to $\mathbb{C}P^r$. The number is also characterized by the equality $\dim M^{U(j-1),x} - \dim M^{U(j),x} = 2(r+n)$.

It is easy to see that Theorem 4.3 remains true in case X_{-1} has several connected components, and perhaps with different $\mathbb{C}P^r$ for different components, as long as all r are even. Therefore if all the first gaps of a multiaxial $U(n)$-manifold M are even, then we have natural splitting

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2}) \oplus \mathbb{S}_{U(n)}(M_{-2}).$$

By the computation in Theorem 4.3, we have

$$\mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2}) = \mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}^{\mathrm{alg}}(X, X_{-1}).$$

By deleting an equivariant regular neighborhood of $M_{-1} = U(n) \times_{U(n-1)} M^{U(1)}$ from M, we get a free $U(n)$-manifold with boundary $(\bar{M}^0, \partial \bar{M}^0)$, and

$$\mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}_{U(n)}(\bar{M}^0, \partial \bar{M}^0).$$

On the other hand, by Lemma 2.1, we have $\mathbb{S}_{U(n)}(M_{-2}) = \mathbb{S}_{U(n-2)}(M^{U(2)})$, where $M^{U(2)}$ is a multiaxial $U(n-2)$-manifold. Moreover, Lemma 3.1 further tells us that, for $x \in M^{U(i)}$, $i > 2$, the first gap of x in $M^{U(2)}$ is $r_3^x = r_1^x + 2$, where r_1^x is the first gap of x in M. This can also be seen from

$$\begin{aligned} \dim M^{U(j-3),x} - \dim M^{U(j-2),x} &= \dim(M^{U(2)})^{U(j-1),x} - \dim(M^{U(2)})^{U(j),x} \\ &= 2(r_3^x + (n-2)), \end{aligned}$$

where we use $n-2$ on the right because $M^{U(2)}$ is a multiaxial $U(n-2)$-manifold. The upshot of this is that all the first gaps of $M^{U(2)}$ remain even, and we have further natural splitting

$$\mathbb{S}_{U(n)}(M_{-2}) = \mathbb{S}_{U(n-2)}(M^{U(2)}) = \mathbb{S}_{U(n-2)}(M^{U(2)}, \mathrm{rel}\, M_{-2}^{U(2)}) \oplus \mathbb{S}_{U(n-2)}(M_{-2}^{U(2)}).$$

Moreover, we have

$$\mathbb{S}_{U(n-2)}(M^{U(2)}, \mathrm{rel}\, M_{-2}^{U(2)}) = \mathbb{S}_{U(n-2)}(\bar{M}^{U(2)}, \partial\bar{M}^{U(2)}) = \mathbb{S}^{\mathrm{alg}}(X_{-2}, X_{-3}),$$

and

$$\mathbb{S}_{U(n-2)}(M_{-2}^{U(2)}) = \mathbb{S}_{U(n-4)}(M^{U(4)}).$$

The splitting continues and gives us the general version of the first part of Theorem 1.1 in the introduction. The mod 4 condition is a rephrasement of the even first gap.

Theorem 5.1. *Suppose M is a multiaxial $U(n)$-manifold, such that the dimension of any connected component of $M^{U(1)}$ is $\dim M - 2n$ mod 4. Then we have natural splitting*

$$\mathbb{S}_{U(n)}(M) = \oplus_{i\geq 0}\mathbb{S}_{U(n-2i)}(\bar{M}^{U(2i)}, \partial\bar{M}^{U(2i)}) = \oplus_{i\geq 0}\mathbb{S}^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}).$$

In general, a multiaxial manifold may have even as well as odd first gaps. Denote by $M_{\mathrm{even}}^{U(1)}$ the union of the connected components of $M^{U(1)}$ of dimension $\dim M - 2n$ mod 4. Denote by $M_{\mathrm{odd}}^{U(1)}$ the union of the connected components of $M^{U(1)}$ of dimension $\dim M - 2(n+1)$ mod 4. Then we have

$$M^{U(i)} = M_{\mathrm{even}}^{U(i)} \cup M_{\mathrm{odd}}^{U(i)}, \quad M_{\mathrm{even}}^{U(i)} = M^{U(i)} \cap M_{\mathrm{even}}^{U(1)}, \quad M_{\mathrm{odd}}^{U(i)} = M^{U(i)} \cap M_{\mathrm{odd}}^{U(1)},$$

such that the components in $M_{\mathrm{even}}^{U(i)}$ have even first gaps, and the components in $M_{\mathrm{odd}}^{U(i)}$ have odd first gaps. This leads to

$$M_{-i,\mathrm{even}} = U(n) \times_{U(n-i)} M_{\mathrm{even}}^{U(i)}, \quad M_{-i,\mathrm{odd}} = U(n) \times_{U(n-i)} M_{\mathrm{odd}}^{U(i)}.$$

We also have the corresponding decompositions

$$X_{-i} = X_{-i,\mathrm{even}} \cup X_{-i,\mathrm{odd}}, \quad \bar{M}^{U(i)} = \bar{M}_{\mathrm{even}}^{U(i)} \cup \bar{M}_{\mathrm{odd}}^{U(i)}.$$

By the same proof as Theorem 5.1, we get the same natural splitting for those with even first gaps

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) \oplus \mathbb{S}^{\mathrm{alg}}(X_{-2,\mathrm{even}})$$

Here the multiaxial $U(n-2)$-manifold $M_{\mathrm{even}}^{U(2)}$ satisfies the condition of Theorem 5.1, so that the second factor can be further split

$$\mathbb{S}^{\mathrm{alg}}(X_{-2,\mathrm{even}}) = \oplus_{i\geq 1}\mathbb{S}^{\mathrm{alg}}(X_{-2i,\mathrm{even}}, X_{-2i-1,\mathrm{even}}).$$

In terms of the multiaxial manifold, this splitting is

$$\mathbb{S}_{U(n-2)}(M_{\mathrm{even}}^{U(2)}) = \oplus_{i\geq 1}\mathbb{S}_{U(n-2i)}(\bar{M}_{\mathrm{even}}^{U(2i)}, \partial\bar{M}_{\mathrm{even}}^{U(2i)}).$$

On the other hand, the first factor

$$\mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) = \mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2,\mathrm{even}}).$$

Let N_{even} and N_{odd} be equivariant neighborhoods of $M_{-1,\mathrm{even}}$ and $M_{-1,\mathrm{odd}}$. Then by the same proof as Theorem 5.1, we have

$$\mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2,\mathrm{even}}) = \mathbb{S}_{U(n)}(\overline{M - N_{\mathrm{even}}}, \partial N_{\mathrm{even}}).$$

Combining everything, we get the following decomposition.

