MODULAR SYMBOLS AND THE TOPOLOGICAL NONRIGIDITY OF ARITHMETIC MANIFOLDS

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ABSTRACT. In this paper we address the existence of smooth manifolds proper homotopy equivalent to nonuniform arithmetic manifolds $M = \Gamma\backslash G/K$ that are not homeomorphic to it. While the manifolds M are properly rigid if $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \leq 2$, we show that the so-called "virtual structure group" has infinite rank as a $\mathbb{Q}$-vector space if $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 4$.

CONTENTS

The Borel conjecture asserts that closed aspherical manifolds are topologically rigid; i.e. if M is closed and aspherical and $f: M' \to M$ is a homotopy equivalence, then f is homotopic to a homeomorphism. This statement is a topological analogue of Mostow rigidity, which in its simplest version asserts that, in the situation where both manifolds are hyperbolic of dimension at least 3, such a map f is in fact homotopic to an isometry. The Borel conjecture has been verified in locally symmetric situations by Farrell-Jones [24] (aside from dimension four cases) and for many non-classical aspherical manifolds by Bartels-Lück [6].

In light of Prasad's extension of Mostow rigidity from uniform to non-uniform lattices [42], it becomes natural to inquire whether there is a proper analogue to the Borel conjecture. Here we define a noncompact manifold M to be *properly rigid* if every proper homotopy equivalence $f: M' \to M$ is properly homotopic to a homeomorphism. Already in 1982, Farrell and Hsiang [23] showed that, for $n > 176$,

if $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ is torsion-free, then $\Gamma\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ is not properly rigid. In our earlier paper [18] we proved the following.

Theorem 0.1. *Let M be a locally symmetric manifold $\Gamma\backslash G/K$, where G is a semisimple Lie group, K is its maximal compact subgroup and Γ is an irreducible lattice.*

(1) *If Γ is not arithmetic then it is properly rigid.*

(2) *If Γ is arithmetic and its $\mathbb{Q}$-rank at most 2, then the twisted Borel conjecture for Γ implies its proper rigidity.*

(3) *If Γ has $\mathbb{Q}$-rank at least 3 then M has a finite-sheeted cover $M' = \Gamma'\backslash G/K$ for which proper rigidity fails.*

The proofs of these statements use a combination of methods from surgery, super-rigidity, and strong approximation for linear groups. The recent paper of Bartels-Lück-Reich-Rüping [7] proves the relevant twisted Borel conjecture that establishes (2) unconditionally. Notice that (3) shows that the Borel conjecture fails systematically in the noncompact proper setting.

There are a number of issues that the above theorem and its proof do not resolve. We will introduce some terminology to address some new ideas.

Definition 0.2. Let M be a noncompact manifold. We say that M is *(properly topologically) strongly nonrigid* if there is a element $f\colon N \to M$ in $S^{Top}(M)$ such that N is not homeomorphic to M. In this case, we say that N is an *exotic version of M*. If $f\colon N \to M$ is a nontrivial element in $S^{Top}(M)$ with N homeomorphic to M, we say that f is *exotic*; i.e. f is not properly homotopic to a homeomorphism. The *trivial element* of $S^{Top}(M)$ is merely the class containing the identity map $i\colon M \to M$.

Let $M = \Gamma\backslash G/K$ be a locally symmetric manifold with Γ irreducible and let $M' = \Gamma'\backslash G/K$ be a finite-sheeted cover which is properly nonrigid. We can ask the following questions.

(a) Are there any exotic versions of M' in $S^{Top}(M')$, or is it the case that every $f\colon N \to M'$ in $S^{Top}(M')$ is merely exotic?

(b) If exotic versions of M' exist, do they have smooth structures?

(c) Suppose that $f\colon N \to M'$ is a nontrivial element of $S^{Top}(M')$. Is there finite-sheeted lift $\widetilde{M'}$ of M' such that the induced lift $\widetilde{f}\colon \widetilde{N} \to \widetilde{M'}$ is trivial $S^{Top}(\widetilde{M'})$?

We can produce a reducible example related to question (c). Let Π be a product of three noncompact surfaces. Then for any cover Π', the set of manifolds proper homotopy equivalent to Π' is finite, but all of these manifolds become homeomorphic upon passing to an appropriate further finite cover. Although the examples here are all smoothable, the method of the authors' previous work [18], which makes use of topological characteristic classes, is definitely topological in nature. Moreover,

all of the examples in [18] actually do become trivial on passing to a further finite cover. Amusingly, they are all nontrivial upon passing to odd-fold covers.

The following theorem addresses questions (a) and (b) in the irreducible case.

Theorem 0.3. *Let $M = \Gamma\backslash G/K$ be an arithmetic manifold with Γ irreducible and $\mathbb{Q}$-rank at least 6. Then for all integers A, there is a finite-sheeted cover N_A of M such that the number of smooth exotic versions of N_A is at least A. As a consequence, the proper topological structure set $S^{Top}(N_A)$ can be made arbitrarily large. In addition, these exotic versions can be distinguished from each other by their first Pontrjagin class.*

The proof of the last statement above implicitly makes use of a refinement of Novikov's theorem on the topological invariance of rational Pontrjagin classes p_i. Because of the rational equivalence of the map $F/O \to F/Top$, in each dimension d, there is a positive integer C_d so that $C_d p_d$ is a topological invariant. Therefore, we can make use of torsion elements in $H^4(N; \mathbb{Z})$ to distinguish topological manifolds.

The goal of this paper is to strengthen the nonrigidity of Theorem 0.1(3) by addressing the questions above. By using higher Pontrjagin classes and modular symbols, we obtain the following substantial improvement, showing that strong nonrigidity holds when the $\mathbb{Q}$-rank is at least 4.

Theorem 0.4. *Let Γ be a lattice in a semisimple algebraic group G over $\mathbb{Q}$ with $\mathbb{Q}$-rank at least 4. Let K be a maximal compact subgroup of G. Then there is $\Gamma' \leq \Gamma$ such that $\Gamma'\backslash G/K$ is strongly nonrigid.*

The proof of Theorem 0.3 is a case-by-case argument while that of Theorem 0.4 is a general argument.

Among the examples presented in this paper is the special case of Hermitian locally symmetric spaces. Such a space M is a Riemannian symmetric space with a parallel complex structure with respect to which the Riemannian metric is Hermitian. Here we exploit the projectivity of the Baily-Borel compactification and the observation by Jost-Yau [32] that the codimension of the singular set is usually reasonably large. With these notions, we can prove that, when the complex dimension is at least 3, the space M is properly nonrigid. The converse is also true.

Theorem 0.5. *If $M = \Gamma\backslash G/K$ is a Hermitian locally symmetric manifold of real dimension at least 5, then the following are equivalent:*

(1) *The $\mathbb{Q}$-rank of Γ is at least 3.*
(2) *The space M is properly topologically nonrigid.*

If these conditions are satisfied, then infinitely many elements in $S^{Top}(M)$ are in fact detected by a rational topological version of the first Pontrjagin class in $H^4(M; \mathbb{Q})$. Also in this case, the space M is properly nonrigid among smooth manifolds (where the conclusion is still homeomorphism).

In other words, for the case of Hermitian symmetric manifolds, Theorem 0.5 strengthens Theorem 0.1 in all of the ways discussed, without requiring that any covers be taken.

Having established the nonvanishing of the proper structure sets, we are interested to know whether they grow, in any sense, when the symmetric space becomes larger. This notion suggests that we can define a set $S^{virt}(M)$, a *virtual structure group* of a manifold M. Neither the methods used for Hermitian symmetric spaces nor the argument of Farrell-Hsiang can be used to bound this object, since these techniques are based on cohomology classes arising from the compact dual, and in this case there are only finitely many. However, using modular symbols we can prove that $S^{virt}(\Gamma\backslash G/K)$ is not only nontrivial, but has infinite rational rank as a $\mathbb{Q}$-vector space. We remark additionally that $S^{Top}(\Gamma\backslash G/K)$ has a natural abelian group structure. Note that in this theorem we do not assume that $\Gamma\backslash G/K$ is irreducible.

Theorem 0.6. *Suppose that $\Gamma\backslash G/K$ is an arithmetic manifold with $\mathbb{Q}$-rank at least 4. Then $S^{virt}(\Gamma\backslash G/K)\otimes\mathbb{Q}$ is an infinite-dimensional $\mathbb{Q}$-vector space.*

The size of the group $S^{Top}(\Gamma\backslash G/K)\otimes\mathbb{Q}$ usually grows sublinearly in the covolume of Γ. The methods of L^2-cohomology imply that linear growth occurs in a tower precisely when $\dim(G/K)$ is a multiple of 8 and $\mathrm{rank}_{\mathbb{C}}G = \mathrm{rank}_{\mathbb{C}}K$. The characterization should either be interpreted as the condition for the existence of discrete series representations, or equivalently for the compact dual of G/K to have nonzero Euler characteristic (which is then the ratio of the order of the Weyl groups). However, even when the growth is sublinear, when one passes up the congruence tower, the ranks of these abelian groups do grow like a positive power of the volume.

Our methods give some information even if the $\mathbb{Q}$-rank is 3. For example, if $\mathcal{O}_{\mathbb{F}}$ is a ring of algebraic integers, then the arithmetic manifold associated to $\mathrm{SL}_n(\mathcal{O}_{\mathbb{F}})$ is nonrigid, with infinitely generated virtual structure set. However, there are different phenomena that arise in general. For example, the virtual structure set vanishes for product lattices in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, as mentioned previously. It would be interesting to clarify the $\mathbb{Q}$-rank 3 situation fully.

The paper is organized as follows. The first section gives some basic remarks about the surgery theory required in this examination. Section 2 discusses $\mathrm{SL}_n(\mathbb{Z})$ from several points of view and gives the proof of Theorem 0.3 in this case. (Note that, when $\mathcal{O}$ is a number ring, we obtain the optimal results of nonrigidity for $\mathrm{SL}_n(\mathcal{O})$ when $n \geq 4$.) The modifications for the more general case are given in section 3. The key idea is to identify deep lattices of a simple algebraic group G over $\mathbb{Q}$ that represents nonzero cycles in appropriate homology groups. Ideas of this sort appear in papers of Lee-Szczarba [34], Charney [19] and Church-Farb [21]. We also quickly show, using a Bockstein argument, that the elements constructed in [18] by strong approximation represents the zero element in its virtual structure set. Section 4 discusses the Hermitian locally symmetric case, while section 5 defines the virtual

structure set of an arithmetic manifold and proves its infinite size when the $\mathbb{Q}$-rank is at least 4. The last section gives a list of the numerical calculations required in section 5.

