The word problem and Applications

1. Review of covering spaces
2. Free products, free products with amalgamation, HNN Extensions.
3. Bass-Serre theory
4. Presentations of groups, statement of the word problem
5. Dehn function.
6. Examples: Free groups, surface groups, Heisenberg group, Baumslag-Solitar group, Gersten’s group and more.
7. Van Kampen’s theorem and when fundamental groups don’t inject. (Relevant to Adian-Rabin theorem)
8. Turing machines
9. Post’s theorem
11. Introduction to Van Kampen diagrams and unsolvable word problem (Theorem of Boone-Novikov)
12. Gromov’s theorem on closed geodesics (application to differential geometry)
13. Higman’s theorem (and Universal groups)
14. Adian-Rabin theorem (Markov properties)
15. Homology of groups and Gordon’s theorem
17. Universal Central extensions and S. Novikov’s theorem
18. Other unsolvable topological problems (including non-c.e. Sets);
19. Sobolev homology?

References:
Rotman, Theory of Groups (GTM)
Stillwell, The word problem, Bull AMS
Algorithmic and classification problems in Group theory (Baumslag and Miller, editors, MSRI Publication)
Serre and Kilmer, Trees (Springer) and
Articles by Gersten and Riley on Dehn Functions
Brown, Cohomology of Groups (GTM)
Weinberger, Computers, Rigidity and Moduli (PUP)
\( \mathbb{R} \times [0,1] \xrightarrow{\phi} \mathbb{R} \)

\( (x, y) \mapsto (x + 2, y) \Rightarrow (x+2, 1+y) \),

\( \mathbb{R} \times [0,1] \xrightarrow{\psi} \mathbb{R} \times \mathbb{R} \)

\( (x, y) \mapsto (x+1, 1+y) \),

\( \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \).
2-fold cover of Moebius Strip is an annulus: the boundary is disconnected.

3-fold cover is MS again.
Connection to group actions on trees.
The picture is a particular case of a Bass-Serre tree associated to the Annel. Free product construction.

T/Γ.
In general

\[ \Gamma \times T \]

Then

\[ \Gamma \text{ can be built by } \]

And F.P. 1 + H N N extension from

\[ \Gamma \} \quad \text{[9, 1, 7]} = \text{s} 4 \text{,} 2 \text{,} \text{2} \]
\[ a^0 = a \]
\[ \langle a, b \mid bab^{-1} = a^2 \rangle \]
\[ b^n a b^{-n} = b a b^{-1} = (b a b^{-1})^{2^{n-1}} = (a^2)^{2^{n-1}} = a^{2^n} \]

So intuitively seeing that \([a^2, a] = e\),

\[ b^n a b^{-n} a b^{-1} b^{-n} a^{-1} = e \]

looks like it involves exponential size use of relations.
Redraw this picture.

Exercises:

The # of squared sides that is exponential.

\[ a, a^b \]
\[ \langle a, b, c \mid b a b^{-1} = a^2 \quad c b c^{-1} = b^2 \rangle \]

Same with \[ a \quad b \quad c \quad b^{-1} \quad c^{-1} \quad b^2 \quad a \quad b^{-2} \quad a^2 \quad b^2 \]
gives Double exponential.

\[ \langle a, b, c, d \mid b c b^{-1} = a^2 \quad c b c^{-1} = b^2 \rangle \]

\[ d c d^{-1} = c^2 \]

Triple exponential.
formal why just using the langley graph.

van kempen diagram.

area of \( V \) of \( \text{cell} \).
Gersten's Group.

\[ \langle x, y \mid x^2 = x^2 \rangle \]

\[ \langle x, z \mid 2x^{-1} = x^2 \rangle \]

Dehn function

BS$_1$
This is the triple exponential group
Dehn function for \( \mathbb{Z}^{2} \),
or \( \mathbb{Z}^{n} \) using Stokes' theorem.

All begin using calculus.
Heisenberg group \( H = \begin{bmatrix} \theta(r) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

\[ = \langle g, h \mid [g, [g, h]] = [h, [g, h]] \rightarrow e \rangle \]

\[ D_H(n) \sim n^3 \]

\[ [g^n, [g^n, h]] = e. \]

\[ B(n) \sim n^4 \text{ elements in } H. \]
Van Kampen diagrams.←
(Hyperbolic groups + Small Cancellation?)

Groups where Dehn function is linear (like a free group)

\[ \exists \text{ greedy algorithm to tell if } w = e. \]

Classical examples

\[ \langle a, b, c | a^2 b^3 c^4 d^5 = 1 \rangle \]

relation doesn't intersect itself
SMALL CANCELLATION GROUPS.
Word problem.