Theorem 5.2. *Suppose M is a multiaxial $U(n)$-manifold. Then we have natural splitting*

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\text{alg}}(X, \operatorname{rel} X_{-2,\text{even}}) \oplus \left(\oplus_{i\geq 1}\mathbb{S}^{\text{alg}}(X_{-2i,\text{even}}, X_{-2i-1,\text{even}})\right).$$

Moreover,

$$\mathbb{S}^{\text{alg}}(X, \operatorname{rel} X_{-2,\text{even}}) = \mathbb{S}_{U(n)}(\overline{M - N_{\text{even}}}, \partial N_{\text{even}})$$

and

$$\mathbb{S}^{\text{alg}}(X_{-2i,\text{even}}, X_{-2i-1,\text{even}}) = \mathbb{S}_{U(n-2i)}(\bar{M}^{U(2i)}_{\text{even}}, \partial\bar{M}^{U(2i)}_{\text{even}}).$$

In the theorem above, $U(n-2i)$ acts freely on $\bar{M}^{U(2i)}_{\text{even}}$, and the structure set is about the ordinary manifold $\bar{M}^{U(2i)}_{\text{even}}/U(n-2i)$. In the first factor $S_{U(n)}(\overline{M - N_{\text{even}}}, \partial N_{\text{even}})$, all the gaps in the multiaxial $U(n)$-manifold $\overline{M - N_{\text{even}}}$ are odd.

Assume M is a multiaxial $U(n)$-manifold, such that all the first gaps are odd. We may use the idea presented before Theorem 4.4. Suppose $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W. Let $Y = W/U(n+1)$ be the orbit space of W. Then $X_{-i} = Y_{-i-1}$. By Lemma 3.1, for any $x \in X_{-1} = Y_{-2}$, the first gap of x in Y is one less than the first gap of x in X. Therefore the first gap of x in Y is even, and the natural splitting of $\mathbb{S}(Y)$ induces the natural splitting

$$\mathbb{S}(X) = \mathbb{S}(X, \operatorname{rel} X_{-1}) \oplus \mathbb{S}(X_{-1}).$$

Since the first gap in the $U(n-1)$-manifold $M^{U(1)}$ is one more than the first gap in M and is therefore also even, we may apply Theorem 5.1 to get further natural splitting

$$\mathbb{S}(X_{-1}) = \oplus_{i\geq 0}\mathbb{S}^{\text{alg}}(X_{-2i-1}, X_{-2i-2}).$$

On the other hand, by the computation in Theorem 4.4, the first factor is

$$\mathbb{S}(X, \operatorname{rel} X_{-1}) = \mathbb{S}^{\text{alg}}(X).$$

Then we get the general version of the second part of Theorem 1.1 in the introduction.

Theorem 5.3. *Suppose M is a multiaxial $U(n)$-manifold, such that the dimension of $M^{U(1)}$ is $\dim M - 2(n+1)$ mod 4. If $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W, then we have natural splitting*

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\text{alg}}(X) \oplus \left(\oplus_{i\geq 0}\mathbb{S}^{\text{alg}}(X_{-2i-1}, X_{-2i-2})\right).$$

Moreover,

$$\mathbb{S}^{\text{alg}}(X_{-2i-1}, X_{-2i-2}) = \mathbb{S}_{U(n-2i-1)}(\bar{M}^{U(2i+1)}, \partial\bar{M}^{U(2i+1)}).$$

We remark that, if $M = W^{U(1)}$ and M is connected, then there is only one first gap r in M, uniquely determined by

$$\dim W - \dim M = 2(r + n + 1).$$

In case r is odd, there is actually no $M^{U(1)}_{\text{even}}$.

There is another case that we can split off $\mathbb{S}(X_{-1})$ from $\mathbb{S}(X)$ with all the first gaps odd but not necessarily unique. If $n = 1$, then multiaxial $U(1)$-manifolds are simply semi-free S^1-manifolds. In this case, $M_{-1} = M^{S^1}$ is the fixed point of the action. Moreover,

$$M_0^{S^1} = M_{\text{odd}}^{S^1}$$

consists of those connected components of codimension being multiples of 4, and

$$M_2^{S^1} = M_{\text{even}}^{S^1}$$

consists of those connected components of codimension being 2 mod 4. Correspondingly, we have equivariant neighborhoods N_0 and N_2 of $M_0^{S^1}$ and $M_2^{S^1}$. Then Theorem 5.2 simply tells us

$$\mathbb{S}_{S^1}(M) = \mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2).$$

Now the fixed points $\overline{M - N_2}^{S^1} = M_0^{S^1}$ has codimension (perhaps different codimension for different components) being a multiple of 4. Therefore we can invoke the replacement theorem [5, Theorem 2.5], which essentially says that the natural map

$$\mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2) \to \mathbb{S}_{S^1}(\overline{M - N_2}^{S^1}) = \mathbb{S}_{S^1}(M_0^{S^1})$$

is split surjective. Since $\mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2, \text{rel}\, \partial N_0)$ is the kernel of the natural map, we get Theorem 1.2 in the introduction.

6 Structure Set of Multiaxial Representation Sphere

Let ρ_n be the defining representation of $U(n)$. Let ϵ be the real 1-dimensional trivial representation. Then for any natural number k, the unit sphere

$$M = S(k\rho_n \oplus j\epsilon) = S(k\rho_n) * S^{j-1}$$

of the representation $k\rho_n \oplus j\epsilon$ is a multiaxial $U(n)$-manifold. In this section, we compute the structure set of this representation sphere.

If $k < n$, then $M = U(n) \times_{U(k)} S(k\rho_k \oplus j\epsilon)$, and the problem is reduced to the $U(k)$-representation sphere $S(k\rho_k \oplus j\epsilon)$. Without loss of generality, therefore, we will always assume $k \geq n$ in the subsequent discussion.

The fixed point subsets are

$$M^{U(i)} = S(k\rho_n^{U(i)} \oplus j\epsilon) = S(k\rho_{n-i} \oplus j\epsilon) = S(k\rho_{n-i}) * S^{j-1}.$$

We have

$$\dim M^{U(i)} = 2k(n-i) - 1 + j, \quad \dim M^{U(i-1)} - \dim M^{U(i)} = 2k.$$

Therefore the first gap is $k - n$. If $k - n$ is even, then we can use Theorem 5.1 to compute the structure set. If $k-n$ is odd, then we may use $M = S(k\rho_{n+1} \oplus j\epsilon)^{U(1)}$, where $k-(n+1)$ is even, so that Theorem 5.3 can be applied.

We first assume $k - n$ is even and compute the top piece $S^{\text{alg}}(X, X_{-1})$ in the decomposition for $S(X) = S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ in Theorem 5.1. Since the representation sphere is the link of the origin in the representation space $k\rho_n \oplus j\epsilon = \mathbb{C}^{kn} \oplus \mathbb{R}^j$, by Lemma 3.2, both X and X_{-1} are connected and simply connected. If the action is neither trivial nor free, then we have $X_{-1} \neq \emptyset$, and the surgery obstruction $\mathbb{L}(\pi_1 X, \pi_1 X_{-1}) = \mathbb{L}(e, e)$ is trivial. Therefore the top piece is the same as the homology

$$\mathbb{S}^{\text{alg}}(X, X_{-1}) = \mathbb{H}(X, X_{-1}; \mathbb{L}).$$

Let

$$Y = S(k\rho_n)/U(n), \quad d = \dim Y = 2kn - 1 - n^2.$$

Then

$$(X, X_{-1}) = (Y, Y_{-1}) * S^{j-1}, \quad \dim X = d + j,$$

and

$$S^{\text{alg}}(X, X_{-1}) = \pi_{d+j} \mathbb{S}^{\text{alg}}(X, X_{-1}) = \pi_{d+j} \mathbb{H}(X, X_{-1}; \mathbb{L}) = H_d(Y, Y_{-1}; \mathbb{L}).$$