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1. The surgery set-up

As given in Wall [48] the surgery exact sequence for a given compact manifold[1] M of dimension n is given by

$$\cdots \to L_{n-1}(\mathbb{Z}\pi_1(M)) \to S^{Top}(M) \to [M\colon F/Top] \to L_n(\mathbb{Z}\pi_1(M)),$$

where the $L_*(\mathbb{Z}\pi_1(M))$ denote the Wall L-groups and $S^{Top}(M)$ is the topological structure set, i.e the collection of pairs (N, f), where N is an n-manifold and $f\colon N \to M$ is a homotopy equivalence, modulo h-cobordism.[2]

Let $N = \Gamma\backslash G/K$ be a locally symmetric space formed as the double quotient of an irreducible Lie group G with its maximal compact subgroup K and an arithmetic lattice Γ. As observed by Block and Weinberger [9], this N can be compactified (via the Borel-Serre compactification) to a π-π manifold $\overline{N}$ with boundary when the $\mathbb{Q}$-rank is at least 3. By a π-π manifold we mean a manifold whose fundamental group coincides with the fundamental group of its boundary. This observation of Block-Weinberger follows from the identification of the homotopy type of the Γ-cover of the boundary with a wedge of $(q-1)$-spheres using the Solomon-Tits theorem [10]. According to Siebenmann's thesis [46], any manifold that is properly homotopy equivalent to M will have the same property. By way of the h-cobordism theorem, one can show that any such manifold has a unique compactification so that the extension of the proper homotopy equivalence to the compactification is a simple homotopy equivalence. We can then identify the structure set $S^{Top}(\overline{N})$ with the proper structure set $S^{Top}(N)$. (Here we suppress the p for proper in the notation when it is clear that the space is noncompact.)

Thus by Wall's π-π theorem [48], the structure set $S^{Top}(\overline{N})$ of $\overline{N}$ is isomorphic to $[\overline{N}, F/Top]$. Since $F/Top_{(2)} \cong \prod_{n=1}^{\infty} K(\mathbb{Z}_2, 4n-2) \times \prod_{m=1}^{\infty} K(\mathbb{Z}, 4m)$ and $F/Top[1/2] \cong BO[1/2]$ (see Madsen-Milgram [36]), we then have

$$
\begin{aligned}
S^{Top}(N)_{(2)} &= S(\overline{N})_{(2)} = [\overline{N}, F/Top]_{(2)} = [N, F/Top]_{(2)} \\
&= \prod_{n=1}^{\infty} H^{4n-2}(N; \mathbb{Z}_2) \times \prod_{m=1}^{\infty} H^{4m}(N; \mathbb{Z}_{(2)})
\end{aligned}
$$

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[1] We ignore issues of orientation.

[2] In our situation, the difference between h-cobordism and homeomorphism can be safely ignored.

and $S^{Top}(M)[1/2] \cong KO^0(X)[1/2]$. To prove that the proper structure set is nontrivial, it suffices to show that some cohomology group $H^{4n-2}(N; \mathbb{Z}_2)$ or $H^{4m}(N; \mathbb{Z}_{(2)})$ is nontrivial.

In [18] we proved that any $M = \Gamma \backslash G/K$ with $\mathbb{Q}$-rank at least 3 has a lift N for which $H^2(N; \mathbb{Z}_2)$ is nontrivial, thereby showing that $S^{Top}(N)$ is also nontrivial. The topological nonrigidity of N, however, does not rule out the case in which each nontrivial element N' of $S^{Top}(N)$ is merely a homeomorphic copy of N equipped with a proper homotopy equivalence $f \colon N' \to N$ that is not proper homotopic to a homeomorphism. If there is N' homotopy equivalent to N but not homeomorphic, we say that N' is an *exotic version of* N. One goal of this paper is to identify exotic versions of arithmetic manifolds $\Gamma \backslash G/K$.

To identify such exotic types, we shall give several different methods, all of which exploit the $H^{4i}(-; \mathbb{Z}_{(2)})$ summand. One method is associated with $i = 1$ and uses a topologically invariant Pontrjagin class p_1 to obtain flexibility. In general these obstructions will be torsion elements and can vanish in the cohomology groups associated to higher covers. Using varying i and various rational cohomology classes we will be able to identify infinitely many obstructions of a fixed level and elements that persist in the cohomology groups associated to deeper sublattices.

2. Lattices commensurable with $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 5$

In this section we will assume that $n \geq 5$. Our arguments can readily be modified to $\mathrm{SL}_n(\mathcal{O})$, where $\mathcal{O}$ is any ring of integers. If R is a ring, denote by $M_n(R)$ the collection of $n \times n$ matrices with entries in R. Denote by $\mathbb{Z}_r$ the cyclic group of order r. If $A \in M_n(\mathbb{R})$ and $r \geq 2$ is a positive integer, we will denote by $[A]_r$ its projection in $M_n(\mathbb{Z}_r)$. Let p be a prime and consider the (principal) congruence subgroup Γ_p of $\mathrm{SL}_n(\mathbb{Z})$ given by

$$\Gamma_p \to \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_p).$$

In other words we have $\Gamma_p = \{A \in \mathrm{SL}_n(\mathbb{Z}) \colon [A]_p = I\}$. Since the $\mathbb{Q}$-rank of $\mathrm{SL}_n(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{R})$ is $n - 1$, it follows that the $\mathbb{Q}$-rank of Γ_p in $\mathrm{SL}_n(\mathbb{R})$ is at least 3. Our previous theorem states that some lift of $M_p = \Gamma_p \backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ is properly nonrigid, but does not give any additional information about M_p itself. Note that, at the moment, we are working with general p but later we will choose p more precisely.

We will restrict ourselves first to the discussion of $n = 5$. The reader will see that the general case is treated in exactly the same manner. As a matter of notation,

let the matrix

$$\begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & e & f & g \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be denoted by $\left\| \begin{smallmatrix} a & b & c & d \\ 1 & e & f & g \end{smallmatrix} \right\|$. Note that all matrices in $\mathrm{SL}_n(\mathbb{Z})$ of the form $\left\| \begin{smallmatrix} a & b & 0 & 0 \\ 1 & e & 0 & 0 \end{smallmatrix} \right\|$ can be identified with the Heisenberg group and, for all primes p, those matrices of the form $\left\| \begin{smallmatrix} pa & pb & 0 & 0 \\ 1 & pe & 0 & 0 \end{smallmatrix} \right\|$ form the *p-multiple Heisenberg group* Heis_p. We distinguish two other subgroups of $\mathrm{SL}_n(\mathbb{Z})$. The subgroup of all elements of the form $\left\| \begin{smallmatrix} 0 & pb & pc & pd \\ 1 & 0 & 0 & 0 \end{smallmatrix} \right\|$ will be denoted A_p, and the subgroup of all elements of the form $\left\| \begin{smallmatrix} pa & pb & pc & pd \\ 1 & pe & 0 & 0 \end{smallmatrix} \right\|$ will be denoted Heis_p^+. Note that Heis_p^+, A_p and Heis_p are all subgroups of Γ_p. In addition, denote by $\langle pb \rangle$, $\langle pc \rangle$ and $\langle pd \rangle$ the infinite cyclic subgroups generated by $\left\| \begin{smallmatrix} 0 & pb & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix} \right\|$, $\left\| \begin{smallmatrix} 0 & 0 & pc & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix} \right\|$ and $\left\| \begin{smallmatrix} 0 & 0 & 0 & pd \\ 1 & 0 & 0 & 0 \end{smallmatrix} \right\|$, respectively, in $\mathrm{SL}_n(\mathbb{Z})$.

There is a surjection $\phi \colon \Gamma_p \to M_n(\mathbb{Z}_p) \cong \mathbb{Z}_p^{n^2}$ of the congruence subgroup to an additive matrix group given by $\phi(A) = \left[\frac{1}{p}(A - I) \right]_p$. In other words, since $A \in \Gamma_p$, it is of the form $A = I + pC$, for some unique square matrix C. Then we define $\phi(A) = C$. Indeed, this map ϕ is a homomorphism because $\phi(AA') = \phi((1+pC)(1+pC')) = \phi(I + p(C + C') + O(p^2)) = [C + C' + O(p)]_p = [C + C']_p = \phi(A) + \phi(A')$. Because of cross terms, the induced map ϕ_* is nontrivial in mod p homology.

Proposition 2.1. *The group $H_3(\Gamma_p; \mathbb{Z})$ contains a nontrivial element of finite order p.*

Proof. The element $\langle pb \rangle$ is nonzero in $H_1(\mathrm{Heis}_p; \mathbb{Z})$ because its image under the map $\phi_* \colon H_1(\Gamma_p; \mathbb{Z}) \to H_1(\mathbb{Z}_p^{n^2}; \mathbb{Z})$ induced by ϕ is nonzero. But $p\langle pb \rangle$ is zero in $H_1(\mathrm{Heis}_p; \mathbb{Z})$, so $\langle pb \rangle$ is 0 in $H_1(\mathrm{Heis}_p; \mathbb{Q})$. In other words, this copy of $\mathbb{Z}$ is not integrally zero but is rationally zero. It is clear that $\mathrm{Heis}_p^+ \cong \mathrm{Heis}_p \times (p\mathbb{Z})^2$. Then $\langle pb \rangle \times \langle pc \rangle \times \langle pd \rangle$ is integrally nonzero but is 0 in $H_3(\mathrm{Heis}_p^+; \mathbb{Q})$. Indeed, the $\langle pb \rangle$ bounds some Q, and so $\langle pb \rangle \times \langle pc \rangle \times \langle pd \rangle$ bounds $\mathbb{T}^2 \times Q$. It is therefore also rationally zero and so has finite order in $H_3(\Gamma_p; \mathbb{Z})$. □

Proposition 2.2. *The group $H^4(\Gamma_p; \mathbb{Z})$ is nontrivial.*

Proof. We have a universal coefficient theorem for cohomology given by

$$H^4(\Gamma_p; \mathbb{Z}) = \mathrm{Hom}(H_4(\Gamma_p; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_3(\Gamma_p; \mathbb{Z}), \mathbb{Z}).$$

The previous proposition gives a nontrivial Ext term.   □

Therefore, to identify a non-equivalent manifold structure in $S^{Top}(M_p)$, we start by studying the collection $N^{Top}(B\Gamma_p)$ of normal invariants.