Problem about groups, not presentations.

Dehn function and geometric meaning of the word problem in terms of isoperimetry and constructing the Cayley graph.

Theorem (Boone - Novikov)

\[ \text{There are groups with unsolvable word problem} \]

\[ \text{There are groups with Dehn function growing faster than any computable function.} \]
Where we are going:

WP is unsolvable.

Most things can’t be told from a presentation (Aelian-Rabin).

R.E. is algebraic action (Higman).

Geometric or topological variations.

(Analytic?)

Closed Geodesics

Novikov’s theorem (Requires some cohomology of groups)

Other unsolvable problems.
Defn: $P$ is a Markov property of a group if $Q \in P$ and $G \in P \Rightarrow K \in P.$ (e.g., nontrivial, nonabelian, etc.)

In a nontrivial property inherited by subgroups.

[Nonexample: Being simple] but still indecidable.

There are still a lot of other non-Markovian "dich Rezniks."
**Theorem:** (Adian - Rabin)

There is no algorithm to tell if a finite presentation describes a Markovian property P group.

**Proof:** Uses HNN construction.
The knot is trivial.

\[ \pi_1(\mathbb{R}^3 - \text{Knot})/\text{Conf} \]

is a complicated group if \( \text{Conf} \) unless the knot is trivial.
LEMMA 3.6 (Main Technical Lemma). Let $K$ be a group given by a presentation on a finite or countably infinite set of generators, say

$$K = \langle x_1, x_2, \ldots \mid R_1 = 1, R_2 = 1, \ldots \rangle.$$ 

For any word $w$ in the given generators of $K$, let $L_w$ be the group with presentation obtained from the given one for $K$ by adding three new generators $a, b, c$ together with defining relations

\begin{align*}
(1) & \quad a^{-1}ba = \begin{cases} c^{-1}b^{-1}c, \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}c, \\ a^{-3}[w, b]a^3 = c^{-3}bc^3, \\ a^{-(3+i)}x_i ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)} & i = 1, 2, \ldots \\ \end{cases} \\
(2) & \quad a^{-1}ba = \begin{cases} c^{-1}b^{-1}c, \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}c, \\ a^{-3}[w, b]a^3 = c^{-3}bc^3, \\ a^{-(3+i)}x_i ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)} & i = 1, 2, \ldots \\ \end{cases} \\
(3) & \quad a^{-1}ba = \begin{cases} c^{-1}b^{-1}c, \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}c, \\ a^{-3}[w, b]a^3 = c^{-3}bc^3, \\ a^{-(3+i)}x_i ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)} & i = 1, 2, \ldots \\ \end{cases} \\
(4) & \quad a^{-1}ba = \begin{cases} c^{-1}b^{-1}c, \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}c, \\ a^{-3}[w, b]a^3 = c^{-3}bc^3, \\ a^{-(3+i)}x_i ba^{(3+i)} = c^{-(3+i)}bc^{(3+i)} & i = 1, 2, \ldots \\ \end{cases} \\
\end{align*}

where $[w, b]$ is the commutator of $w$ and $b$. Then

1. if $w \neq_K 1$ then $K$ is embedded in $L_w$ by the inclusion map on generators;
2. the normal closure of $w$ in $L_w$ is all of $L_w$; in particular, if $w =_K 1$ then $L_w \cong 1$, the trivial group;
3. $L_w$ is generated by the two elements $b$ and $ca^{-1}$.

If the given presentation of $K$ is finite, then the specified presentation of $L_w$ is also finite.
Definition. A **quadruple** is a 4-tuple of one of the following three types:

- \( q_i s_j s_k q_l \),
- \( q_i s_j Rq_l \),
- \( q_i s_j Lq_l \).

A **Turing machine** \( T \) is a finite set of quadruples no two of which have the same first two letters. The **alphabet** of \( T \) is the set \( \{ s_0, s_1, \ldots, s_M \} \) of all \( s \)-letters occurring in its quadruples.

Definition. An **instantaneous description** \( \alpha \) is a positive word of the form \( \alpha = \sigma q_i \tau \), where \( \sigma \) and \( \tau \) are \( s \)-words and \( \tau \) is not empty.