Proposition 6.1. *If $k \geq n$, then for $Y = S(k\rho_n)/U(n)$, we have*

$$H_{\dim Y}(Y, Y_{-1}; \mathbb{L}) = \mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}},$$

where $A_{n,k}$ is the number of n-tuples $(\mu_1, \dots, \mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n, \quad \sum \mu_i \text{ is even},$$

and $B_{n,k}$ is the number of n-tuples satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n, \quad \sum \mu_i \text{ is odd}.$$

Proof. The homology can be computed by a spectral sequence

$$E^2_{p,q} = H_p(Y, Y_{-1}; \pi_q \mathbb{L}(e)) = \begin{cases} H_p(Y, Y_{-1}; \mathbb{Z}), & \text{if } q = 0 \bmod 4, \\ H_p(Y, Y_{-1}; \mathbb{Z}_2), & \text{if } q = 2 \bmod 4, \\ 0, & \text{if } q \text{ is odd}. \end{cases}$$

Since the top pure stratum $Y - Y_{-1}$ is a manifold, by the Poincaré duality, we have $H_p(Y, Y_{-1}; R) = H^{d-p}(Y - Y_{-1}; R)$. The homotopy type of $Y - Y_{-1}$ is well known to be the complex Grassmanian $G(n, k)$. Therefore we have

$$E^2_{p,q} = \begin{cases} H^{d-p}(G(n,k); \mathbb{Z}), & \text{if } q = 0 \bmod 4, \\ H^{d-p}(G(n,k); \mathbb{Z}_2), & \text{if } q = 2 \bmod 4, \\ 0, & \text{if } q \text{ is odd}. \end{cases}$$

Using the CW structure of $G(n, k)$ given by the Schubert cells, which are all even dimensional, $E^2_{p,q}$ vanishes when either q or $d - p$ is odd. This implies that all the differentials in $E^2_{p,q}$ vanish, so that the spectral sequence collapses, and we get

$$H_d(Y, Y_{-1}; \mathbb{L}) = \left(\oplus_{q \leq d,\ q=0(4)} H^q(G(n,k); \mathbb{Z})\right) \oplus \left(\oplus_{q \leq d,\ q=2(4)} H^q(G(n,k); \mathbb{Z}_2)\right).$$

Since $G(n,k)$ is a closed manifold, we always have $q \le \dim G(n,k) \le \dim Y = d$. Of course this is nothing but $q \le 2n(k-n) \le 2kn - 1 - n^2 = d$. Therefore the requirement $q \le d$ is automatically satisfied in the summation above, and we have

$$H_d(Y, Y_{-1}; \mathbb{L}) = \mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}},$$

where $A_{n,k}$ is the number of Schubert cells in $G(n,k)$ of dimension 0 mod 4, and $B_{n,k}$ is the number of Schubert cells of dimension 2 mod 4. The description of $A_{n,k}$ and $B_{n,k}$ in the proposition is the well known numbers of such Schubert cells. □

The unitary group $U(n)$ acts trivially on $S(k\rho_n \oplus j\epsilon)$ only when $n = 0$ and $j > 0$. In this case, we have $S^{\text{alg}}(X, X_{-1}) = S^{\text{alg}}(X) = S(S^{j-1})$. (Here the first S in $S(S^{j-1})$ means the structure set, not the sphere.) By Poincaré conjecture, the structure set of the sphere is trivial. This means that we should require $n > 0$ in the notation $\mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}}$.

The action is free only when $n = 1$ and $j = 0$. In this case, we have $S^{\text{alg}}(X, X_{-1}) = S^{\text{alg}}(X) = S(\mathbb{C}P^{k-1})$. The homology is still $\mathbb{Z}^{A_{1,k}} \oplus \mathbb{Z}_2^{B_{1,k}}$. But the surgery obstruction is $L_{2(k-1)}(\pi_1 X, \pi_1 X_{-1}) = L_{2(k-1)}(\pi_1 X) = L_{2(k-1)}(e) = \mathbb{Z}$. Here we recall that $k - 1 = k - n$ is assumed even. Since this piece of surgery obstruction is simply the summand $H^0(G(1,k); \mathbb{Z})$ in the computation of the homology, this reduces the number of copies of $\mathbb{Z}$ by 1. The computation is exactly the fake complex projective space studied in [14, Section 14C].

If $k - n$ is even, then Proposition 6.1 and the subsequent discussion about the exceptions can be applied to the pieces $S^{\text{alg}}(X_{-2i}, X_{-2i-1})$ in the decomposition for $S(X) = S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ in Theorem 5.1, simply by replacing n with $n - 2i$. The exception is that, in case n is odd and $j = 0$, the $U(1)$-action on $M^{U(n-1)}$ is free, so that $X_{-n} = \emptyset$. The exception happens to the last piece $S^{\text{alg}}(X_{-n+1}, X_{-n}) = S^{\text{alg}}(X_{-n+1}) = S^{\text{alg}}(\mathbb{C}P^{k-1})$, and the number of copies of $\mathbb{Z}$ is reduced by 1. This concludes the first part of Theorem 1.4.

If $k - n$ is odd, then Proposition 6.1 can be applied to all pieces except the top one in the decomposition for $S(X)$ in Theorem 5.3, simply by replacing n with $n - 2i - 1$. The exception is that, in case n is even and $j = 0$, the last piece is $S^{\text{alg}}(X_{-n+1}) = S^{\text{alg}}(\mathbb{C}P^{k-1})$, and the number of copies of $\mathbb{Z}$ should be reduced by 1. The top piece $S^{\text{alg}}(X)$ may be computed by the surgery fibration

$$\mathbb{S}^{\text{alg}}(X) \to \mathbb{H}(X; \mathbb{L}) \to \mathbb{L}(\pi_1 X).$$

Since X is simply connected, $\mathbb{L}(\pi_1 X)$ is the usual surgery specturm $\mathbb{L}$, and the assembly map is induced by the map from X to the single point. Therefore

$$S^{\text{alg}}(X) = \tilde{H}_{d+j}(X; \mathbb{L}) = \begin{cases} H_d(Y; \mathbb{L}), & \text{if } j > 0, \\ \tilde{H}_d(Y; \mathbb{L}), & \text{if } j = 0. \end{cases}$$

The reduced homology is given by Proposition A.1 of the appendix by Jared Bass. Since $k - n$ is odd, we have

$$\tilde{H}_d(Y; \mathbb{L}) = \mathbb{Z}^{A_{n,k-1}} \oplus \mathbb{Z}_2^{B_{n,k-1}}.$$

The unreduced homology is modified from the reduced one accordingly to Corollary A.2 of the appendix. This concludes the second part of Theorem 1.4.