Proposition 2.3. *For p sufficiently large, there is a map $\nu\colon B\Gamma_p \to F/O$ so that $p_1[1/2](\nu)$ is nontrivial, where $p_1[1/2]$ is defined via the diagram as follows:*

$$
\begin{array}{ccccc}
F/O & \longrightarrow & BSO & \xrightarrow{\quad p_1 \quad} & K(\mathbb{Z}, 4) \\
 & & \downarrow & & \downarrow \\
 & & BSTop & \xrightarrow{\quad p_1[1/2] \quad} & K(\mathbb{Z}[1/2], 4)
\end{array}
$$

Remark 2.4. The fact that $H^4(BSTop; \mathbb{Z}[1/2]) \to H^4(BSO; \mathbb{Z}[1/2])$ is an isomorphism follows from the calculations that $\pi_i(Top/PL) = 0$ for $i \neq 3$ and $\pi_3(Top/PL) = \mathbb{Z}_2$ and that $\pi_i(PL/O) = 0$ for $i < 7$. See Kirby-Siebenmann [33]. The lift of $p_1 \in H^4(BSO; \mathbb{Z})$ to $H^4(BSTop; \mathbb{Z}[1/2])$ will be called $p_1[1/2]$.

Proof. (of Proposition) We know that $B\Gamma_p$ is a finite complex of some finite dimension k, independent of p. If $p > k$ then $\pi_i(BF) \otimes \mathbb{Z}_{(p)} = 0$ for all $i \leq k+1$. Assume that p is sufficiently large so that $\pi_i(PL/O) \otimes \mathbb{Z}_{(p)} = 0$ for all $i \leq k+1$. Then, upon localizing at p, we have the equivalences

$$
\begin{array}{ccc}
(F/O)_{(p)}^{\leq k+1} & \longrightarrow & BSO_{(p)}^{\leq k+1} \\
\downarrow & & \downarrow \\
(F/Top)_{(p)}^{\leq k+1} & \longrightarrow & BSTop_{(p)}^{\leq k+1}
\end{array}
$$

where the superscript $\leq k+1$ refers to the stage in the Postnikov decomposition. Note that Bott periodicity (formulated in terms of the Chern character inducing an isomorphism $\pi_{2n}(BU) \to \mathbb{Z}$) implies that, for $p > k+1$, the map $\oplus p_i \colon BSO^{\leq k+1} \to \prod K(\mathbb{Z}, 4i)^{\leq k+1}$ is a $\mathbb{Z}_{(p)}$-equivalence. The result now follows.   □

As a consequence, for p large, we can find a smooth normal invariant of a proper homotopy equivalence (by the π-π theorem) such that the underlying tangent bundles can be distinguished using $p_1[1/2]$, a topological invariant.

By Theorem 3 of Anderson-Hodgkin [1], there is no map $K(\mathbb{Z}, 4) \to F/Top$ that is nontrivial on the finite skeleta. For this reason, we need to restrict ourselves to a finite complex mapping into $K(\mathbb{Z}, 4)$ and then split it (after multiplying by some appropriate constant).

Theorem 2.5. *Let $n \geq 5$ and $G = \mathrm{SL}_n(\mathbb{R})$. Let Γ_p be the principal p-congruence subgroup of G. Then for p sufficiently large, the manifold $M = \Gamma_p \backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ is properly strongly nonrigid; i.e. there is a N proper homotopy equivalent to M that is not homeomorphic to M.*

Proof. Construct $\Gamma = \Gamma_p$ as above. Mostow's rigidity theorem implies that all self-equivalences of arithmetic manifolds come from isometries. In particular the Pontrjagin class p_1 is preserved under pullback. Therefore the constructed manifolds in the previous discussion are genuinely non-homeomorphic to the standard $\Gamma \backslash G / K$. □

Remark 2.6. The Bass-Milnor-Serre Theorem states that $\mathrm{SL}_n(\mathbb{Z})$ satisfies the Congruence Subgroup Property with $n \geq 3$; i.e. given a subgroup $\Gamma \leq \mathrm{SL}_n(\mathbb{R})$ of finite index, we can find a principal congruence subgroup $\Gamma_k \leq \Gamma$ of finite index. Note that whenever $k | \ell$, we have $\Gamma_\ell \leq \Gamma_k$. Therefore we can find a sublattice $\Gamma_\ell \leq \Gamma$ of finite index such that ℓ is divisible by a sufficiently large prime. The results of this section can then be extended to lattices commensurable with $\mathrm{SL}_n(\mathbb{Z})$. Recall that two lattices Δ and Δ' are commensurable if their intersection is of finite index in both of them.

The argument in the proof of Theorem 2.5 fails when $n = 4$, but we will deal with this case in a later section.

3. Proof for large $\mathbb{Q}$-rank

Now we consider the case of general arithmetic lattices in a semisimple Lie group G. We rely on a theorem of Farb and Shalen [22] to prove the existence of an appropriate abelian sublattice isomorphic to $\mathbb{Z}^3$.

Theorem 3.1. *(Farb-Shalen) Let Γ be a lattice in a simple algebraic group G over $\mathbb{Q}$. Let $d = \mathrm{rank}_{\mathbb{Q}}^G(\Gamma) \geq 6$ (can be reduced to 5 if we omit type D_5). Then Γ contains two commuting subgroups A and B which are isomorphic to irreducible lattices with $\mathbb{Q}$-ranks 2 and $d - 3$.*

Proof. By the Margulis Arithmeticity Theorem, we can assume that Γ is the $\mathbb{Z}$-points of such a group G. If G is simple, then the Dynkin diagrams for the root system Φ of G show that, when $d \geq 7$, one can remove a vertex of the diagram to obtain a graph with two components: one with two vertices and one with at least $d - 3$ vertices. Let G_1 and G_2 be the root subgroups corresponding to these components of the Dynkin diagram. Then G_1 commutes with G_2. The group of $\mathbb{Q}$-points of G_1 has $\mathbb{Q}$-rank at least 2, and the group of $\mathbb{Q}$-points of G_2 has $\mathbb{Q}$-rank at least $d - 3$. Results by Borel and Harish-Chandra (see for example Zimmer [53]) show that $A = \Gamma \cap G_1$ and $B = \Gamma \cap G_2$ are arithmetic lattices in G_1 and G_2. Then A and B have the necessary properties. □

Proposition 3.2. *Let Γ be a lattice in a semisimple algebraic group G over $\mathbb{Q}$ with $\mathbb{Q}$-rank at least 6. There is a nontrivial element in $H_3(\Gamma; \mathbb{Z})$ which is rationally zero.*

Proof. According to Farb-Shalen, we have two commuting subgroups represented by irreducible lattices Γ_1 and Γ_2 with $\mathbb{Q}$-ranks at least 2, corresponding to two

disjoint subgraphs of the Dynkin diagram of G. Since the rank of Γ_1 is 2, we can identify an abelian sublattice in Γ_1 isomorphic to $\mathbb{Z}^2$. Since the other component is associated with a lattice of $\mathbb{Q}$-rank at least 2, then Γ_2 has a finite abelianization by property T (see the general discussion in [8]). Therefore there is a nonzero element in $H_1(\Gamma_2; \mathbb{Z})$ of finite order. The three-dimensional class in $H_3(\Gamma; \mathbb{Z})$ that emerges from this computation is rationally trivial. $\square$

The techniques of the previous section give the following:

Theorem 3.3. *Let Γ be a lattice in a semisimple algebraic group G over $\mathbb{Q}$ with $\mathbb{Q}$-rank at least 6. Let K be a maximal compact subgroup of G. Then there is $\Gamma' \leq \Gamma$ of finite index such that $\Gamma' \backslash G/K$ is strongly nonrigid.*

Proof. Let Γ be the integer points of a rational algebraic group. We will consider, as in the previous section, a congruence subgroup associated to a large prime. As before, it has a homomorphism onto the Lie algebra of a finite field. Going deeper by powers of p, we obtain similar homomorphisms for these deeper subgroups as well. It is clear that this tower converges to the trivial group, so given a free abelian group in our lattice, one can find a level so that the intersection of that group with the relevant lattice has as its image a free abelian p-group with the same number of factors. $\square$

Theorem 3.4. *If $\operatorname{rank}_{\mathbb{Q}}^{G} \Gamma \geq 5$ there is a nontrivial $N \in S^{Top}(\Gamma \backslash G/K)$ which is smooth.*

Proof. As before, the three-dimensional torsion homology class gives rise to an element in $H^4(-; \mathbb{Z})$. If it is p-torsion for a large enough prime (depending on dimension) we are guaranteed (as in Proposition 2.3) a lifting to F/O, giving us a smooth normal invariant detected by a topologically invariant Pontrjagin class. $\square$

It is meaningful to ask whether the elements constructed here become trivial upon passing to further covers. We believe that they do, and only know of very occasional elements constructed by finite-order methods that continue to be nontrivial upon passing to finite covers. We note here that the elements constructed in our previous paper [18], upon passing to appropriate finite covers, all become trivial.

Proposition 3.5. *The elements constructed in [18] by strong approximation represent the zero element in $S^{virt}(\Gamma \backslash G/K)$; i.e. if $N = \Gamma \backslash G/K$ is an arithmetic space constructed by strong approximation and if (P, f) is a nontrivial element in $S^{Top}(N)$, then there is a finite lift $N' = \Gamma' \backslash G/K$ of N for which the corresponding Γ'-lift (P', f') of (P, f) is the trivial element in $S^{Top}(N')$.*

Proof. By the strong approximation of Lubotzky, there is a subgroup $\Gamma' \leq \Gamma$ with an epimorphism $\phi \colon \Gamma' \to \mathbb{Z}_2$. For each $k \geq 1$, we say that ϕ *lifts to stage k* if there

is a commutative diagram

$$\Gamma' \xrightarrow{\phi_k} \mathbb{Z}_{2^k}$$
$$\phi \searrow \quad \downarrow$$
$$\mathbb{Z}_2$$

Our proof in [18] shows that there is a minimum n for which ϕ does not lift to stage n. Consider then the lift to $\Gamma' \to \mathbb{Z}_{2^{n-1}}$. This lift gives an element u in $H^1(\Gamma'; \mathbb{Z}_{2^{n-1}})$ whose Bockstein element (associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^{n-1}} \to 0$) is nontrivial in $H^2(\Gamma'; \mathbb{Z}_2)$. Otherwise it would lift nontrivially higher. In other words, if $\beta_{\Gamma'} \colon H^1(\Gamma'; \mathbb{Z}_{2^{n-1}}) \to H^2(\Gamma'; \mathbb{Z}_2)$ is the Bockstein map, then $\beta_{\Gamma'}(u) \neq 0$.