**What's on the tape? \( \text{'and 'where's The machine'} \)**

Definition. Let \( T \) be a Turing machine. An ordered pair \( (\alpha, \beta) \) of instantaneous descriptions is a **basic move** of \( T \), denoted by

\[ \alpha \rightarrow \beta, \]

if there are (possibly empty) positive \( s \)-words \( \sigma \) and \( \sigma' \) such that one of the following conditions hold:

(i) \( \alpha = \sigma q_i s_j \sigma' \) and \( \beta = \sigma q_i s_k \sigma', \) where \( q_i s_j s_k q_l \in T; \)

(ii) \( \alpha = \sigma q_i s_j s_k \sigma' \) and \( \beta = \sigma q_i s_l \sigma', \) where \( q_i s_j Rq_l \in T; \)

(iii) \( \alpha = \sigma q_i s_j \) and \( \beta = \sigma q_i s_l s_0, \) where \( q_i s_j Rq_l \in T; \)

(iv) \( \alpha = \sigma s_k q_i s_j \sigma' \) and \( \beta = \sigma q_i s_l s_j \sigma', \) where \( q_i s_j Lq_l \in T; \)

(v) \( \alpha = q_i s_j \sigma' \) and \( \beta = q_i s_0 s_j \sigma', \) where \( q_i s_j Lq_l \in T. \)

If \( \alpha \) describes the tape at a given time, the state \( q_i \) of \( T \), and the symbol \( s_j \) being scanned, then \( \beta \) describes the tape, the next state of \( T \), and the symbol being scanned after the machine's next move. The proviso in the definition of a Turing machine that no two quadruples have the same first two symbols means that there is never ambiguity about a machine's next move: if \( \alpha \rightarrow \beta \) and \( \alpha \rightarrow \gamma \), then \( \beta = \gamma. \)

Some further explanation is needed to interpret basic moves of types (iii) and (v). Tapes are finite, but when the machine comes to an end of the tape, the tape is lengthened by adjoining a blank square. Since \( s_0 \) means blank, these two rules thus correspond to the case when \( T \) is scanning either the last symbol on the tape or the first symbol.
Turing Machine versus Mona Lisa.
Suppose that a semigroup $\Gamma$ has a presentation

$$\Gamma = (X | \alpha_j = \beta_j, j \in J).$$

If $\omega$ and $\omega'$ are positive words on $X$, then it is easy to see that $\omega = \omega'$ in $\Gamma$ if and only if there is a finite sequence

$$\omega \equiv \omega_1 \rightarrow \omega_2 \rightarrow \cdots \rightarrow \omega_t \equiv \omega',$$

where $\omega_i \rightarrow \omega_{i+1}$ is an elementary operation; that is, either $\omega_i \equiv \sigma \alpha_j \tau$ and $\omega_{i+1} \equiv \sigma \beta_j \tau$ for some $j$, where $\sigma$ and $\tau$ are positive words on $X$ or $\omega_{i+1} \equiv \sigma \beta_j \tau$ and $\omega_t \equiv \sigma \alpha_j \tau$.

**Post's Theorem**

**Definition.** If $T$ is a Turing machine having stopping state $q_0$, then its associated semigroup $\Gamma(T)$ has the presentation:

$$\Gamma(T) = (q, h, s_0, s_1, \ldots, s_M, q_0, q_1, \ldots, q_N | R(T)),$$

where the relations $R(T)$ are

$$q_i s_j = q_i s_k \quad \text{if} \quad q_i s_j s_k q_i \in T,$$

for all $\beta = 0, 1, \ldots, M$:

$$q_i s_j s_\beta = s_j q_i s_\beta \quad \text{if} \quad q_i s_j R q_i \in T;$$

$$q_i s_j h = s_j q_i s_0 h \quad \text{if} \quad q_i s_j R q_i \in T;$$

$$s_\beta q_i s_j = q_i s_\beta s_j \quad \text{if} \quad q_i s_j L q_i \in T;$$

$$h q_i s_j = h q_i s_0 s_j \quad \text{if} \quad q_i s_j L q_i \in T;$$

$$q_0 s_\beta = q_0,$$

$$s_\beta q_0 h = q_0 h,$$

$$h q_0 h = q.$$
**Theorem:** In $T(P)$ let $\omega \equiv h$ if $\omega$ is a stopping state for the machine $P$.

$\Rightarrow$ is pretty obvious.

$\Leftarrow$ Focus on "special words $hah$" instaneous description

Let $\omega$ and $\omega'$ be words on $\{s_0, s_1, \ldots, s_M, q_0, q_1, \ldots, q_N\}$ with $\omega \neq q$ and $\omega' \neq q$. If $\omega \rightarrow \omega'$ is an elementary operation, then $\omega$ is $h$-special if and only if $\omega'$ is $h$-special.