7 Suspension of Multiaxial Representation Sphere

Let us first review the suspension map for the structure set of the complex projective space

$$*S(\rho_1)\colon S_{U(1)}(S(k\rho_1)) = S(\mathbb{C}P^{k-1}) \to S_{U(1)}(S((k+1)\rho_1)) = S(\mathbb{C}P^k).$$

Equivariantly, we have

$$S((k+1)\rho_1) = S(k\rho_1) \times D(\rho_1) \cup S(\rho_1).$$

The orbit space under the action of $U(1) = S^1$ is $\mathbb{C}P^k = E \cup pt$, where $pt = \mathbb{C}P^0$ is the base point and E is a disk bundle over $\mathbb{C}P^{k-1}$

$$D^2 \to E = (S(k\rho_1) \times D(\rho_1))/S^1 \xrightarrow{p} S(k\rho_1)/S^1 = \mathbb{C}P^{k-1}.$$

The fibre of the bundle is the unit disk of the "last" ρ_1 that we use to suspend. We know that $\mathbb{C}P^{k-1}$ has one cell B^{2i} at each even dimension $2i \leq 2(k-1)$, consisting of points of form $[0, \dots, 0, 1, *, \dots, *]$ of length k, with 1 in the $(k-i)$-th position. Then $E^{2(i+1)} = p^{-1}(B^{2i})$ is the corresponding cell of dimension $2(i+1)$ in $\mathbb{C}P^k$, consisting of points of form $[0, \dots, 0, 1, *, \dots, *, *]$ of length $k+1$, which is obtained by adding one more term (i.e., the last ρ_1) at the end. We also note that the base point $E^0 = pt$ is $[0, \dots, 0, 1]$.

The classical surgery sequence for the structure set (including homological manifolds) of the complex projective space is

$$0 \to S(\mathbb{C}P^k) \to [\mathbb{C}P^k; \mathbb{L}] = \oplus_{i\leq k} L_{2i}(e) \to L_{2k}(e) \to 0.$$

Here we view the normal invariants $[\mathbb{C}P^k; \mathbb{L}] = H^0(\mathbb{C}P^k; \mathbb{L})$ as the cohomology given by the spectrum $\mathbb{L}$. The suspension map on the normal invariant is then induced by the projection of the canonical unit disk bundle

$$H^0(\mathbb{C}P^{k-1}; \mathbb{L}) \xrightarrow{p^*} H^0(E; \mathbb{L}) \to H^0(\mathbb{C}P^k; \mathbb{L}).$$

If we apply the generalised Poincaré duality, then we get the surgery sequence in terms of the homology assembly map

$$0 \to S(\mathbb{C}P^k) \to H_{2k}(\mathbb{C}P^k; \mathbb{L}) \to L_{2k}(e) \to 0.$$

We have (from spectral sequence computation, for example)

$$H_{2k}(\mathbb{C}P^k; \mathbb{L}) = \oplus_{i\leq k} H_{2(k-i)}(E^{2(k-i)}, \partial E^{2(k-i)}; L_{2i}(e)) = \oplus_{i\leq k} L_{2i}(e),$$

or one copy of $L_{2i}(e) = \mathbb{Z}$ or $\mathbb{Z}_2$ associated to each cell $E^{2(k-i)}$. We have the similar computation

$$H_{2(k-1)}(\mathbb{C}P^{k-1}; \mathbb{L}) = \oplus_{i\leq k-1} H_{2(k-1-i)}(B^{2(k-1-i)}, \partial B^{2(k-1-i)}; L_{2i}(e)) = \oplus_{i\leq k-1} L_{2i}(e).$$

Then under the Poincaré duality, the map on cohomology induced by the projection p becomes the map on the homology that takes the copy of $L_{2i}(e)$ associated to $B^{2(k-1-i)}$

to the same copy of $L_{2i}(e)$ associated to $E^{2(k-i)}$. Combinatorially, we see that, in the homological computation, the suspension takes the copy of $L_{2i}(e)$ associated to the cell $B^{2i} = \{[0,\dots,0,1,*,\dots,*]\}$ to the copy of $L_{2i}(e)$ associated to the cell $E^{2(i+1)} = \{[0,\dots,0,1,*,\dots,*,*]\}$.

The picture we saw for the suspension of complex projective spaces carries to the suspension of multiaxial $U(n)$-spheres. To simplify the discussion, we assume $j = 0$. We have

$$S((k+1)\rho_n) = S(k\rho_n) \times D(\rho_n) \cup S(\rho_n), \quad S((k+1)\rho_n)/U(n) = E \cup pt,$$

where E is a stratified system of bundles over $S(k\rho_n)/U(n)$

$$D(\rho_n)/G_x \to E = (S(k\rho_n) \times D(\rho_n))/U(n) \xrightarrow{p} S(k\rho_n)/U(n).$$

A CW structure of the orbit space $S(k\rho_n)/U(n)$ is given by Jared Bass in the proof of Proposition A.1. We have one cell $B(m_1,\dots,m_r)$ corresponding to each $n \times k$ row echelon form, with the label satisfying

$$k \geq m_1 > \cdots > m_r > 0, \quad r \leq n.$$

Using the same CW structure for $S((k+1)\rho_n)/U(n)$, we have

$$p^{-1}(B(m_1,\dots,m_r)) = \begin{cases} B(m_1+1,\dots,m_r+1) \cup B(m_1+1,\dots,m_r+1,1), & \text{if } r < n, \\ B(m_1+1,\dots,m_n+1), & \text{if } r = n. \end{cases}$$

Geometrically, the preimage p^{-1} means adding one more column. For $r < n$, this means

$$p^{-1}\begin{bmatrix} \lambda_1 & \cdots & * & \cdots \\ & \ddots & \vdots & \\ & & \lambda_r & \cdots \\ & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & * & \cdots & * \\ & \ddots & \vdots & & \vdots \\ & & \lambda_r & \cdots & * \\ & & & & \lambda_{r+1} \\ & & & & \end{bmatrix}$$

Here $\lambda_i > 0$ for $1 \leq i \leq r$ and $\lambda_{r+1} \geq 0$. The generic case is $\lambda_{r+1} > 0$, which corresponds to the cell $B(m_1+1,\dots,m_r+1,1)$. The reduced case is $\lambda_{r+1} = 0$, which gives $B(m_1+1,\dots,m_r+1)$ and is part of the boundary of $B(m_1+1,\dots,m_r+1,1)$. If $r = n$, then there is no $(r+1)$st row, and we only have the reduced case.

Like the complex projective space, in the homological computation of the normal invariant, the suspension takes the cell $B(m_1,\dots,m_r)$ to the generic cell $B(m_1+1,\dots,m_r+1,1)$ for $r < n$ and takes the cell $B(m_1,\dots,m_r)$ to the only cell $B(m_1+1,\dots,m_n+1)$ for $r = n$.

Suppose $k - n$ is even. By Theorem 1.4, we have

$$\mathcal{S}_{U(n)}(S(k\rho_n)) = \mathbb{Z}^{\sum_{2i<n} A_{n-2i,k}} \oplus \mathbb{Z}_2^{\sum_{2i<n} B_{n-2i,k}}.$$

By the proof of Proposition 6.1, each copy of $\mathbb{Z}$ in the factor $\mathbb{Z}^{A_{n-2i,k}}$ is associated with a Schubert cell of $G(n-2i,k)$ of dimension 0 mod 4. In fact, we may also use the CW structure of Jared Bass to directly compute the homology in Proposition 6.1. The difference from Proposition A.1 is that we no longer require $m_n > 1$ for the generators of the homology. Then each copy of $\mathbb{Z}$ in $\mathbb{Z}^{A_{n-2i,k}}$ is associated with a cell $B(m_1,\ldots,m_{n-2i})$ of $S(k\rho_n)/U(n)$ with codimension (and allowing $m_{n-2i}=1$)

$$2b_k(m_1,\ldots,m_{n-2i}) = \dim S(k\rho_{n-2i})/U(n-2i) - \dim B(m_1,\ldots,m_{n-2i}) = 0 \bmod 4.$$

Here

$$\begin{aligned} b_k(m_1,\ldots,m_r) &= \frac{1}{2}[\dim S(k\rho_r)/U(r) - \dim B(m_1,\ldots,m_r)] \\ &= kr - \frac{1}{2}r(r-1) - (m_1+\cdots+m_r). \end{aligned}$$

Similarly, each copy of $\mathbb{Z}_2$ is associated with a cell $B(m_1,\ldots,m_{n-2i})$ with odd $b_k(m_1,\ldots,m_{n-2i})$.

