The word problem and Applications

1. Review of covering spaces
2. Free products, free products with amalgamation, HNN Extensions.
3. Bass-Serre theory
4. Presentations of groups, statement of the word problem
5. Dehn function.
6. Examples: Free groups, surface groups, Heisenberg group, Baumslag-Solitar group, Gersten's group and more.
7. Van Kampen's theorem and when fundamental groups don't inject. (Relevant to Adian-Rabin theorem)
8. Turing machines Skip to A-R thm
9. Post's theorem

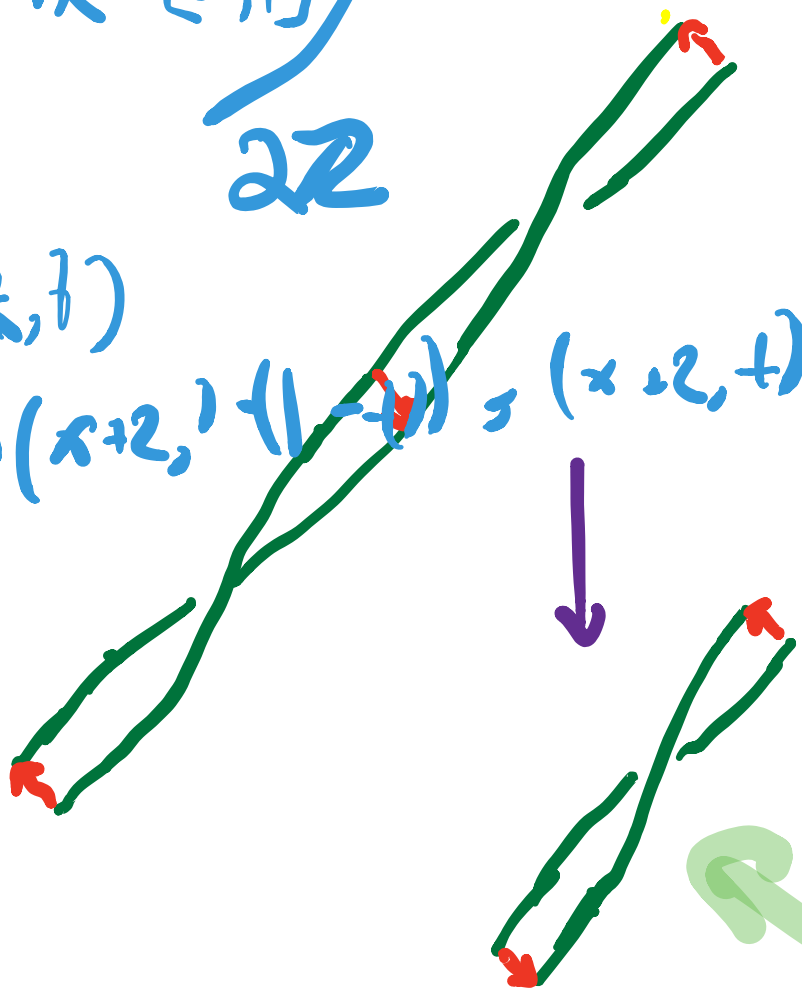
11. Introduction to Van Kampen diagrams and unsolvable word problem (Theorem of Boone-Novlkov)
12. Gromov's theorem on closed geodesics (application to differential geometry)
13. Higman' s theorem (and Universal groups)
14. Adian-Rabin theorem (Markov properties)
15. Homology of groups and Gordon's theorem
16. Hopf exact sequence and Kervaire's theorem.
17. Universal Central extensions and S. Novikov's theorem
18. Other unsolvable topological problems (including non-c.e. Sets);
19. Sobolev homology?

References:
Rotman, Theory of Groups (GTM)
Stillwell, The word problem, Bull AMS
Algorithmic and classification problems in Group theory (Baumslag and Miller, editors, MSRI Publication)
Serre and Kilmer, Trees (Springer) and
Articles by Gersten and Riley on Dehn Functions
Brown, Cohomology of Groups (GTM)
Weinberger, Computers, Rigidity and Moduli (PUP)

$$\frac{\mathbb{R}\times[0,1]}{2\mathbb{Z}}$$

(x,t)