Now let Δ be the kernel of the map $\phi_{n-1} \colon \Gamma' \to \mathbb{Z}_{2^{n-1}}$. Since the restriction map $\phi_{n-1}|_\Delta \colon \Delta \to \mathbb{Z}_{2^{n-1}}$ is zero, it lifts further to $\mathbb{Z}_{2^n}$. If v represents the map $\phi_{n-1}|_\Delta$ in $H^1(\Delta; \mathbb{Z}_{2^{n-1}})$ and the Bockstein map is given by $\beta_\Delta \colon H^1(\Delta; \mathbb{Z}_{2^{n-1}}) \to H^2(\Delta; \mathbb{Z}_2)$, then the commutative diagram

$$H^1(\Gamma'; \mathbb{Z}_{2^{n-1}}) \longrightarrow H^1(\Delta; \mathbb{Z}_{2^{n-1}})$$
$$\beta_{\Gamma'} \downarrow \qquad\qquad \downarrow \beta_\Delta$$
$$H^2(\Gamma'; \mathbb{Z}_2) \longrightarrow H^2(\Delta; \mathbb{Z}_2)$$

gives $\beta_\Delta(v) = 0$ by naturality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. The Hermitian symmetric case

In this section we completely classify properly nonrigid Hermitian locally symmetric manifolds in high real dimension. Hermitian locally symmetric manifolds are Riemmanian symmetric spaces with a parallel complex structure with respect to which the Riemannian metric is Hermitian. In terms of the classification of Riemannian symmetric spaces, the Hermitian symmetric spaces are the four infinite series AIII, BDI with $p = 2$ or $q = 2$, DIII and CI, and two exceptional spaces, namely EIII and EVII. See Helgason [29] for additional information on Hermitian symmetric spaces.

	Manifold	Complex dimension
AIII	$\mathrm{SL}(p+q, \mathbb{C})$	pq
BDI $(q = 2)$	$\mathrm{SL}(p+2, \mathbb{C})$	p
DIII	$\mathrm{SL}(2n, \mathbb{C})$	$\frac{1}{2}n(n-1)$
CI	$\mathrm{Sp}(2n, \mathbb{C})$	$\frac{1}{2}n(n+1)$
EIII	$E_6^{\mathbb{C}}$	16
EVII	$E_7^{\mathbb{C}}$	27

In the following theorem, we demonstrate that a Hermitian locally symmetric manifold, when it is properly rigid, is properly rigid for fairly simple reasons.

Theorem 4.1. *If $M = \Gamma \backslash G / K$ is a Hermitian locally symmetric manifold of real dimension at least 5, then the following are equivalent:*

(1) *The $\mathbb{Q}$-rank of Γ is at least 3.*
(2) *The space M is properly topologically nonrigid.*

If these conditions are satisfied, then infinitely many elements in $S^{Top}(M)$ are in fact detected by a rational topological version of the first Pontrjagin class in $H^4(M;\mathbb{Q})$. Also in this case, the space M is properly nonrigid among smooth manifolds (where the conclusion is still homeomorphism).

Proof. Suppose that $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \leq 2$. Then the topological rigidity of $M = \Gamma\backslash G/K$ can be inferred from the work of Farrell-Jones [24] and Bartels-Lück-Reich-Rüping [7] by the following argument. In Block-Weinberger [9], Theorem 1.3 shows that proper rigidity of the relevant assembly map depends on an isomorphism statement for the manifold at infinity. By Borel-Serre [10] there is an exact sequence $1 \rightarrow F_\infty \rightarrow \pi_1(\partial M) \rightarrow \Gamma \rightarrow 1$, where F_∞ is the free group of countably infinitely many generators and ∂M is the Borel-Serre boundary. We identify $H_*(B\pi_1(\partial M), \mathbb{L}(e))$ with $H_*(B\Gamma; \mathbb{H}(BF_\infty; \mathbb{L}(e)))$ and then further identify this group with $H_*(B\Gamma; \mathbb{L}(F_\infty))$ by the Borel conjecture for free groups (infinite rank makes no difference by a limit argument). Finally the isomorphism $H_*(B\Gamma; \mathbb{L}(F_\infty)) \cong L_*(\pi_1(\partial M))$ is exactly the conclusion of the Farrell-Jones isomorphism conjecture in the nontrivial coefficient system $\mathbb{Z}F_\infty$.

If $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 3$, then we are in the π-π situation, so it suffices to prove that $H^4(\Gamma\backslash G/K; \mathbb{Z})$ is nontrivial. When $X = \Gamma\backslash G/K$ is a Hermitian symmetric space, one can take its Baily-Borel compactification $\overline{X}$, constructed by adding to X a singular set Σ, to obtain a singular projective variety. It has a nontrivial even-dimensional fundamental class $[\overline{X}] \in H_d(\overline{X}; \mathbb{Z})$ (see Baily-Borel [5] and Borel-Ji [14]). In fact, by taking hyperplane sections, one can find nontrivial elements in homological dimensions $d - 2k$ for all k. It is known that the complex codimension of $\Sigma \hookrightarrow \overline{X}$ in our case is at least 3 (see Jost-Yau [32]), in which case there is an isomorphism $H_*(\overline{X}; \mathbb{Z}) \hookrightarrow H_*(\overline{X}, \Sigma; \mathbb{Z})$ above the dimension of Σ. In particular we have an isomorphism if $* = d - 4$. By excision, the above group is isomorphic to $H_*(M_\Sigma, \partial M_\Sigma; \mathbb{Z})$, where M_Σ is the complement of a tubular neighborhood of Σ and ∂M_Σ is its boundary. The boundary is in fact the Borel-Serre boundary and M_Σ is a manifold. We then have $H_{d-4}(\overline{X}; \mathbb{Z}) \cong H_{d-4}(M_\Sigma, \partial M_\Sigma; \mathbb{Z}) \cong H^4(M_\Sigma; \mathbb{Z}) \cong H^4(\Gamma\backslash G/K; \mathbb{Z})$, so that $H^4(\Gamma\backslash G/K; \mathbb{Q})$ is nontrivial. Of course this result implies that $\Gamma\backslash G/K$ is properly nonrigid and its topological structure set is infinite. Moreover, as in Proposition 2.3 and Theorem 3.4, infinitely many of these elements are smoothable. $\square$

In addition, when (1) and (2) are satisfied, there are infinitely exotic versions N of M such that, for all $\Delta \leq \Gamma$ of finite index, the lifts N_Δ and M_Δ are (still) not homeomorphic.

Remark 4.2. Now that we know that the structure set is infinite for a reason based on p_1, we can actually realize a subset containing a lattice in $H^4(\Gamma; \mathbb{Z})/tors$ coming from smooth exotic versions of $\Gamma\backslash G/K$. It is easy to see that the automorphisms of Γ, lying in $\mathrm{SL}(H^4(\Gamma; \mathbb{Z})/tors)$, must have infinitely many orbits in our set, and

therefore we produce infinitely many manifolds, not just proper homotopy equivalences.

5. The virtual structure set

Having examined the proper rigidity or nonrigidity of locally symmetric spaces in a number of different cases, we now wish to define a notion of a *virtual structure set* associated to the covers of a manifold of the form $\Gamma \backslash G/K$. Suppose that Γ and Γ' are both torsion-free lattices of G with $\Gamma' \leq \Gamma$. Suppose that the index $[\Gamma : \Gamma']$ is finite. There is an induced covering map $\Gamma' \backslash G/K \to \Gamma \backslash G/K$, called a *lift of* $\Gamma \backslash G/K$, which induces a map $S^{Top}(\Gamma' \backslash G/K) \to S^{Top}(\Gamma \backslash G/K)$ of topological structure sets. There is then a transfer map $S^{Top}(\Gamma \backslash G/K) \to S^{Top}(\Gamma' \backslash G/K)$. Suppose Λ denotes the inverse system of inclusions $\Lambda \colon \cdots \to \Gamma_3 \to \Gamma_2 \to \Gamma_1 = \Gamma$ with each $\Gamma_i \leq G$ torsion-free and each index $[\Gamma_i : \Gamma_{i+1}]$ finite. We will say that Λ is a *finite index sequence* of torsion-free lattices in G. In this case, the induced direct system $S^{Top}(\Gamma \backslash G/K) = S^{Top}(\Gamma_1 \backslash G/K) \to S^{Top}(\Gamma_2 \backslash G/K) \to S^{Top}(\Gamma_3 \backslash G/K) \to \cdots$ of transfer maps gives rise to a limit object $S_\Lambda^{virt}(\Gamma \backslash G/K) \equiv \varinjlim S^{Top}(\Gamma_i \backslash G/K)$, called the *virtual structure set of* $\Gamma \backslash G/K$ *with respect to* Λ. Note that $S_\Lambda^{virt}(\Gamma \backslash G/K)$ has a group structure because each $S^{Top}(\Gamma_i \backslash G/K)$ can be given a group structure for which all the transfer maps are homomorphisms.

There are several natural examples of sequences Λ that can be studied.

(1) Let Δ_k be the intersection of all subgroups of Γ of index at most k. Define by $S^{virt}(\Gamma \backslash G/K)$ the structure set associated to the particular sequence $\Lambda \colon \cdots \to \Delta_3 \to \Delta_2 \to \Delta_1 \to \Gamma$. If $\Lambda' \colon \cdots \to \Gamma_3 \to \Gamma_2 \to \Gamma_1 \to \Gamma$ is a finite index sequence of torsion-free lattices of G, then there is an increasing sequence n_i of positive integers with maps $\Gamma_i \to \Delta_{n_i}$ such that all relevant compositions commute. Therefore Λ can be regarded as the terminal object in the collection of all finite index sequences of torsion-free lattices in G. We then suppress the mention of Λ in the notation $S^{virt}(\Gamma \backslash G/K)$, which we then call the *universal virtual structure set*. Our attention will be mainly be spent on the determination of this set. In the arithmetic case, this set is an invariant of $G(\mathbb{Q})$.