By Theorem 1.4, we also have

$$S_{U(n)}(S((k+1)\rho_n)) = \mathbb{Z}^{A_{n,k}+\sum_{2i-1<n} A_{n-2i+1,k+1}} \oplus \mathbb{Z}_2^{B_{n,k}+\sum_{2i-1<n} B_{n-2i+1,k+1}}$$

Each copy of $\mathbb{Z}$ in $\mathbb{Z}^{A_{n-2i+1,k+1}}$ is associated with a cell $B(m_1,\ldots,m_{n-2i+1})$ of $S((k+1)\rho_n)/U(n)$ with even $b_{k+1}(m_1,\ldots,m_{n-2i+1})$, and each copy of $\mathbb{Z}_2$ in $\mathbb{Z}_2^{B_{n-2i+1,k+1}}$ is associated with a cell $B(m_1,\ldots,m_{n-2i+1})$ with odd $b_{k+1}(m_1,\ldots,m_{n-2i+1})$.

For $i>0$, like the suspension of the complex projective space, the suspension takes the copy of $\mathbb{Z}$ or $\mathbb{Z}_2$ in $S_{U(n)}(S(k\rho_n))$ associated with $B(m_1,\ldots,m_{n-2i})$ to the same copy of $\mathbb{Z}$ or $\mathbb{Z}_2$ in $S_{U(n)}(S((k+1)\rho_n))$ associated with $B(m_1+1,\ldots,m_{n-2i}+1,1)$. Note that by

$$b_{k+1}(m_1+1,\ldots,m_r+1,1) = b_k(m_1,\ldots,m_r) + k - r,$$

$b_k(m_1,\ldots,m_{n-2i})$ and $b_{k+1}(m_1+1,\ldots,m_{n-2i}+1,1)$ have the same parity. Therefore, $\mathbb{Z}$ is sent to $\mathbb{Z}$ and $\mathbb{Z}_2$ is sent to $\mathbb{Z}_2$.

For $i=0$, the factors $\mathbb{Z}^{A_{n,k}}$ and $\mathbb{Z}_2^{B_{n,k}}$ in $S_{U(n)}(S((k+1)\rho_n))$ come from the homology in Proposition A.1. Therefore each copy of $\mathbb{Z}$ or $\mathbb{Z}_2$ in the factors is associated with a cell $B(m_1,\ldots,m_n)$ of $S((k+1)\rho_n)/U(n)$ *satisfying* $m_n>1$. On the other hand, each copy of $\mathbb{Z}$ or $\mathbb{Z}_2$ in the factors $\mathbb{Z}^{A_{n,k}}$ and $\mathbb{Z}_2^{B_{n,k}}$ of $S_{U(n)}(S(k\rho_n))$ is associated with a cell $B(m_1,\ldots,m_n)$ of $S(k\rho_n)/U(n)$ *without requiring* $m_n>1$. The suspension takes the cell $B(m_1,\ldots,m_n)$ of $S(k\rho_n)/U(n)$ to the cell $B(m_1+1,\ldots,m_n+1)$ of $S((k+1)\rho_n)/U(n)$. We note that $m_n+1>1$ is satisfied, and

$$b_k(m_1,\ldots,m_n) = b_{k+1}(m_1+1,\ldots,m_n+1).$$

So we have interpreted the suspension as a map that sends specific cells in $S_{U(n)}(S(k\rho_n))$ to specific cells in $S_{U(n)}(S((k+1)\rho_n))$. Next we give explicit description of the effect of this map in sending the factors $\mathbb{Z}^{A_{*,*}}$ or $\mathbb{Z}_2^{B_{*,*}}$ in $S_{U(n)}(S(k\rho_n))$ to similar factors in $S_{U(n)}(S((k+1)\rho_n))$.

We use the definition of $A_{n,k}$ and $B_{n,k}$ in terms of the n-tuples $(\mu_1,\ldots,\mu_n)$ in Proposition 6.1. The numbers also include $A_{0,k}=1$ and $B_{0,k}=0$, which count the number of cells in $G(0,k)=pt$. We claim the following relation

$$A_{n,k} = A_{n-1,k-1} + A'_{n,k-1}, \quad A'_{n,k} = \begin{cases} A_{n,k}, & \text{if } n \text{ is even,} \\ B_{n,k}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } k>n\geq 1.$$

The reason is that, in counting the number of n-tuples $(\mu_1,\ldots,\mu_n)$ for $A_{n,k}$, we consider two possibilities:

1. In case $\mu_1 = 0$, we delete μ_1 and get an $(n-1)$-tuple $\mu'_i = \mu_{i+1}$ satisfying
$$0 \le \mu'_1 \le \cdots \le \mu'_{n-1} \le k-n = (k-1)-(n-1), \quad \sum \mu'_i = \sum \mu_i \text{ is even.}$$
The total number of such $(n-1)$-tuples is $A_{n-1,k-1}$.
2. In case $\mu_1 > 0$, we get an n-tuple $\mu'_i = \mu_i - 1$ satisfying
$$0 \le \mu'_1 \le \cdots \le \mu'_n \le (k-1)-n, \quad \sum \mu'_i = \sum \mu_i - n \text{ is even.}$$
The total number of such n-tuples is $A_{n,k-1}$ when n is even, and is $B_{n,k-1}$ when n is odd.

The case of $n = 1$ can be directly verified. By similar reason, we have
$$B_{n,k} = B_{n-1,k-1} + B'_{n,k-1}, \quad B'_{n,k} = \begin{cases} B_{n,k}, & \text{if } n \text{ is even,} \\ A_{n,k}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } k > n \ge 1.$$

For $k-n$ even, we find the following interpretation of the suspension.

1. For $i > 0$, the suspension takes the factors $\mathbb{Z}^{A_{n-2i,k}}$ and $\mathbb{Z}_2^{B_{n-2i,k}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ isomorphically onto the first components of the factors $\mathbb{Z}^{A_{n-2i+1,k+1}} = \mathbb{Z}^{A_{n-2i,k}} \oplus \mathbb{Z}^{A'_{n-2i+1,k}}$ and $\mathbb{Z}_2^{B_{n-2i+1,k+1}} = \mathbb{Z}_2^{B_{n-2i,k}} \oplus \mathbb{Z}_2^{B'_{n-2i+1,k}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$.
2. For $i = 0$, the suspension takes the factors $\mathbb{Z}^{A_{n,k}}$ and $\mathbb{Z}_2^{B_{n,k}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ isomorphically onto the factors $\mathbb{Z}^{A_{n,k}}$ and $\mathbb{Z}_2^{B_{n,k}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$.