$\to (x+2, 1-(1-t)) = (x+2, t).$

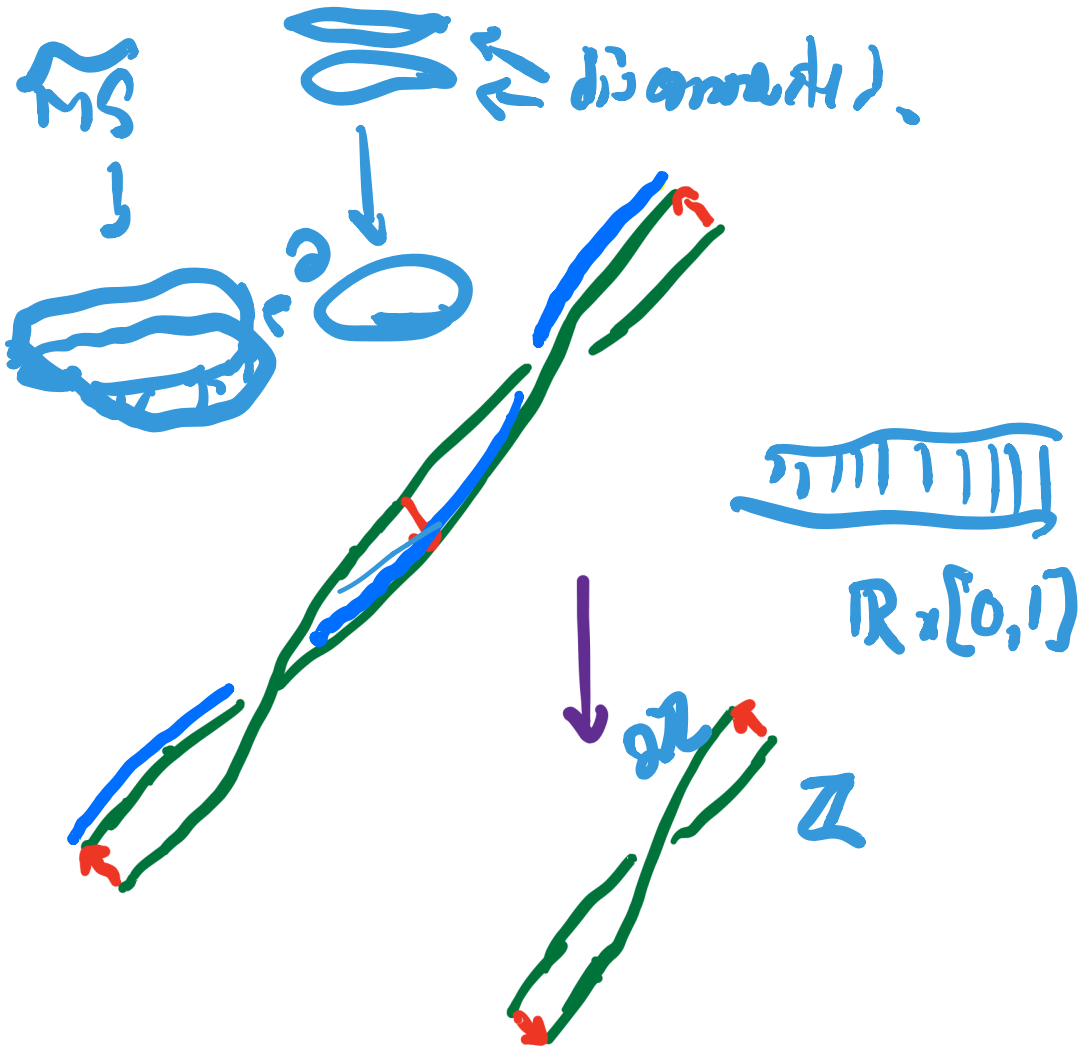


$\mathbb{R}\times[0,1]$

$(x,t) \xrightarrow{+1} (x+1, 1-t).$

$\mathbb{Z}\times\mathbb{R} \to \mathbb{R}.$

Mobius Strip.



2-fold cover of Moebius Strip is an annulus: the boundary is disconnected.

upstairs.

3-fold cover is MS again.

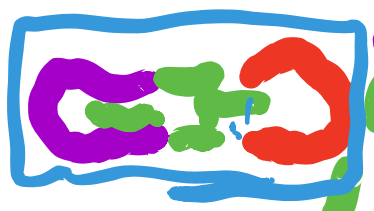
$\Gamma = A *_B C$

$A *_B$

Connection to group actions on trees.

$X = \mathbb{Z}_4 * \mathbb{Z}_6$

$\mathbb{Z}_2$



$\tilde{X} \to$ Tree

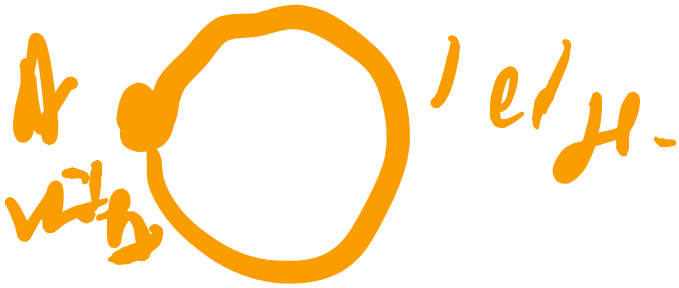
$\Gamma \to$ Action on tree

This picture is a particular case of a Bass-Serre tree associated to the Amal. Free product construction.

T/Γ.