(2) Let Γ be a lattice in a linear algebraic group. We can then embed Γ as a subgroup of $\mathrm{SL}_N(\mathbb{Z})$ for large N. In $\mathrm{SL}_N(\mathbb{Z})$ we have the sequence of principal m-congruence subgroups given by $\Delta(m) = \ker(\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}_m))$, and we can then consider the corresponding congruence subgroups of Γ given by $\Gamma_m = \Gamma \cap \Delta(m)$. An interesting sequence is then given by $\Lambda \colon \cdots \to \Gamma_{m_3} \to \Gamma_{m_2} \to \Gamma_{m_1}$, where, for all i, there is a prime p_i for which $m_i = p_i m_{i+1}$.

(3) Using the notation in (2), fix a positive prime $p \in \mathbb{Z}$ and consider the sequence $\Lambda_p \colon \cdots \to \Gamma_{p^3} \to \Gamma_{p^2} \to \Gamma_p$.

Example 5.1. Let Λ as in (2) above. When the $\mathbb{Q}$-rank of Γ is large, then $S_\Lambda^{virt}(\Gamma\backslash G/K)$ contains an infinitely generated torsion group (as we showed in Section 3).

Example 5.2. Let Π be a product of a product of three punctured tori. We mentioned in the introduction that the universal virtual structure set $S^{virt}(\Pi)$ vanishes. However, using the notation in (3) above, we note that the virtual structure set $S_{\Lambda_2}^{virt}(\Pi)$ is trivial while, when p is an odd prime, the structure set $S_{\Lambda_p}^{virt}(\Pi)$ is an infinite sum of copies of $\mathbb{Z}_2$.

At this stage we turn to a discussion of the final goal of the paper: the determination of the universal virtual structure set $S^{virt}(\Gamma\backslash G/K)$ in a variety of cases. We begin with some terminology and a theorem of Ash.

Notation 5.3. Suppose that G is a semisimple algebraic group defined over $\mathbb{Q}$ with maximal compact subgroup $K \subseteq G(\mathbb{R})$ and $\Gamma_0 \subseteq G(\mathbb{Q})$ a torsion-free arithmetic subgroup. Let Γ be a subgroup of Γ_0 of finite index and let $e_P(\Gamma)$ be the number of distinct double cosets $(P \cap \Gamma_0)\backslash\Gamma_0/\Gamma$ for each fixed $\mathbb{Q}$-parabolic subgroup P of G and let d be the real dimension of the $\mathbb{R}$-unipotent radical of such a P.

Our main tool in computing the virtual structure set is the following theorem of Ash.

Theorem 5.4. *(Ash* [3] *and Schwermer* [45]*) In the situation given above, we have*

$$\dim H^d(\Gamma;\mathbb{R})/H_{(2)}^d(\Gamma;\mathbb{R}) \geq e_P(\Gamma).$$

In addition, the term $e_P(\Gamma)$ increases as Γ decreases, so the number of linearly independent non-square-integrable cohomology classes will tend to infinity.

We note that, if X is a space and Y is a finite-sheeted cover of X, then the rational cohomology of X injects into the rational cohomology of Y, so any nonzero elements in the former persist into the latter. Our discussion in Section 1 indicates that, using the result of Ash above, the virtual structure set of an arithmetic manifold $\Gamma\backslash G/K$ will be infinite if we can find a $\mathbb{Q}$-parabolic whose unipotent radical has real dimension d divisible by 4. Our analysis involves a calculation for each Lie group via its Dynkin diagram. These computations are relegated to Section 6 of the paper. In the following lemma and thereafter, the term "horospherical" refers to a unipotent radical of a parabolic subgroup.

Lemma 5.5. *Let G be a connected, semisimple algebraic group over $\mathbb{Q}$. If $\mathrm{rank}_\mathbb{Q} G \geq 4$, then G has a nontrivial horospherical $\mathbb{Q}$-subgroup U such that $\dim_\mathbb{R} U$ is divisible by 4.*

Proof. Suppose G does not have such a horospherical $\mathbb{Q}$-subgroup. Write $G = G_1 \times \cdots \times G_k$ with $k \geq 2$, where each G_i is almost simple. We may assume that each G_i is isotropic (since $\mathbb{Q}$-anisotropic factors do not affect the $\mathbb{Q}$-rank).

Case 1. Assume that $k = 1$. Inspection of the calculations verifies that every almost-simple group of $\mathbb{Q}$-rank ≥ 4 has an appropriate horospherical $\mathbb{Q}$-subgroup. More precisely, the only almost-simple $\mathbb{Q}$-groups G of $\mathbb{Q}$-rank ≥ 3 for which there

does not exist a nontrivial horospherical $\mathbb{Q}$-subgroup whose dimension is divisible by 4 are:

(1) type C_n or $^{1,2}D_n$ of $\mathbb{Q}$-rank 3, with $d = 2$ (i.e. constructed from a quaternion division algebra), or

(2) type E_7 of $\mathbb{Q}$-rank 3.

Case 2. Assume that $k = 2$. From Case 1, we know that $\mathrm{rank}_{\mathbb{Q}}G_i \leq 3$ for all i. Suppose that $\mathrm{rank}_{\mathbb{Q}}G_1 = 3$. Let m be the dimension of a nontrivial horospherical $\mathbb{Q}$-subgroup U_2 of G_2. Then $m \not\equiv 0 \bmod 4$. Also, we know that G_1 is one of the groups listed in Case 1, so, by inspection of the calculations, we see that it has a horospherical subgroup U_1 such that $\dim_{\mathbb{R}} U_1 \equiv -\dim_{\mathbb{R}} U_2 \bmod 4$. Then $\dim(U_1 \times U_2)$ is divisible by 4.

We may now assume $\mathrm{rank}_{\mathbb{Q}}G_1 = \mathrm{rank}_{\mathbb{Q}}G_2 = 2$. As was mentioned above, inspection of the calculations establishes that there is a nontrivial horospherical subgroup U_i of G_i such that $\dim_{\mathbb{R}} U_i$ is even. Then either $\dim_{\mathbb{R}} U_1$ or $\dim_{\mathbb{R}} U_2$ or $\dim(U_1 \times U_2)$ is divisible by 4.

Case 3. Assume that $k = 3$. There must exist $k \in \{1, 3\}$ such that, for each i, the dimension of every nontrivial horospherical $\mathbb{Q}$-subgroup of G_i is congruent to $k \bmod 4$. Since groups of rank at least 2 have a horospherical subgroup of even dimension, we have $\mathrm{rank}_{\mathbb{Q}}G_i = 1$. Therefore $\mathrm{rank}_{\mathbb{Q}}G = k = 3$, contradicting the fact that $\mathrm{rank}_{\mathbb{Q}}G \geq 4$.

Case 4. Assume that $k \geq 4$. Every collection of at least four integers has a nonempty subset whose sum is divisible by 4. $\qquad\square$

Remark 5.6. The calculations show that any group of rank at least 2 has a nontrivial horospherical $\mathbb{Q}$-subgroup of even dimension. Therefore, if $G = \mathrm{Res}_{\mathbb{F}/\mathbb{Q}}H$ is a restriction of scalars with $\mathrm{rank}_{\mathbb{Q}}G \geq 2$, and G does not have a horospherical $\mathbb{Q}$-subgroup of dimension divisible by 4, then $\mathbb{F}$ must have odd degree over $\mathbb{Q}$.

The lemma directly proves the following theorem about the virtual structure set.

Theorem 5.7. *Let G be a connected, semisimple algebraic group over $\mathbb{Q}$ and Γ an arithmetic lattice in G. If $\mathrm{rank}_{\mathbb{Q}}G \geq 4$, then $S^{virt}(\Gamma\backslash G/K) \otimes \mathbb{Q}$ has infinite $\mathbb{Q}$-dimension. If $\mathrm{rank}_{\mathbb{Q}}G \geq 3$ and Γ is irreducible, then the same result holds as long as G is not of the types listed in Case 1 of the previous lemma. If $\Gamma = \mathrm{SL}_n(\mathcal{O})$ for a number ring $\mathcal{O}$, or if Γ is the result of restriction by scalars for an extension $\mathbb{Q} \to \mathbb{F}$, where $[\mathbb{F} : \mathbb{Q}]$ is even, then the conclusion holds.*

Remark 5.8. Let T be the punctured torus, which has $\mathbb{Q}$-rank 1. The triple product $T \times T \times T$ has $\mathbb{Q}$-rank 3 (e.g. using coarse volume growth) but the virtual structure set is trivial. So without the irreducibility criterion, this theorem is false. However it may be the only example.

One may ask, in the case when Ash's theorem can be applied to prove that $S^{virt}(M) \otimes \mathbb{Q}$ has infinite as a $\mathbb{Q}$-vector space, how quickly the rank grows as a function of Γ. In other words, suppose that $G = \Gamma\backslash G/K$ has $\mathbb{Q}$-rank ≥ 4. Using the notation

of (1) in Section 5, we let Δ_k be the intersection of all subgroups of Γ of index at most k. Let $a_k(M)$ the rank of $S^{Top}(\Delta_k \backslash G/K)$ as a vector space over $\mathbb{Q}$. It is a consequence of the paper of Ash [2] that in fact the quantity $a_k(M)$ grows as a power of the index $[\Gamma : \Delta_k]$ (or volume). One may then try to identify the manifolds $M = \Gamma \backslash G/K$ for which the growth rate is linear.