The statements above also include $\oplus j\epsilon$. In case k, n odd and $j = 0$, the factor $\mathbb{Z}^{A_{1,k}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ should be changed to $\mathbb{Z}^{A_{1,k}-1}$. The lost copy of $\mathbb{Z}$ corresponds to the base point. This does not affect the description of the suspension. In case n odd and $j > 0$, the factor $\mathbb{Z}_2^{B_{n,k}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$ should be changed to $\mathbb{Z}_2^{B_{n,k}+1}$. The extra copy of $\mathbb{Z}_2$ also corresponds to the base point and does not affect the description of the suspension.

Now we turn to the case $k-n$ is odd. The suspension maps
$$S_{U(n)}(S(k\rho_n)) = \mathbb{Z}^{A_{n,k-1} + \sum_{2i+1<n} A_{n-2i-1,k}} \oplus \mathbb{Z}_2^{B_{n,k-1} + \sum_{2i+1<n} B_{n-2i-1,k}}$$
to
$$S_{U(n)}(S((k+1)\rho_n)) = \mathbb{Z}^{\sum_{2i<n} A_{n-2i,k+1}} \oplus \mathbb{Z}_2^{\sum_{2i<n} B_{n-2i,k+1}}.$$
In more details, the suspension takes the factors $\mathbb{Z}^{A_{n-2i-1,k}}$ and $\mathbb{Z}_2^{B_{n-2i-1,k}}$ in $S_{U(n)}(S(k\rho_n))$ to the factors $\mathbb{Z}^{A_{n-2i,k+1}}$ and $\mathbb{Z}_2^{B_{n-2i,k+1}}$ in $S_{U(n)}(S((k+1)\rho_n))$, by taking $B(m_1, \dots, m_{n-2i-1})$ to $B(m_1+1, \dots, m_{n-2i-1}+1, 1)$. This has the same interpretation in terms of the row echelon form as the case $k-n$ is even. Moreover, the suspension takes the factors $\mathbb{Z}^{A_{n,k-1}}$ and $\mathbb{Z}_2^{B_{n,k-1}}$ in $S_{U(n)}(S(k\rho_n))$ to the factors $\mathbb{Z}^{A_{n,k+1}}$ and $\mathbb{Z}_2^{B_{n,k+1}}$ in $S_{U(n)}(S((k+1)\rho_n))$, by taking $B(m_1, \dots, m_n)$ (where $m_n > 1$) to $B(m_1+1, \dots, m_n+1)$. In terms of the row echelon form, this is
$$\begin{bmatrix} & \lambda_1 & \cdots & * & \cdots & * \\ & & \ddots & \vdots & & \vdots \\ & & & \lambda_n & \cdots & * \end{bmatrix} \xrightarrow{*S(\rho_n)} \begin{bmatrix} & \lambda_1 & \cdots & * & \cdots & * & * \\ & & \ddots & \vdots & & \vdots & \vdots \\ & & & \lambda_n & \cdots & * & * \end{bmatrix}$$

Here a column occupied by $*$ means a non-echelon column. Therefore the row echelon forms on the right have $m_n > 2$.

Translated into $(\mu_1, \ldots, \mu_r)$, we find the following interpretation of the suspension in case $k-n$ is odd. The statements also include $\oplus j\epsilon$.

1. The suspension takes the factors $\mathbb{Z}^{A_{n-2i-1,k}}$ and $\mathbb{Z}_2^{B_{n-2i-1,k}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ isomorphically onto the first components of the factors $\mathbb{Z}^{A_{n-2i,k+1}} = \mathbb{Z}^{A_{n-2i-1,k}} \oplus \mathbb{Z}^{A'_{n-2i,k}}$ and $\mathbb{Z}_2^{B_{n-2i,k+1}} = \mathbb{Z}_2^{B_{n-2i-1,k}} \oplus \mathbb{Z}_2^{B'_{n-2i,k}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$.

2. The suspension takes the factors $\mathbb{Z}^{A_{n,k-1}}$ and $\mathbb{Z}_2^{B_{n,k-1}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ isomorphically onto the last components in the factors $\mathbb{Z}^{A_{n,k+1}} = \mathbb{Z}^{A_{n-1,k}} \oplus \mathbb{Z}^{A'_{n-1,k-1}} \oplus \mathbb{Z}^{A_{n,k-1}}$ and $\mathbb{Z}_2^{B_{n,k}} = \mathbb{Z}^{B_{n-1,k}} \oplus \mathbb{Z}^{B'_{n-1,k-1}} \oplus \mathbb{Z}^{B_{n,k-1}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$.

In the second part, we use $A_{n,k+1} = A_{n-1,k} + A'_{n,k} = A_{n-1,k} + A'_{n-1,k-1} + A_{n,k-1}$ and the similar decomposition for $B_{n,k+1}$. In the decomposition of $A_{n,k+1}$, which counts all the $n \times (k+1)$ row echelon forms, $A_{n-1,k}$ counts those with $m_n = 1$, $A'_{n-1,k-1}$ counts those with $m_n = 2$, and $A_{n,k-1}$ counts those with $m_n > 2$.

In case k odd, n even and $j = 0$, the factor $\mathbb{Z}^{A_{1,k}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ should be changed to $\mathbb{Z}^{A_{1,k}-1}$. The lost copy of $\mathbb{Z}$ corresponds to the base point and does not affect the description of the suspension. In case n odd and $j > 0$, the factor $\mathbb{Z}_2^{B_{n,k-1}}$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ should be changed to $\mathbb{Z}_2^{B_{n,k-1}+1}$. The extra copy of $\mathbb{Z}_2$ also corresponds to the base point. The suspension of the base point is

$$\begin{bmatrix} & \lambda_1 \end{bmatrix} \xrightarrow{*S(\rho_n)} \begin{bmatrix} & \lambda_1 & * \\ & & \lambda_2 \end{bmatrix}$$

Since $b_k(1) = k-1$, $b_{k+1}(2,1) = 2k-2$, and k is odd, we see that $b_k(1)$ and $b_{k+1}(2,1)$ have the same parity. Since the result of the suspension has $m_2 = 1$, the suspension takes the extra copy of $\mathbb{Z}_2$ in $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ isomorphically onto a copy of $\mathbb{Z}_2$ in the first component of the factor $\mathbb{Z}_2^{B_{n,k}} = \mathbb{Z}^{B_{n-1,k}} \oplus \mathbb{Z}^{B'_{n-1,k-1}} \oplus \mathbb{Z}^{B_{n,k-1}}$ in $S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon))$.

8 Multiaxial $Sp(n)$-manifold

The symplectic group $Sp(n)$ consists of $n \times n$ quaternionic matrices that preserve the standard hermitian form on $\mathbb{H}^n$

$$\langle x, y \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

We call an $Sp(n)$-manifold multiaxial, if any isotropy group is conjugate to a symplectic subgroup $Sp(i)$, and lower strata are locally flat submanifolds of higher strata. As illustrated by the discussion in [7, 8], all our discussion about multiaxial $U(n)$-manifolds is still valid.