$x \to x+1$ $\qquad$ $\mathbb{Z}$

$(x \to \frac{1}{2} - x)$

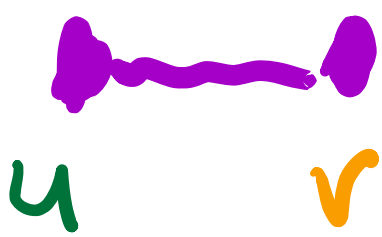


$\pi_1^*(e) = \mathbb{Z}$

群作用 : $\mathbb{R}$

$u \quad x \to -x$

$v \quad x \to 1-x$



In general

$$\Gamma \times T \circlearrowleft$$

Technical: $\boxed{\gamma(e) = e \Rightarrow \gamma|_e = \mathrm{id}.}$

Then

Γ can be built by Amalg. F.P. + HNN extensions from

$[g, h] = ghg^{-1}h^{-1}$

$h \quad 2$

$a^b = a^{-}$

$\langle a, b \mid bab^{-1} = a^2 \rangle$

$ba = a^2 b$

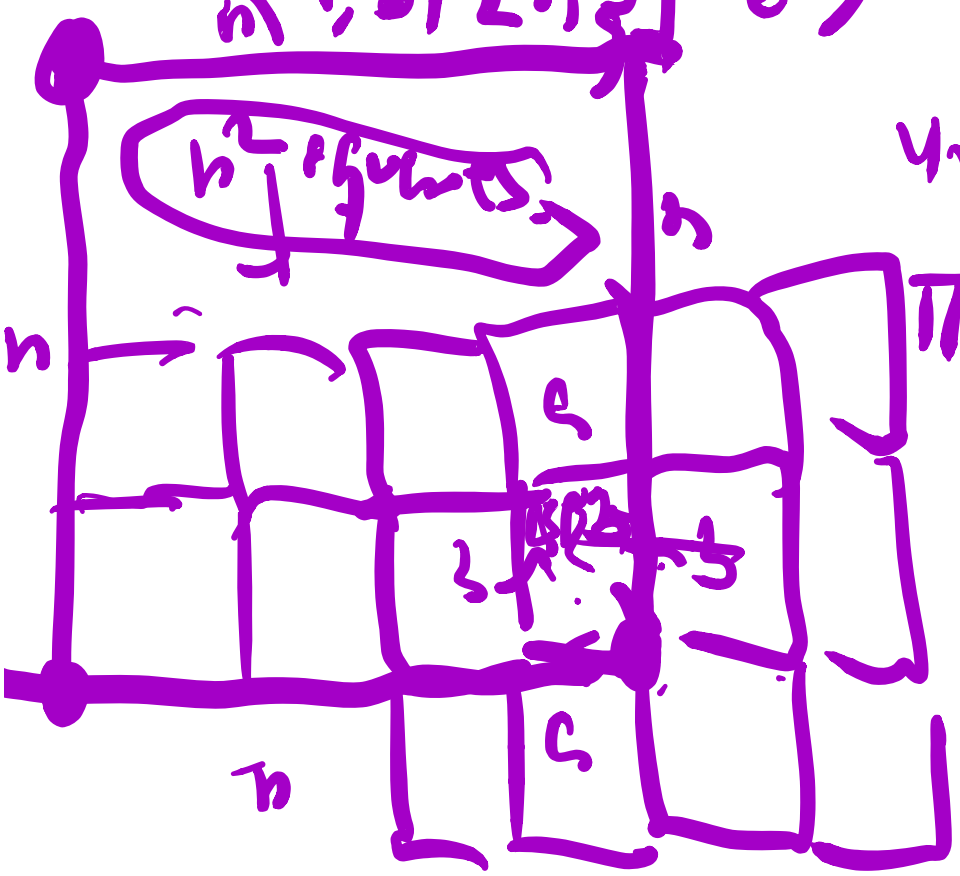
$$b^n a b^{-n} = b a^{2^{n-1}} b^{-1}$$
$$= (bab^{-1})^{2^{n-1}}$$
$$= (a^2)^{2^{n-1}} = a^{2^n}$$

So intuitively seeing that $[a^{b^n}, a] \neq e$.

$$b^n a b^{-n} a b^n a^{-1} b^{-n} a^{-1} = e$$

looks like it involves an exponential size use of relations.