Proposition 5.9. *Let $M = \Gamma \backslash G/K$ have $\mathbb{Q}$-rank at least 3, and let M^u be the compact dual of M. Then $a_k(M)$ grows linearly in the index $[\Gamma : \Delta_k]$ iff the Euler characteristic $\chi(M^u)$ is nonzero and $\dim M \equiv 0 \bmod 8$.*

Proof. This theorem is a consequence of Lück's approximation theorem relating L^2-cohomology of universal covers of compact spaces and the ordinary cohomology of their finite covers, together with the fact (first established by Cheeger and Gromov [20], but proved more conceptually by Gaboriau [26]) that, for nonuniform lattices, the L^2-cohomology of the universal covers of their Borel-Serre compactification is the same as that of the symmetric space. The details are as follows. The only nontrivial L^2-cohomology group arises in the middle dimension (see e.g. Olbrich [39]). Therefore the Betti numbers other than the middle one grow sublinearly. Consequently, linear growth occurs in a dimension that is a multiple of 4 iff the middle dimension is a multiple of four; i.e. $\dim(G/K)$ is a multiple of 8, and the Euler characteristic is nonzero. However, by Gauss-Bonnet, the Euler characteristic of M and of the compact dual are proportional (via the ratios of their volumes), completing the proof.   □

6. Computations

There is a method by which one can easily calculate the real dimension of the unipotent radical of a $\mathbb{Q}$-parabolic. The dimension of the unipotent radical associated to the Borel subgroup is the number of positive roots in the root system. For a smaller parabolic, we just need to subtract the number of positive roots that are in the Levi subgroup of the parabolic. The diagrams in Tits' *Classification of algebraic semisimple groups* [47] can be used to identify the Levi subgroup of each parabolic subgroup. The table on page 66 of Humphreys [30] tells us the number of positive roots there are in each simple factor of the Levi subgroup.

The complete list of Dynkin diagrams of semisimple groups over algebraically closed fields is given below. Each diagram determines a strict isogeny class of semisimple groups over any given field k.

	Dynkin diagram	Positive roots μ
A_n		$\binom{n+1}{2}$
B_n		n^2
C_n		n^2
D_n		$n^2 - n$
E_6		36
E_7		63
E_8		120
F_4		24
G_2		6

There is a natural one-to-one correspondence between the conjugacy classes of parabolic subgroups of G and the subsets of the set Δ of all simple roots. In the following, we denote by n the absolute (real) rank and r the relative (rational) rank of the considered group, i.e. the total number of vertices and the number of distinguished orbits of the diagram. Note that $0 \leq r \leq n$. See Platonov-Rapinchuk [41] for a general treatment.

In the following pages, we examine each type of Lie group G and calculate the real rank of the unipotent radical for various classes parabolic subgroup. We enumerate the Lie groups according to the subtypes given by Tits: ^1A_n, ^2A_n, B_n, C_n, ^1D_n, ^2D_n, $^{3,6}D_4$, $^1E_6$, $^2E_6$, E_7, E_8, F_4, G_2. The reader should refer to [47] for the subscripting and superscripting in this notation. For each of these types we use the Satake-Tits diagrams which classify real forms of the complex Lie algebra corresponding to the Dynkin diagrams. Our list of such diagrams is taken from Helgason [29], Satake [44], Onishchik-Vinberg [40] and Tits [47]. The nodes colored black correspond to the simple roots of the anisotropic kernel. We list the possible parabolic subgroups according to rational rank, which is given by the number of open circles (i.e. non-black nodes) in each diagram. A parabolic $\mathbb{Q}$-subgroup is determined by choosing any subset of these circles. Since we only want proper parabolics, our calculations always involve a nonempty subset. We then compute the real dimension of the associated unipotent radical by calculating the positive roots of the original diagram and then subtracting the positive roots of the subdiagrams that result upon deleting the open nodes.

Recall that, by Theorem 5.4 and Section 1, it suffices to find a parabolic whose unipotent radical has real dimension divisible by 4. In the case of $\mathbb{Q}$-rank at least 4, if we find one whose dimension is divisible by 4, then we calculate no further. Otherwise we consider cases of larger parabolics by filling in various open nodes and calculating the real dimension of unipotent radical corresponding to the new Satake-Tits diagram. The number of positive roots of particular Lie group G is denoted by μ.

The calculations below are done only for absolutely simple groups over $\mathbb{Q}$. For a restriction of scalars, the answers are to be multiplied by the degree of the field extension.

Type ^1A_n: Here we have the special linear group $\mathrm{SL}_{r+1}(D)$, where D is a central division algebra of degree d over k. These constants satisfy $d(r+1) = n+1$ and $d \geq 1$.
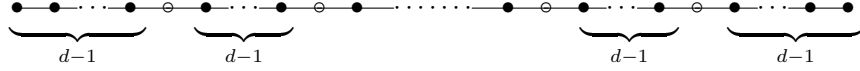
(1) rank 1:

$$\dim_{\mathbb{R}} U = \mu(A_n) - 2\mu(A_{d-1})$$
$$= \binom{n+1}{2} - 2\binom{d}{2}$$
$$= \binom{2d}{2} - 2\binom{d}{2}$$
$$= d^2 \equiv \begin{cases} 1 \bmod 4 & \text{if } d \text{ odd,} \\ 0 \bmod 4 & \text{if } d \text{ even.} \end{cases}$$

(2) rank 2:

$$\dim_{\mathbb{R}} U = \mu(A_n) - 3\mu(A_{d-1})$$
$$= \binom{n+1}{2} - 3\binom{d}{2}$$
$$= \binom{3d}{2} - \binom{d}{2} - \binom{2d}{2}$$
$$= 2d^2 \equiv \begin{cases} 2 \bmod 4 & \text{if } d \text{ odd,} \\ 0 \bmod 4 & \text{if } d \text{ even.} \end{cases}$$

If we color in one of the two white circles, we have

$$\dim_{\mathbb{R}} U = \mu(A_n) - \mu(A_{d-1}) - \mu(A_{2d-1})$$
$$= \binom{n+1}{2} - \binom{d}{2} - \binom{2d}{2}$$
$$= \binom{3d}{2} - 3\binom{d}{2}$$
$$= 3d^2 \equiv \begin{cases} 3 \bmod 4 & \text{if } d \text{ odd,} \\ 0 \bmod 4 & \text{if } d \text{ even.} \end{cases}$$

(3) rank $r \geq 3$:   if $d = 1$ and if $d = 2$

$$\begin{aligned}
\dim_{\mathbb{R}} U &= \mu(A_n) - \mu(A_{(k+1)d-1}) - (r-k)\mu(A_{d-1}) \\
&= \binom{n+1}{2} - \binom{(k+1)d}{2} - (r-k)\binom{d}{2} \\
&= \binom{d(r+1)}{2} - \binom{(k+1)d}{2} - (r-k)\binom{d}{2} \\
&= \frac{d^2}{2}(r^2 + r - k(k+1)) \\
&= \frac{d^2}{2}(r+k+1)(r-k).
\end{aligned}$$

This quantity is $0 \bmod 4$ when $r \equiv k \bmod 8$ or $r \equiv (7-k) \bmod 8$. For each r, one can choose an appropriate k so that $\dim_{\mathbb{R}} U \equiv 0 \bmod 8$.

Type ^2A_n: Here we have the special unitary group $\mathrm{SU}_{(n+1)/d}(D,h)$, where D is a central division algebra of degree d over a quadratic extension k' of k with an involution of the second kind σ such that $k = \{x \in k' : x^\sigma = x\}$ and h is a nondegenerate Hermitian form of index r relative to σ. The constants satisfy $d \mid (n+1)$ and $d \geq 1$. Also $2rd \leq n+1$.

or
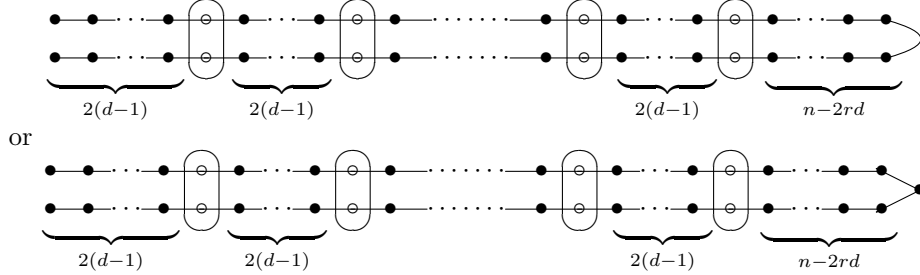
When $n+1 = 2rd$, the right end of the second picture is replaced with an open circle.

(1) rank 1:   for $d = 1$ and for $d = 2$

$$\begin{aligned}
\dim_{\mathbb{R}} U &= \mu(A_n) - 2\mu(A_{d-1}) - \mu(A_{n-2d}) \\
&= \binom{n+1}{2} - 2\binom{d}{2} - \binom{n-2d+1}{2} \\
&= 2dn + 2d - 3d^2 \\
&= \begin{cases} 0 & \text{if } d \text{ even,} \\ 1 & \text{if } d \text{ and } n \text{ odd,} \\ 3 & \text{if } d \text{ odd and } n \text{ even.} \end{cases}
\end{aligned}$$

(2) rank $r \geq 2$:

 for $d = 1$ and for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(A_n) - 2r\mu(A_{d-1}) - \mu(A_{n-2rd})$$

$$= \binom{n+1}{2} - 2r\binom{d}{2} - \binom{n-2rd+1}{2}$$

$$= \begin{cases} r(2n - 2r + 1) & \text{if } d = 1, \\ 4rn - 8r^2 & \text{if } d = 2. \end{cases}$$

When $d = 2$ we have $\dim_{\mathbb{R}} U = 4rn - 8r^2 \equiv 0 \bmod 4$. For $d = 1$, the quantity $r(2n - 2r + 1)$ may not be divisible by 4. In this latter case, we take the larger parabolic associated to ●—○—●—●—●——⋯, for which $\dim_{\mathbb{R}} U = \mu(A_n) - 2\mu(A_1) - A_{n-4}$. This calculation has been done in the rank 1 case (with $d = 2$), and is congruent to $0 \bmod 4$.

Type B_n or C_n: Here B_n is the special orthogonal group $\mathrm{SO}_{2n+1}(k, q)$, where q is a quadratic form of index r, and defect 1 in case char $k = 2$. Here C_n is the special unitary group $\mathrm{SU}_{2n/d}(D, h)$, where D is a division algebra of degree d over k, and h is a nondegenerate antihermitian sesquilinear form of index r relative to an involution σ of the first kind such that D^σ (the space of symmetric elements) has dimension $\binom{d}{2}$. When $d = 1$, the group is simply $\mathrm{Sp}_{2n}(k)$. For C_n, we have $d = 2^a | 2n$ and $d \geq 1$. If $d = 1$, then $n = r$.