The role played by $U(1) = S^1$ is replaced by $Sp(1) = S^3$, the group of quaternions of unit length. If S^3 acts freely on a sphere, then the dimension of the sphere is 3 mod 4, and the quotient is homotopic to $\mathbb{H}P^r$. The quaternionic version of Lemma 3.1 still holds because $M^{Sp(j)} = M^{T^j}$ and all tori in $Sp(n)$ of the same dimension are conjugate, and the first gap is given by

$$\dim M^{Sp(j-1),x} - \dim M^{Sp(j),x} = 4(r_1^x + n).$$

Since $\mathbb{H}P^r$ is always connected and simply connected, Lemma 3.2 remains true for multiaxial $Sp(n)$-manifolds.

For even r, $\mathbb{H}P^r$ is a manifold of signature one. Therefore the results in Section 4 still hold after replacing $\mathbb{C}P^r$ by $\mathbb{H}P^r$. As a consequence, the splitting theorems in Section 5 for the structure sets of multiaxial manifolds remain true for $Sp(n)$.

Theorem 8.1. *Suppose M is a multiaxial $Sp(n)$-manifold, such that the dimension of any connected component of $M^{Sp(1)}$ is $\dim M - 4n$ mod 8. Then we have natural splitting*

$$\mathbb{S}_{Sp(n)}(M) = \oplus_{i\geq 0}\mathbb{S}_{Sp(n-2i)}(\bar{M}^{Sp(2i)}, \partial\bar{M}^{Sp(2i)}) = \oplus_{i\geq 0}\mathbb{S}^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}).$$

Theorem 8.2. *Suppose M is a multiaxial $Sp(n)$-manifold, such that the dimension of $M^{Sp(1)}$ is $\dim M - 4(n+1)$ mod 8. If $M = W^{Sp(1)}$ for a multiaxial $Sp(n+1)$-manifold W, then we have natural splitting*

$$\mathbb{S}_{Sp(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X) \oplus \left(\oplus_{i\geq 0}\mathbb{S}^{\mathrm{alg}}(X_{-2i-1}, X_{-2i-2})\right).$$

Moreover,

$$\mathbb{S}^{\mathrm{alg}}(X_{-2i-1}, X_{-2i-2}) = \mathbb{S}_{Sp(n-2i-1)}(\bar{M}^{Sp(2i+1)}, \partial\bar{M}^{Sp(2i+1)}).$$

Theorem 5.2 can also be extended. Moreover, we have the quaternonic version of Theorem 1.2 (?????).

Theorem 8.3. *Suppose the quaternionic sphere S^3 acts semifreely on a topological manifold M^m, such that the fixed points M^{S^3} is a locally flat submanifold. Let $M_0^{S^3}$ and $M_2^{S^3}$ be the unions of those connected components of M^{S^3} that are respectively of dimensions m mod 8 and $m+4$ mod 8. Let N be the complement of (the interior of) an equivariant tube neighborhood of M^{S^3}, with boundaries $\partial_0 N$ and $\partial_2 N$ corresponding to the two parts of the fixed points. Then*

$$S_{S^3}(M) = S(M_0^{S^3}) \oplus S(N/S^3, \partial_2 N/S^3, \mathrm{rel}\, \partial_0 N/S^3).$$

We can also compute the structure sets of multiaxial $Sp(n)$-representation spheres. The dimensions of the Schubert cells of quaternionic Grassmannians $G_{\mathbb{H}}(n,k)$ are multiples of 4, so that the analogue of Proposition 6.1 gives copies of $L_{4i}(e) = \mathbb{Z}$, regardless of the parity. Since the total number of Schubert cells in $G_{\mathbb{H}}(n,k)$ is $A_{n,k} + B_{n,k} = \binom{k}{n}$, we have

$$H_d(S(k\rho_n)/Sp(n), S(k\rho_n)_{-1}/Sp(n); \mathbb{L}) = \mathbb{Z}^{\binom{k}{n}}, \quad k \geq n,$$

where

$$d = \dim S(k\rho_n)/Sp(n) = 4kn - 1 - n(2n+1).$$

On the other hand, the CW structure by Jared Bass can also be applied to the orbit space $S(k\rho_n)/Sp(n)$. The reason is that the unique representative by row echelon form is a consequence of the fact that $GL(n,\mathbb{C}) = U(n)N$, where $U(n)$ is the maximal compact subgroup of the semisimple Lie group $SL(n,\mathbb{C})$ and N is the upper triangular matrix with positive diagonal entries. This is a special example of the Iwasawa decomposition. When the decomposition is applied to the semisimple Lie group $SL(n,\mathbb{H})$, for which $Sp(n)$ is the maximal compact subgroup, we get $GL(n,\mathbb{H}) = Sp(n)N$. Therefore the orbit space $S(k\rho_n)/Sp(n)$ has cells $B(m_1,\dots,m_r)$ similar to the orbit space $S(k\rho_n)/U(n)$, except that

$$\dim B(m_1,\dots,m_r) = 4(m_1+\cdots+m_r) - 3r - 1.$$

This leads to the analogue of Proposition A.1

$$\tilde{H}_d(S(k\rho_n)/Sp(n);\mathbb{L}) = \mathbb{Z}^{\binom{k-1}{n}}, \quad k \geq n.$$

For the case $k-n$ is odd, this is the top piece

$$S^{\mathrm{alg}}(S(k\rho_n)/Sp(n)) = \tilde{H}_d(S(k\rho_n)/Sp(n);\mathbb{L})$$

in the decomposition of the structure set $S_{Sp(n)}(S(k\rho_n))$. If $k-n$ is odd and $j>0$, then the top piece is

$$\begin{aligned} S^{\mathrm{alg}}((k\rho_n \oplus j\epsilon)/Sp(n)) &= \tilde{H}_{d+j}(X;\mathbb{L}) = H_d(S(k\rho_n)/Sp(n);\mathbb{L}) \\ &= \tilde{H}_d(S(k\rho_n)/Sp(n);\mathbb{L}) \oplus H_0(Y;\pi_d\mathbb{L}). \end{aligned}$$

The extra homology at the base point is

$$H_0(Y;\pi_{4kn-1-n(2n+1)}\mathbb{L}) = L_{4kn-1-n(2n+1)}(e) = \begin{cases} \mathbb{Z}, & \text{if } n = 1 \bmod 4, \\ \mathbb{Z}_2, & \text{if } n = 3 \bmod 4, \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Finally, we need to consider the case the last piece in the decomposition is $S(\mathbb{H}P^{k-1})$, which happens when k,n odd and $j=0$, or k odd, n even and $j=0$. In this case, the number of copies of $\mathbb{Z}$ should be reduced by 1.

In summary, the quaternionic analogue of Theorem 1.4 is the following.

Theorem 8.4. *Suppose $k \geq n$ and ρ_n is the canonical representation of $Sp(n)$.*

1. *If $k-n$ is even, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{0\leq 2i<n}\binom{k}{n-2i}},$$

with the only exception that there is one less $\mathbb{Z}$ in case n is odd and $j=0$.

2. *If $k-n$ is odd, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\binom{k-1}{n}+\sum_{0\leq 2i+1<n}\binom{k}{n-2i-1}},$$

with the following exceptions: (i) Ihere is one less $\mathbb{Z}$ in case n is even and $j=0$; (ii) There is one more $\mathbb{Z}$ in case $n=1$ mod 4 and $j>0$; (iii) There is one more $\mathbb{Z}_2$ in case $n=3$ mod 4 and $j>0$.