$\langle a, b \mid [a, b] = e \rangle$



$\forall n \in \mathbb{N}$

πn^2 relations

$\{[a^n, b^n]\}$

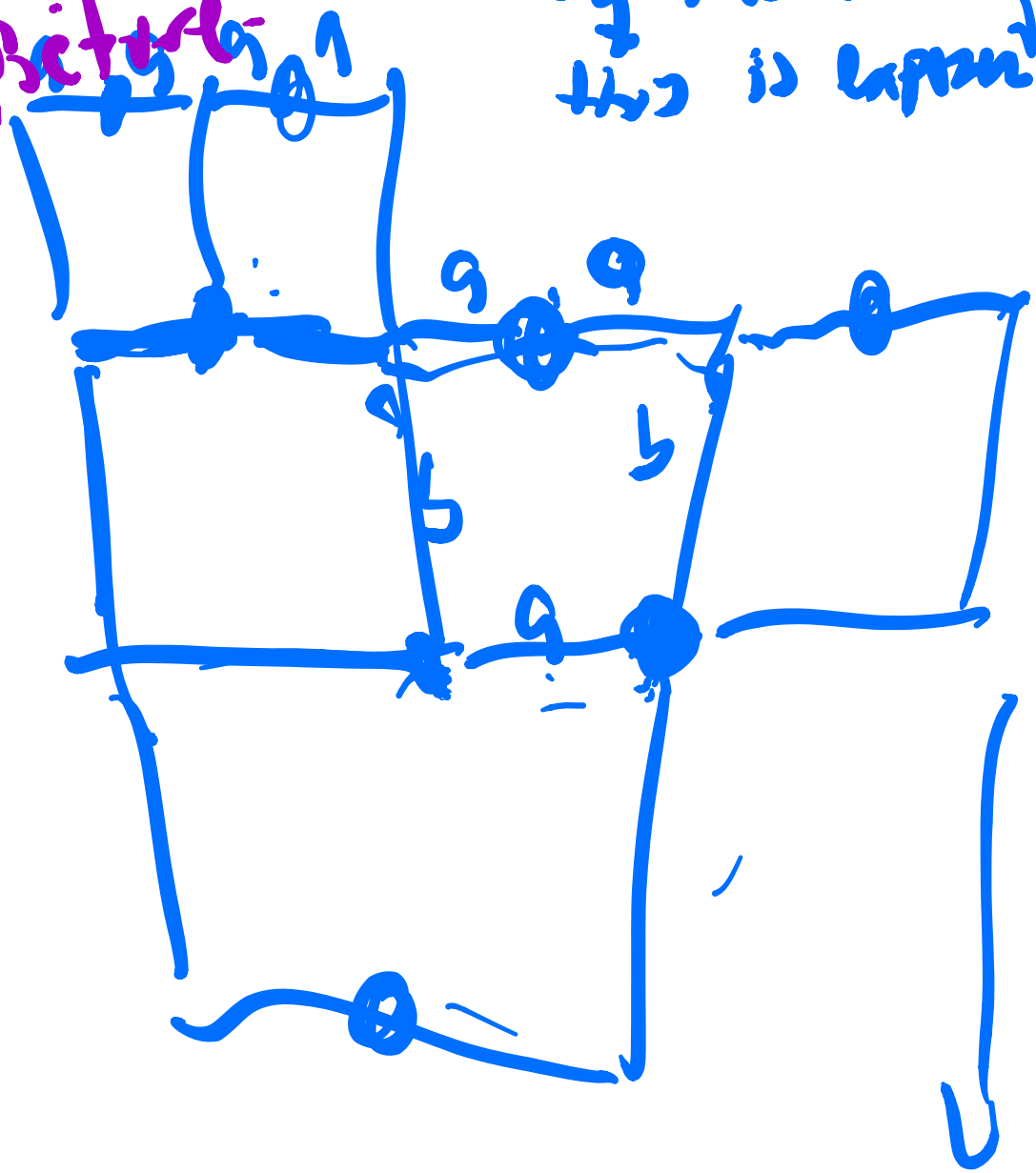
Dehn function

$[a, a^{b^n}]$

Exercise:

The # of squares inside this is exponential!

Redraw this picture



$$\langle \underline{a, b, c} \mid bab^{-1} = a^2 \quad cbc^{-1} = b^2 \rangle$$

Same with $c^n b c^{-n} a c^n b^{-1} c^{-n}$

$$= b^{2^n} a b^{-2^n} = a^{2^{2^n}}$$

gives Double exponential.

BS BS

$$\langle a, b, c, \underline{d} \mid bab^{-1} = a^2 \quad cbc^{-1} = b^2$$

$$dcd^{-1} = c^2 \rangle$$

BS

Triple exponential.

Formal Why Just Using the Langley graph.

Voronoi diagram.

GERSTEN'S GROUP.