●—●—⋯—●—○—●—●—⋯—●—○—●—●——⋯⋯——●—●—○—●—⋯—●—○—●—●—⋯—●—●⇐○

underbraces: $d-1$ $\quad$ $d-1$ $\quad\quad$ $d-1$ $\quad$ $n-rd$
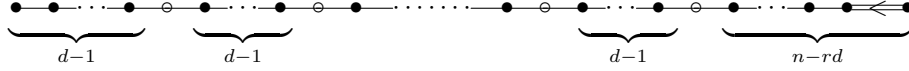
When $n = rd$, the right end becomes ●———●⇐○

(1) rank 1: $\quad$ ○—●—●—●——⋯ for $d = 1$ and ●—○—●—●—●——⋯ for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(B_n) - \mu(A_{d-1}) - \mu(B_{n-d})$$

$$= n^2 - \binom{d}{2} - (n - d)^2$$

$$= 2dn - d^2 - \binom{d}{2}$$

$$= \begin{cases} 1 & \text{if } d \text{ and } n \text{ odd}, \\ 3 & \text{otherwise}. \end{cases}$$

(2) rank 2:
(a) $\quad$ ○—○—●—●—●——⋯ for $d = 1$

$\dim_{\mathbb{R}} U = \mu(B_n) - \mu(B_{n-2}) = n^2 - (n-2)^2 = 4n - 4 \equiv 0 \bmod 4$.

(b) $\quad$ ●—○—●—○—●—●—●——⋯ if $d = 2$

$\dim_{\mathbb{R}} U = \mu(B_n) - 2\mu(A_1) - \mu(B_{n-4}) = n^2 - 2(1) - (n-4)^2 = 8n - 18 \equiv 2 \bmod 4$.

Since we do not have $0 \bmod 4$, we compute larger parabolics by turning white nodes black. The case in which we fill in the leftmost white node

has been computed in (1). The remaining case is

$$\bullet\!-\!\bullet\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots$$

for which $\dim_{\mathbb{R}} U = \mu(B_n) - \mu(A_3) - \mu(B_{n-4}) = n^2 - 6 - (n-4)^2 = 8n - 22 \equiv 2 \bmod 4$.

(3) rank 3: We can assume $d = 2$, since we have 0 above for $d = 1$ (see 2a).

$$\circ\!-\!\circ\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\cdots \text{ for } d=1 \text{ and } \bullet\!-\!\circ\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots \text{ for } d=2$$
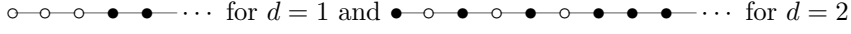
   (a) For $d = 1$, we immediately resort to the larger parabolic by blackening the third white dot. This case is computed in 2a to yield 0 mod 4.

   (b) For $d = 2$, we have $\dim_{\mathbb{R}} U = \mu(B_n) - 3\mu(A_1) - \mu(B_{n-6}) = n^2 - 3(1) - (n-6)^2 = 12n - 39 \equiv 1 \bmod 4$. Since we do not have 0 mod 4, we compute the dimension related to the higher parabolics that are previously unconsidered.

      (i) $\quad\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots$

$\dim_{\mathbb{R}} U = \mu(B_n) - \mu(A_5) - \mu(B_{n-6}) = n^2 - 15 - (n-6)^2 = 12n - 51 \equiv 1 \bmod 4$.

      (ii) $\quad\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots \text{ or } \bullet\!-\!\bullet\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots$
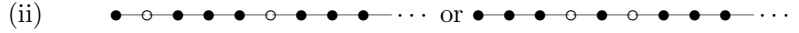
$\dim_{\mathbb{R}} U = \mu(B_n) - \mu(A_1) - \mu(A_3) - \mu(B_{n-6}) = n^2 - 1 - 6 - (n-1)^2 = 12n - 43 \equiv 1 \bmod 4$.

(4) rank 4: The $d = 1$ can be handled as in (3a). We may then assume $d = 2$. In this case, we immediately consider the larger parabolic obtained by blackening the three leftmost white nodes:

$$\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots$$

$\dim_{\mathbb{R}} U = \mu(B_n) - \mu(A_7) - \mu(B_{n-8}) = n^2 - \binom{8}{2} - (n-8)^2 = 16n - 92 \equiv 0 \bmod 4$.

Type ^1D_n: Over $\mathbb{Q}$ there are only the possibilities $d = 1$ and 2. If char $k \neq 2$, we have the special unitary group $\mathrm{SU}_{2n/d}(D, h)$, where D is a central division algebra of degree 2 over k, and h is a nondegenerate Hermitian form of discriminant 1 and index r, relative to an involution σ of the first kind such that D^σ (the space of symmetric elements) has dimension $\binom{d}{2}$. Here $d = 2^a | 2n$ and $d \geq 1$. Also $rd \leq n$ and $n \neq rd + 1$.
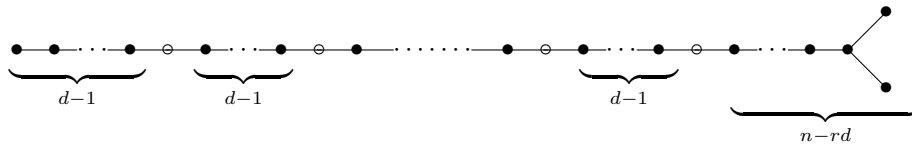
When $n - rd \leq 2$, the right end has one of the following forms. Note that the case $n = rd + 1$ does not occur. In the cases (i) $(n, d) = (r, 1)$, (ii) $(n, d) = (2r, 2)$, (iii) $n = rd$ and $d \geq 3$, (iv) $n = rd + 2$, we have the following, respectively:

(1) rank 1:   ○—●—●—●—⋯ for $d = 1$ or ●—○—●—●—⋯ for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(D_n) - \mu(A_{d-1}) - \mu(D_{n-d})$$

$$= (n^2 - n) - \binom{d}{2} - ((n-d)^2 - (n-d))$$

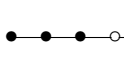$$= 2nd - d^2 - d - \binom{d}{2}$$

$$= \begin{cases} 1 & \text{if } d = 2, \\ 2 & \text{if } n \text{ even and } d = 1, \\ 0 & \text{if } n \text{ and } d \text{ odd.} \end{cases}$$

(2) rank 2:   ○—○—●—●—●—●—⋯ for $d = 1$ and ●—○—●—○—●—●—●—⋯ for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(D_n) - 2\mu(A_{d-1}) - \mu(D_{n-2d})$$

$$= (n^2 - n) - 2\binom{d}{2} - ((n-2d)^2 - (n-2d))$$

$$= 4nd - 4d^2 - 2d - 2\binom{d}{2}$$

$$\equiv 2d - 2\binom{d}{2} \mod 4$$

$$\equiv \begin{cases} 2 \mod 4 & \text{if } d = 1, \\ 2 \mod 4 & \text{if } d = 2. \end{cases}$$

Since we do not have 0 mod 4, we consider the cases of larger parabolics.
●—○—●—●—●—⋯ for $d = 1$ and ●—●—●—○—●—●—⋯ for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(D_n) - \mu(A_{2d-1}) - \mu(D_{n-2d})$$

$$= (n^2 - n) - \binom{2d}{2} - ((n-2d)^2 - (n-2d))$$

$$\equiv \begin{cases} 1 \mod 4 & \text{if } d = 1, \\ 2 \mod 4 & \text{if } d = 2. \end{cases}$$

(3) rank 3: Given the situation in rank 1, we may assume that d or n is even.
○—○—○—●—●—●—⋯ for $d = 1$ and ●—○—●—○—●—○—●—●—●—⋯ for $d = 2$
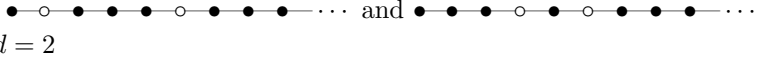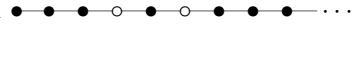
$$\dim_{\mathbb{R}} U = \mu(D_n) - 3\mu(A_{d-1}) - \mu(D_{n-3d})$$

$$= (n^2 - n) - 3\binom{d}{2} - ((n-6)^2 - (n-6))$$

$$= 6dn - 9d^2 - 3d - 3\binom{d}{2}$$

$$\equiv \begin{cases} 0 \mod 4 & \text{if } d = 1 \text{ (here } n \text{ even)}, \\ 3 \mod 4 & \text{if } d = 2. \end{cases}$$

Given this calculation, we now compute the dimension for larger parabolics
when $d = 2$. We only list the cases that have not been previously computed.

(a)   ●—●—●—●—●—○—●—●—● ——— $\cdots$ for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(D_n) - \mu(A_5) - \mu(D_{n-6})$$

$$= (n^2 - n) - \binom{6}{2} - ((n-6)^2 - (n-6))$$

$$= 12n - 21 \equiv 3 \mod 4.$$

(b)   ●—○—●—●—●—○—●—●—● ——— $\cdots$ and ●—●—●—○—●—○—●—●—● ——— $\cdots$
for $d = 2$

$$\dim_{\mathbb{R}} U = \mu(D_n) - \mu(A_1) - \mu(A_3) - \mu(D_{n-6})$$

$$= (n^2 - n) - \binom{2}{2} - \binom{4}{2} - ((n-6)^2 - (n-6))$$

$$= 12n - 49 \equiv 3 \mod 4.$$

(4) rank ≥ 4. We may assume that $d = 2$.   ●—○—●—○—●—○—●—○—●—● ——— $\cdots$

$$\dim_{\mathbb{R}} U = \mu(D_n) - \mu(D_{n-8}) - 4\mu(A_1)$$

$$= (n^2 - n) - ((n-8)^2 - (n-8)) - 4\binom{2}{2}$$

$$= 16n - 76 \equiv 0 \mod 4.$$

Type ^2D_n: The same as ^1D_n except that all forms in question have now discriminant $\neq 1$. Here $d = 2^a | 2n$ and $d \geq 1$. Also $rd \leq n - 1$.
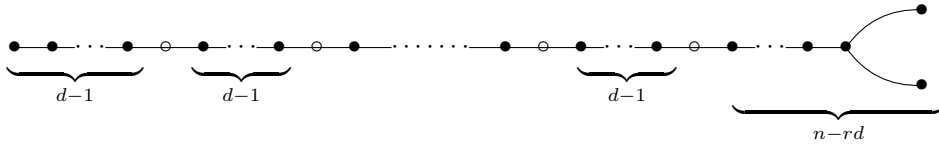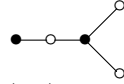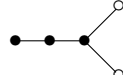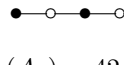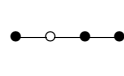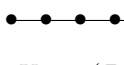
When $n = rd + 1$, then $d = 1$ or 2, and the right end becomes respectively:

Calculations are the same as for ^1D_n except that the Dynkin diagram is slightly different if $n = rd + 1$, where r is the rank under consideration. We have the following cases. Recall that the two rightmost white nodes are always clustered as one.
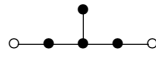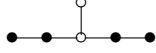
(1) When $d = 1$, we have $r = n - 1$. Now note that $\mu(A_1) \equiv 1 \mod 4$, $\mu(A_2) \equiv 3 \mod 4$ and $\mu(A_3) \equiv 2 \mod 6$. Therefore $\dim_{\mathbb{R}} U = \mu(D_n) - \mu(A_j) \equiv 0 \mod 4$ for an appropriate choice of $j \in \{0, 1, 2, 3\}$.