If $\omega = hah$ is $h$-special, $\omega' \neq q$, and $\omega \rightarrow \omega'$ is an elementary operation of one of the first five types, then $\omega' \equiv h\hat{s}h$, where either $\alpha \rightarrow \beta$ or $\beta \rightarrow \alpha$ is a basic move of $T$.

The last move is to the right

$$hah \rightarrow hah$$

But $\exists!$ rightward motion or back motion is irrelevant (i.e., is undone by the forward)

$$q_{i-1} \leftarrow q_i \rightarrow q_{i+1}$$
\[ \Rightarrow \alpha_{i-1} = \alpha_{i+1} \]

**Theorem:** In \( T(p) \) let \( u(h) = q \)

iff \( u \) is a stopping state for the machine \( r \).

The rest of the work is modifying \( T(p) \)

and supplying for an embedding in \( B(T) \).

**B(T) =**

\[ \langle \text{generators: } q, q_0, \ldots, q_N, s_0, \ldots, s_M, r_i, i \in I, x, t, k \rangle \]

\[ \text{relations: for all } i \in I \text{ and all } \beta = 0, \ldots, M, \]

\[ xs_\beta = s_\beta x^2, \quad r_i s_\beta = s_\beta x r_i x, \quad r_i^{-1} F_i^\# q_i G_i r_i = H_i^\# q_i K_i, \quad tr_i = r_i t, \quad tx = xt, \quad kr_i = r_i k, \quad kx = xk, \quad k(q^{-1} t q) = (q^{-1} t q)k. \]
We won't do all the combinatorics
but will note the various HNN-extensions

- $B_0 = \{x \mid x \neq 1\} \cong \mathbb{Z}$

- $B_1 = \{B_0, s_0 \cdots s_n \mid \Delta_1\}$

  is an HNN extension with $s_i$ all
  stable letters.

- $B_2 = \{B, *Q, r_i \mid \Delta_2\}$

  $B_2$ is HNN of $B_1 + Q$ with $r_i$ as
  stable letter.

- $B_3 = \{B_2 + \mid \Delta_3\}$

  $B_3$ is HNN of $B_2$ with $+$ as stable letter

- $B^* is HNN of B_3 with stable letter $*$.\
Key Tool is the Van Kampen diagram.

(See e.g. [Riley] or Office Hours with a GEOMETRIC GROUP THEORIST)

This is used for proving “Britton’s lemma” which ends up reducing the group case to the semigroup.
Subgroups of finitely presented groups

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(Received 20 February 1961)

The main theorem of this paper states that a finitely generated group can be embedded in a finitely presented group if and only if it has a recursively enumerable set of defining relations. It follows that every countable Abelian group, and every countable locally finite group can be so embedded; and that there exists a finitely presented group which simultaneously embeds all finitely presented groups. Another corollary of the theorem is the known fact that there exist finitely presented groups with recursively insoluble word problem. A by-product of the proof is a genetic characterization of the recursively enumerable subsets of a suitable effectively enumerable set.

See [Rotman] or [Stillwell]
First Application to a variational problem

Theorem: Let $M$ be a closed Riemannian manifold with $\pi_1 M$ having unsolvable word problem then $M$ has only many closed contractible geodesics.
Remark: (a) \( \iff \) periodic solutions to a certain nonlinear ODE with additional geo properties. 
(b) Lusternich-Fet theorem 
(c) You are never entitled to contractible ones that
are local minima of $E_{\text{energy}}$.

These are exactly what are produced here.
2nd application: Novikov's theorem

Theorem: There is no algorithm to decide if $M^n \cong S^n$ for manifolds of dim $\geq 5$.

Unknown for $n=4$ There is an algorithm for $n=3$. (Rubinstein - Thompson 1990's)

Unfortunately this will need some algebraic preliminaries.
(Cohomology of groups, Universal Central Extensions and Kervaire's theorem.

Definition: $H^1(\pi)$, $H^2(\pi)$.

$H^2(\pi) \hookrightarrow$ Central extension

$H^2_2(\pi) \twoheadrightarrow$ Center of Universal Central Extension
Gordon's Theorem
There is no algorithm to tell if \( H_2(\Gamma) = 0 \).

Barnsley-Dyer-Heller
Given a finite complex \( X \)
there is an associated \( G(x) \)
such that \( H_4(G(x)) \subset H_4(x) \)

By explicit construction.
And has some "obvious" modifications.
Idea of Miniker.

Let $A$ be a group where $A = e$ or $w + A$ is nontrivial.

$A \times w, A$ is similar.

(We'll use Baumslag-Dyer-Heller approach)

$\mathbb{Z} \times A * \mathbb{Z} \times A'$

$\sum_{a \in A} t = t'$

with $A'$