$yxy^{-1} \cdot x \cdot yx^{-1}y^{-1}$

$\langle x, y \mid x^{x^y} = x^2 \rangle$

$\langle x, z \mid zxz^{-1} = x^2 \rangle$ *

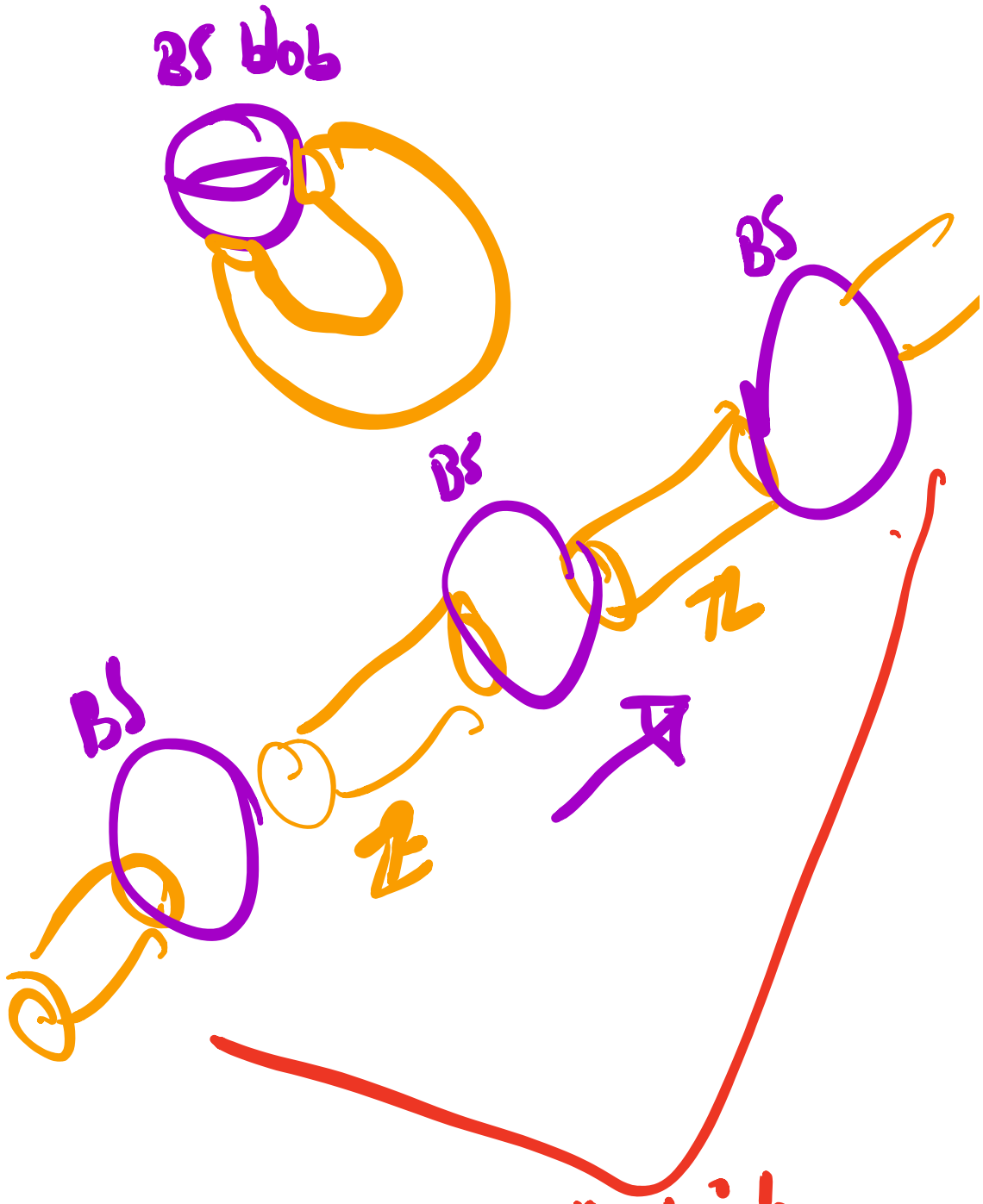
$z = txt^{-1}$

$\{x, t \mid x^{x^t} = x^2\}$

Dehn function

$BS_{1,2}$

$> 2^{2^{2^{2^{\cdot^{\cdot^{2}}}}}}$



This is the triple exponential group

Dehn function for $\mathbb{Z}^2$.

or $\mathbb{Z}^n$ using Stokes' theorem

All legit using calculus.

Heisenberg group $H = \begin{array}{c} \begin{array}{ccc} O(r) & O(r^2) & \end{array} \\ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \end{array} O(r)$

$$= \langle g, h \mid [g,[g,h]] = [h,[g,h]] = e \rangle$$

$D_H(n) \sim n^3$

$$\underbrace{[g^n, [g^n, h^n]]}_{[g,h]^{n^2}} = e.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$B(n) \sim n^4$ elements in H.

Van Kampen diagrams.

(Hyperbolic groups + Small Cancellation?)

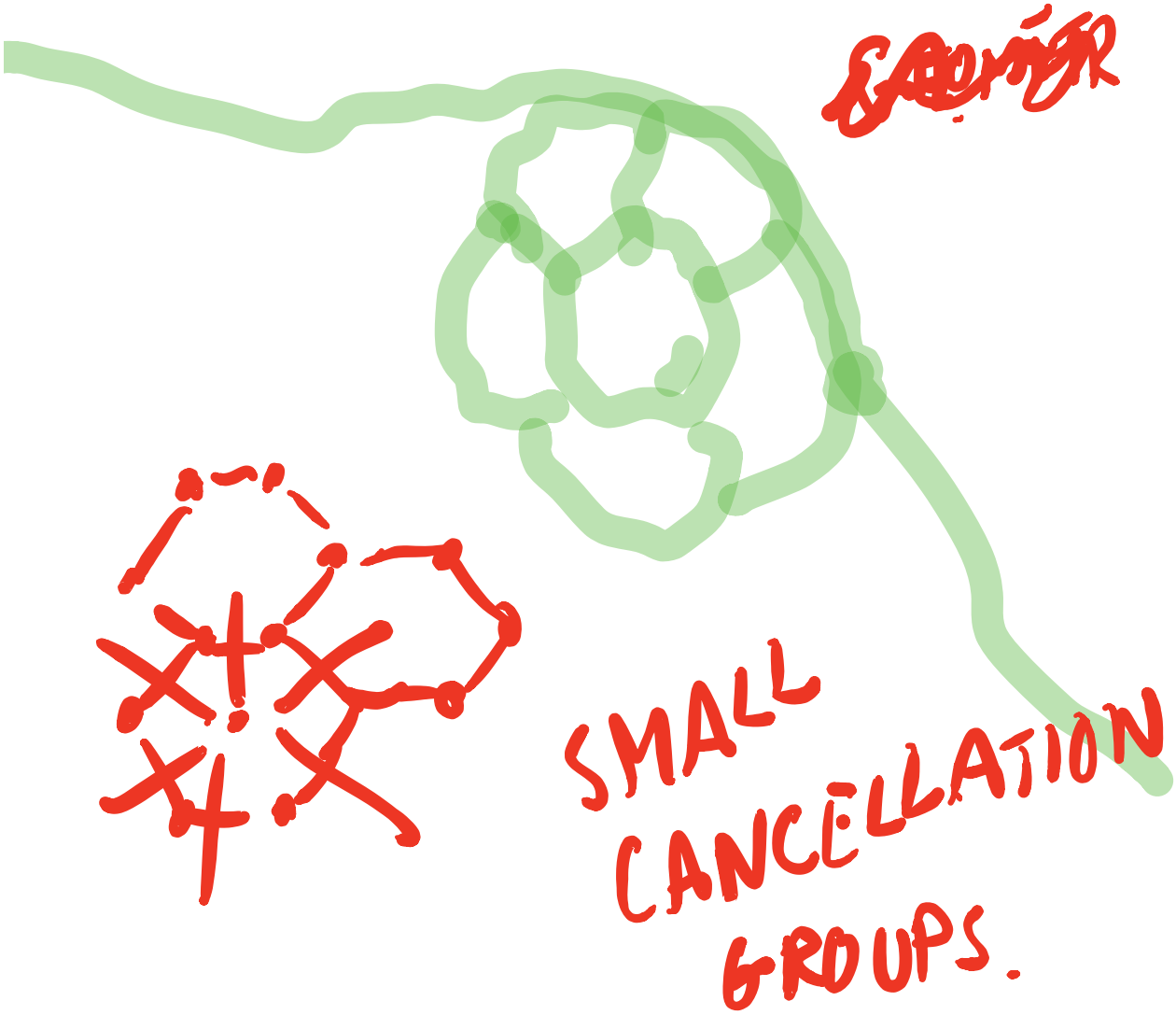
Groups where Dehn function is linear (like a free group)