(2) When $d = 2$, we have $2r = n - 1$, so that n is odd.

(a) $r = 2$ and $n = 5$.

$\dim_{\mathbb{R}} U = \mu(D_5) - 2\mu(A_1) = 20 - 2 = 18 \equiv 2 \bmod 4.$

Since we do not have $0 \bmod 4$, we compute the dimension for the larger parabolic

$\dim_{\mathbb{R}} U = \mu(D_5) - \mu(A_3) = 20 - 6 = 14 \equiv 2 \bmod 4.$

(b) $r = 3$ and $n = 7$.

$\dim_{\mathbb{R}} U = \mu(D_7) - 3\mu(A_1) = 42 - 3 = 39 \equiv 3 \bmod 4.$

Since we do not have $0 \bmod 4$, we compute the dimension for larger parabolics.

(i)   and

$\dim_{\mathbb{R}} U = \mu(D_7) - \mu(A_3) - \mu(A_1) = 42 - 6 - 1 = 35 \equiv 1 \bmod 4.$

(ii)

$\dim_{\mathbb{R}} U = \mu(D_7) - \mu(A_5) = 42 - 15 = 27 \equiv 2 \bmod 4.$

(c) Let $r \geq 4$. Then let j be the least residue of $r \bmod 4$, and select the parabolic associated to $D_{2r+1} - jA_3 - (k - 2j)A_1$. Then we have
$\dim_{\mathbb{R}} U = (2r+1)^2 - (2r+1) - j\binom{3}{2} - (r - 2j) = 4k^2 + k - j \equiv 0 \bmod 4.$

Type $^{3,6}D_4$: The subtype involving D_n when $n = 4$ requires special consideration.

(1) rank 1:

$\dim_{\mathbb{R}} U = \mu(D_4) - 3\mu(A_1) = 12 - 3 = 1 \equiv 1 \bmod 4.$

(2) rank 2:   Note that here we have the Borel subgroup.

$\dim_{\mathbb{R}} U = \mu(D_4) = 12 \equiv 0 \bmod 4.$

Type $^1E_6$: Here $^1E_{6,2}^{28}$ is the form which can be realized as a collineation group of a Cayley plane, and $^1E_{6,2}^{16}$ is the form constructed by means of an associative division algebra of degree 3. There is nothing of rank 1.

(1) first group of rank 2

$\dim_{\mathbb{R}} U = \mu(E_6) - \mu(D_4) = 36 - 12 = 24 \equiv 0 \bmod 4.$

(2) second group of rank 2

$\dim_{\mathbb{R}} U = \mu(E_6) - 2\mu(A_2) = 36 - 2(3) = 30 \equiv 2 \bmod 4.$

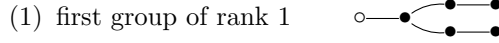Since we do not have $0 \bmod 4$, we consider larger parabolics.

(a)

$\dim_{\mathbb{R}} U = \mu(E_6) - \mu(A_5) = 36 - \binom{6}{2} = 21 \equiv 1 \bmod 4.$

(b)
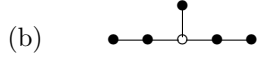
$$\dim_{\mathbb{R}} U = \mu(E_6) - 2\mu(A_2) - \mu(A_1) = 36 - 2(3) - 1 = 29 \equiv 1 \bmod 4.$$

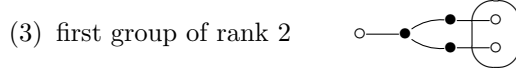(3) In higher rank cases, we can always pass to the larger parabolic associated to the first group of rank 2 (Case 1).

Type $^2E_6$: There are five subtypes in this case.

(1) first group of rank 1

$$\dim_{\mathbb{R}} U = \mu(E_6) - \mu(A_5) = 36 - \binom{6}{2} = 21 \equiv 1 \bmod 4.$$

(2) second group of rank 1

$$\dim_{\mathbb{R}} U = \mu(E_6) - \mu(D_4) = 36 - 12 = 24 \equiv 0 \bmod 4.$$

(3) first group of rank 2

Here we resort to the larger parabolic associated with the diagram in Case 2 above and obtain 0 mod 4.
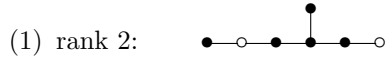
(4) second group of rank 2

This calculation has been done in the case of $^1E_6$ in the second group of rank 2, in which we obtain 1 or 2 mod 4.
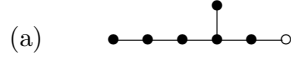
(5) higher rank:
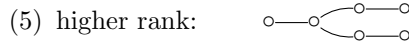
$$\dim_{\mathbb{R}} U = \mu(E_6) = 36 \equiv 0 \bmod 4.$$

Type E_7: Here $E_{7,3}^{28}$ is the form constructed by means of a Cayley division algebra. There are no subtypes of rank 1.
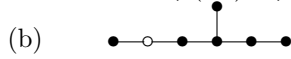
(1) rank 2:
$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(A_1) - \mu(D_4) = 63 - 1 - (4^2 - 4) = 50 \equiv 2 \bmod 4.$$
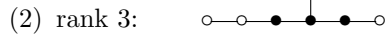
Since we do not have 0 mod 4, we consider larger parabolics.

(a)
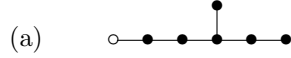$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(D_6) = 63 - (6^2 - 6) = 33 \equiv 1 \bmod 4.$$

(b)
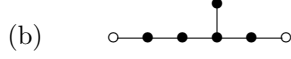$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(A_1) - \mu(D_5) = 63 - 1 - (5^2 - 5) = 42 \equiv 2 \bmod 4.$$

(2) rank 3:
$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(D_4) = 63 - (4^2 - 4) = 51 \equiv 3 \bmod 4.$$

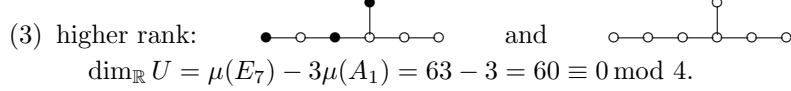Since we do not have 0 mod 4, we consider larger parabolics.

(a)
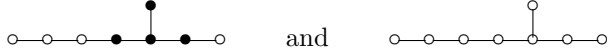$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(E_6) = 63 - 36 = 27 \equiv 3 \bmod 4.$$

(b)

$$\dim_{\mathbb{R}} U = \mu(E_7) - \mu(D_5) = 63 - (5^2 - 5) = 43 \equiv 3 \bmod 4.$$

(c) The remaining case is handled in (1).

(3) higher rank: and

$$\dim_{\mathbb{R}} U = \mu(E_7) - 3\mu(A_1) = 63 - 3 = 60 \equiv 0 \bmod 4.$$
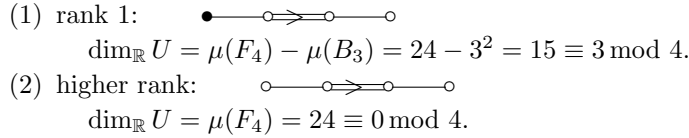
Type E_8: Here $E_{8,4}^{28}$ is the form constructed by means of a Cayley division algebra. There are only two subtypes under consideration:

         and

$$\dim_{\mathbb{R}} U = \mu(E_8) - \mu(D_4) = 120 - (4^2 - 4) = 108 \equiv 0 \bmod 4.$$

Type F_4: A group of this type is always the automorphism group of an exceptional simple Jordan algebra J. There are only two subtypes under consideration.

(1) rank 1:

$$\dim_{\mathbb{R}} U = \mu(F_4) - \mu(B_3) = 24 - 3^2 = 15 \equiv 3 \bmod 4.$$

(2) higher rank:

$$\dim_{\mathbb{R}} U = \mu(F_4) = 24 \equiv 0 \bmod 4.$$

Type G_2: A group of type G_2 is always the automorphism group of a Cayley algebra.

rank 2: $\dim_{\mathbb{R}} U = \mu(G_2) = 6 \equiv 2 \bmod 4.$

Since we do not have $0 \bmod 4$, we compute the case of the larger parabolic for which $\dim_{\mathbb{R}} U = \mu(G_2) - \mu(A_1) = 6 - 1 = 5 \equiv 1 \bmod 4.$

Conclusion: When the $\mathbb{Q}$-rank is at least 4, there is always a parabolic subgroup for which the real dimension of the unipotent radical is divisible by 4. When $\mathbb{Q}$-rank is 3, there are some exceptions.

Remark 6.1. There is a rank-3 form of E_7 that does not have a unipotent radical whose dimension is divisible by 4. This $\mathbb{Q}$-form of G could be split over $\mathbb{R}$. So both the Hermitian symmetric real form and the split real form of E_7 have arithmetic lattices in which no nontrivial horospherical $\mathbb{Q}$-subgroup has dimension divisible by 4. In the Hermitian case, the structure set is nontrivial and infinite, but we do not know more.

References

1. D. W. Anderson and Luke Hodgkin, *The K-theory of Eilenberg-MacLane complexes*, Topology **7** (1968), 317–329.
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