Finally, the discussion on the suspension

$$*S(\rho_n)\colon S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) \to S_{Sp(n)}(S((k+1)\rho_n \oplus j\epsilon))$$

can be carried out just like Section 7 and conclude that the suspension is injective. The discussion is actually simpler, without the need to pay attention to the distinction between $\mathbb{Z}$ and $\mathbb{Z}_2$ caused by the parity.

A Homology of a Sphere Mod the Axial Action

Following earlier notation, we say

$$Y = S(k\rho_n)/U(n), \quad d = \dim Y = 2kn - 1 - n^2.$$

Through an explicit CW decomposition, we will compute the reduced homology $\tilde{H}_d(Y;\mathbb{L})$.

Proposition A.1. *If $k \geq n$, then for $Y = S(k\rho_n)/U(n)$, we have*

$$\tilde{H}_{\dim Y}(Y;\mathbb{L}) = \mathbb{Z}^{a_{n,k}} \oplus \mathbb{Z}_2^{b_{n,k}},$$

where $a_{n,k}$ is the number of n-tuples $(\mu_1,\ldots,\mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k-n-1, \quad \sum \mu_i + kn \text{ is even,}$$

and $b_{n,k}$ is the number of n-tuples satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k-n-1, \quad \sum \mu_i + kn \text{ is odd.}$$

In the case $k-n$ is odd, which is what we are really interested in, we note that $\sum \mu_i + kn$ and $\sum \mu_i$ have the same parity, so that $a_{n,k} = A_{n,k-1}$ and $b_{n,k} = B_{n,k-1}$ from Proposition 6.1. In the case $k-n$ is even, $\sum \mu_i + kn$ and $\sum \mu_i + n$ have the same parity.

Proof. An element in $S(k\rho_n)$ is a k-tuple $\xi = (v_1,\ldots,v_k)$ of vectors in ρ_n satisfying $\|\xi\|^2 = \|v_1\|^2 + \cdots + \|v_k\|^2 = 1$, with the $U(n)$-action $g\xi = (gv_1,\ldots,gv_k)$. We may regard ξ as a complex $k \times n$-matrix. We claim that we can find a unique representative for ξ of in the row echelon form

$$\bar{\xi} = \begin{bmatrix} \lambda_1 & \cdots & * & \cdots & * & \cdots & * & \cdots \\ & & \lambda_2 & \cdots & * & \cdots & * & \cdots \\ & & & & \lambda_3 & \cdots & * & \cdots \\ & & & & & \ddots & \vdots & \cdots \\ & & & & & & \lambda_r & \cdots \\ & & & & & & & \end{bmatrix},$$

where the empty spaces are occupied by 0, $*$ and dots mean complex numbers, $\lambda_i > 0$, and the total length of all the entries is 1, as it was for ξ. To get $\bar{\xi}$, apply the Gram-Schmidt process to the columns of ξ to obtain an orthonormal basis for $\mathbb{C}^n$ (adding extra vectors if necessary). If we then apply to ξ the unitary matrix taking this new basis to the standard basis, we get $\bar{\xi}$ as desired. The orbit space Y is the collection of all matrices $\bar{\xi}$ of the above form.

If λ_j appears m_j places from the right end of the matrix (i.e., λ_j lies in the $k - m_j + 1$ column), then we say that the matrix has *shape* $(m_1,\ldots,m_r)$. Note that r is the rank of the matrix ξ. For any $r \leq n$, $k \geq m_1 > \cdots > m_r > 0$, all $\bar{\xi}$ of the shape $(m_1,\ldots,m_r)$ form a cell $B(m_1,\ldots,m_r)$ of dimension

$$\dim B(m_1,\ldots,m_r) = 2(m_1 + \cdots + m_r) - r - 1.$$

Geometrically, the cell is the subset of a sphere of the above dimension determined by r coordinates being nonnegative. The boundary of this cell consists of those shapes $(m'_1, \ldots, m'_{r'})$ satisfying $r' \leq r$ and $m'_i \leq m_i$, with at least one inequality being strict. In homological computation, only those shapes of one dimension less matter. This only occurs when

$$m_r = 1, \quad r' = r - 1, \quad m'_i = m_i \text{ for } 1 \leq i < r.$$

Therefore, the only nontrivial boundary map of the cellular chain complex is

$$\partial B(m_1, \ldots, m_{r-1}, 1) = B(m_1, \ldots, m_{r-1}).$$

The homology is then freely generated by the shapes that are neither $(m_1, \ldots, m_{r-1}, 1)$ nor $(m_1, \ldots, m_{r-1})$ in the equality above. These are exactly the shapes satisfying $r = n$ (meaning ξ has full rank) and $m_n > 1$, and the shape (1) (meaning $r = 1$ and $m_1 = 1$). The shape (1) is the base point of Y.

The reduced homology $\tilde{H}_*(Y; \mathbb{L})$ is the limit of a spectral sequence with

$$E_2^{p,q} = \tilde{H}_p(Y; \pi_q \mathbb{L}) = \begin{cases} \tilde{H}_p(Y; \mathbb{Z}), & \text{if } q = 0 \bmod 4, \\ \tilde{H}_p(Y; \mathbb{Z}_2), & \text{if } q = 2 \bmod 4, \\ 0, & \text{if } q \text{ is odd.} \end{cases}$$

Note that the reduced homology $\tilde{H}_p Y$ is freely generated by shapes satisfying $r = n$ and $m_n > 1$. Since the dimensions of such cells have the same parity as $n+1$, $\tilde{H}_p Y$ is nontrivial only if p has the same parity as $n + 1$. This implies that $E_2^{p,q}$ already collapses and

$$\tilde{H}_d(Y; \mathbb{L}) = (\oplus_{q=0(4)} \tilde{H}_{d-q}(Y; \mathbb{Z})) \oplus (\oplus_{q=2(4)} \tilde{H}_{d-q}(Y; \mathbb{Z}_2)).$$

We have

$$\oplus_{q=0(4)} \tilde{H}_{d-q}(Y; \mathbb{Z}) = \mathbb{Z}^{a_{n,k}},$$

where $a_{n,k}$ is the number of shapes $(m_1, \ldots, m_n)$ satisfying

$$m_n > 1, \quad 2(m_1 + \cdots + m_n) - n - 1 = d = 2kn - 1 - n^2 \bmod 4.$$

Let $\mu_i = m_{n-i+1} - (i+1)$, so this condition can be interpreted in terms of the nondecreasing sequence of nonnegative integers $(\mu_1, \ldots, \mu_n)$, as in the statement of the proposition. Through a similar computation we get the description of $b_{n,k}$ for the case $q = 2 \bmod 4$. $\square$

For the unreduced homology $H_d(Y; \mathbb{L})$, we also need to consider the basepoint. So we need to further take the direct sum with the homology at the base, $H_0(Y; \pi_d \mathbb{L}) = L_d(e)$. In our case of interest, when $k - n$ is odd, we have $d = n^2 + 1 \bmod 4$. This yields the following.

Corollary A.2. *For $k - n$ odd, the unreduced homology $H_{\dim Y}(Y; \mathbb{L})$ is given by Proposition A.1 with an additional factor of*

$$H_0(Y; \pi_d \mathbb{L}) = L_d(e) = \begin{cases} \mathbb{Z}_2, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

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