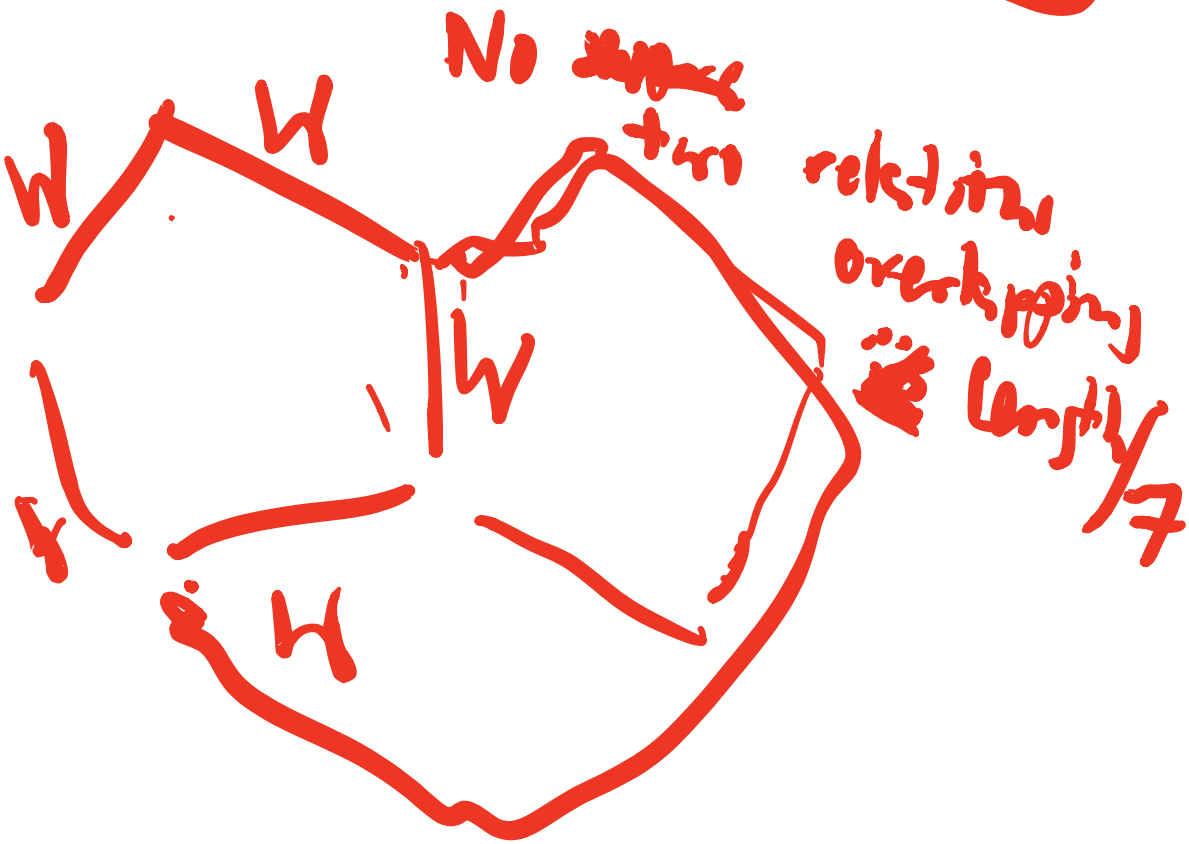
$\exists$ greedy algorithm to tell if $w = e$.

Classical examples

$\langle a, b, c, d \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle$

relation doesn't interset itself


SMALL
CANCELLATION
GROUPS.


H
H
H
H
H
No
two
relative
overlapping
length

Word problem.

In long long subword.

→ Problem about groups, not presentations.

→ Dehn function and geometric meaning of the word problem in terms of isoperimetry and constructing the Cayley graph.

Theorem (Boone · Novikov)

∃ There are groups with unsolvable word problem

⟺ There are groups with Dehn function ∃ growing faster than any computable function.

Where we are going:

WP is unsolvable.

Most things can't be told from a presentation (Adian-Rabin).

RE is algebraic action (Higman).

Geometric + Topological variations.

(+ Analytic?).

Closed Geodesics

Novikov's theorem (Requires some cohomology of groups)

Other unsolvable problems.

<u>Defn</u>: P is a Markov property of groups

if $e \notin P$ and $G \in P$ $G \subset K$

$\Rightarrow K \in P$. (e.g. nontrivial, nonabelian, etc.)

i.e. a nontrivial property inherited by supergroups.

[Nonexample: Being simple]
but still undecidable.

There are still a lot of other non-Markovian "dim Rabin +"

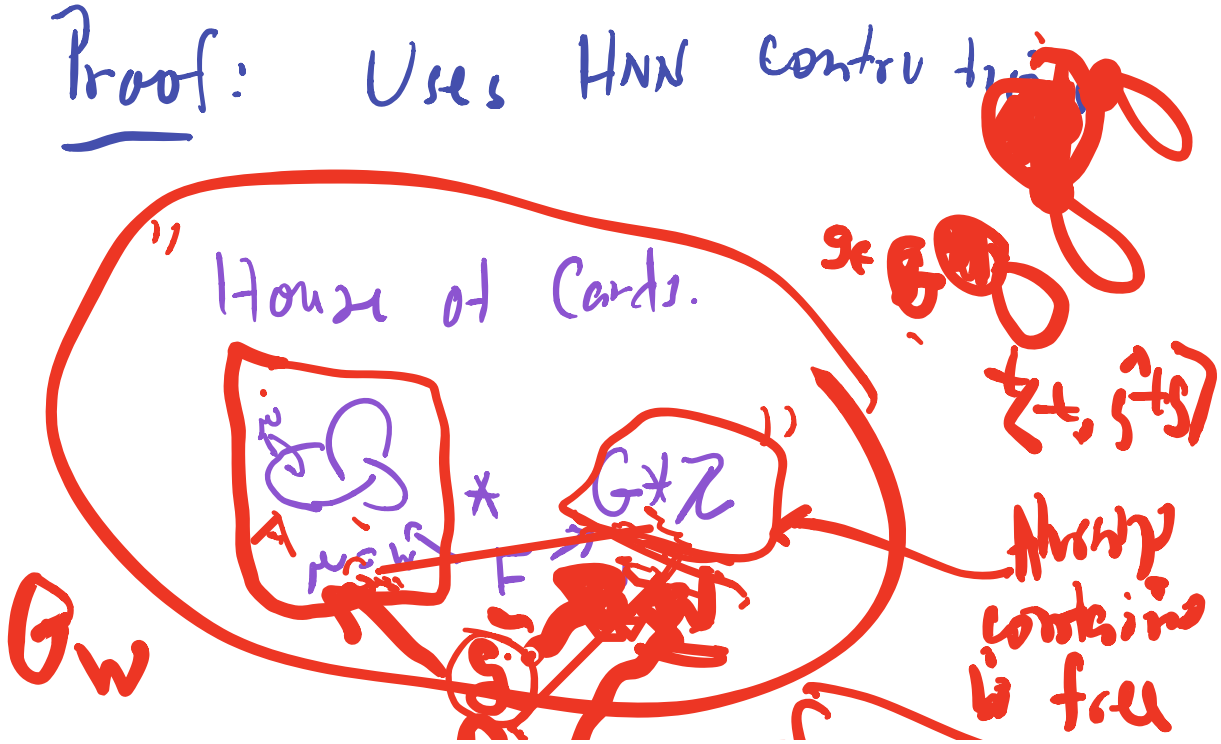
THEOREM: (ADIAN - RABIN)

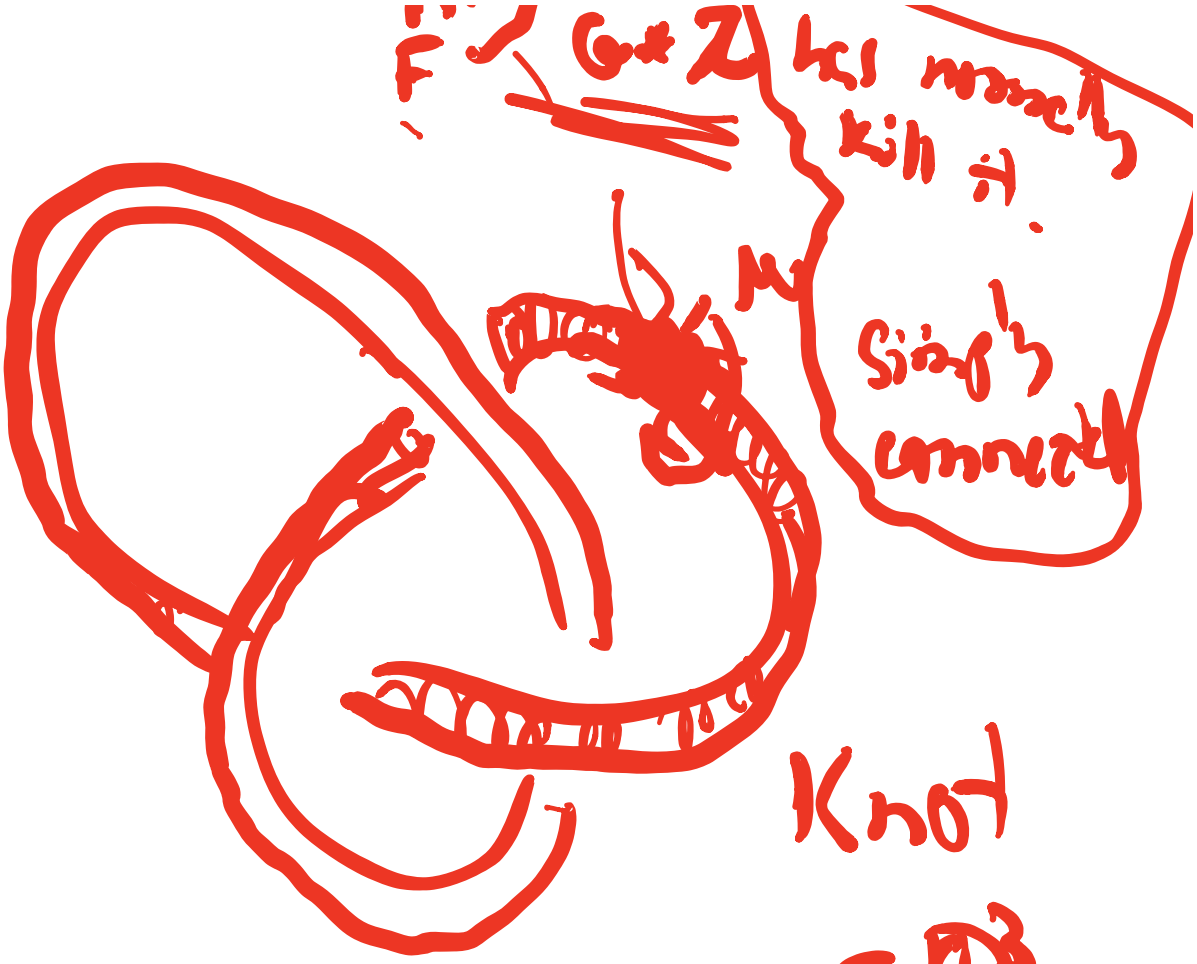
There is no algorithm to tell if a finite presentation descri, a property P group.

Markov.

$\langle A, j^{+}s, j^{-}t, s^{-}t \rangle$

Proof: Uses HNN construction





Knot

$\subset \mathbb{R}^3$

$\pi_1(\mathbb{R}^3 - \text{Knot})/\langle w \rangle = e$

is a complicated group $\supset F_2$ unless the knot is trivial

Groups.

LEMMA 3.6 (Main Technical Lemma). *Let K be a group given by a presentation on a finite or countably infinite set of generators, say*

$$K = < x_1, x_2, \ldots \mid R_1 = 1, R_2 = 1, \ldots > .$$

For any word w in the given generators of K, let L_w be the group with presentation obtained from the given one for K by adding three new generators a, b, c together with defining relations

$$\begin{array}{lrcl}
(1) & a^{-1}ba & = & c^{-1}b^{-1}cbc \\
(2) & a^{-2}b^{-1}aba^2 & = & c^{-2}b^{-1}cbc^2 \\
(3) & a^{-3}[w,b]a^3 & = & c^{-3}bc^3 \\
(4) & a^{-(3+i)}x_iba^{(3+i)} & = & c^{-(3+i)}bc^{(3+i)} \quad i = 1, 2, \ldots
\end{array}$$

$\supset F_\infty \subset F_2\langle b, c\rangle$

where $[w, b]$ is the commutator of w and b. Then

(1) if $w \neq_K 1$ then K is embedded in L_w by the inclusion map on generators;

(2) the normal closure of w in L_w is all of L_w; in particular, if $w =_K 1$ then $L_w \cong 1$, the trivial group;

(3) L_w is generated by the two elements b and ca^{-1}.

If the given presentation of K is finite, then the specified presentation of L_w is also finite.

$L_w \cong (\underline{K * \langle a, b\rangle}) * \langle b, c\rangle$

$(K * G)$ LHS = RHS

Update. $G \hookrightarrow G_w$ If $w \neq e$

$|G_w| = 1$ if $w = e$.

Definition. A ***quadruple*** is a 4-tuple of one of the following three types:

$$q_i s_j s_k q_l,$$

$$q_i s_j R q_l,$$

$$q_i s_j L q_l.$$

A ***Turing machine*** T is a finite set of quadruples no two of which have the same first two letters. The ***alphabet*** of T is the set $\{s_0, s_1, \ldots, s_M\}$ of all s-letters occurring in its quadruples.

Definition. An ***instantaneous description*** α is a positive word of the form $\alpha = \sigma q_i \tau$, where σ and τ are s-words and τ is not empty.

"What's on the tape?!" (and "where's the machine")

Definition. Let T be a Turing machine. An ordered pair (α, β) of instantaneous descriptions is a ***basic move*** of T, denoted by

$$\alpha \to \beta,$$

if there are (possibly empty) positive s-words σ and σ' such that one of the following conditions hold:

(i) $\alpha = \sigma q_i s_j \sigma'$ and $\beta = \sigma q_l s_k \sigma'$, where $q_i s_j s_k q_l \in T$;

(ii) $\alpha = \sigma q_i s_j s_k \sigma'$ and $\beta = \sigma s_j q_l s_k \sigma'$, where $q_i s_j R q_l \in T$;

(iii) $\alpha = \sigma q_i s_j$ and $\beta = \sigma s_j q_l s_0$, where $q_i s_j R q_l \in T$;

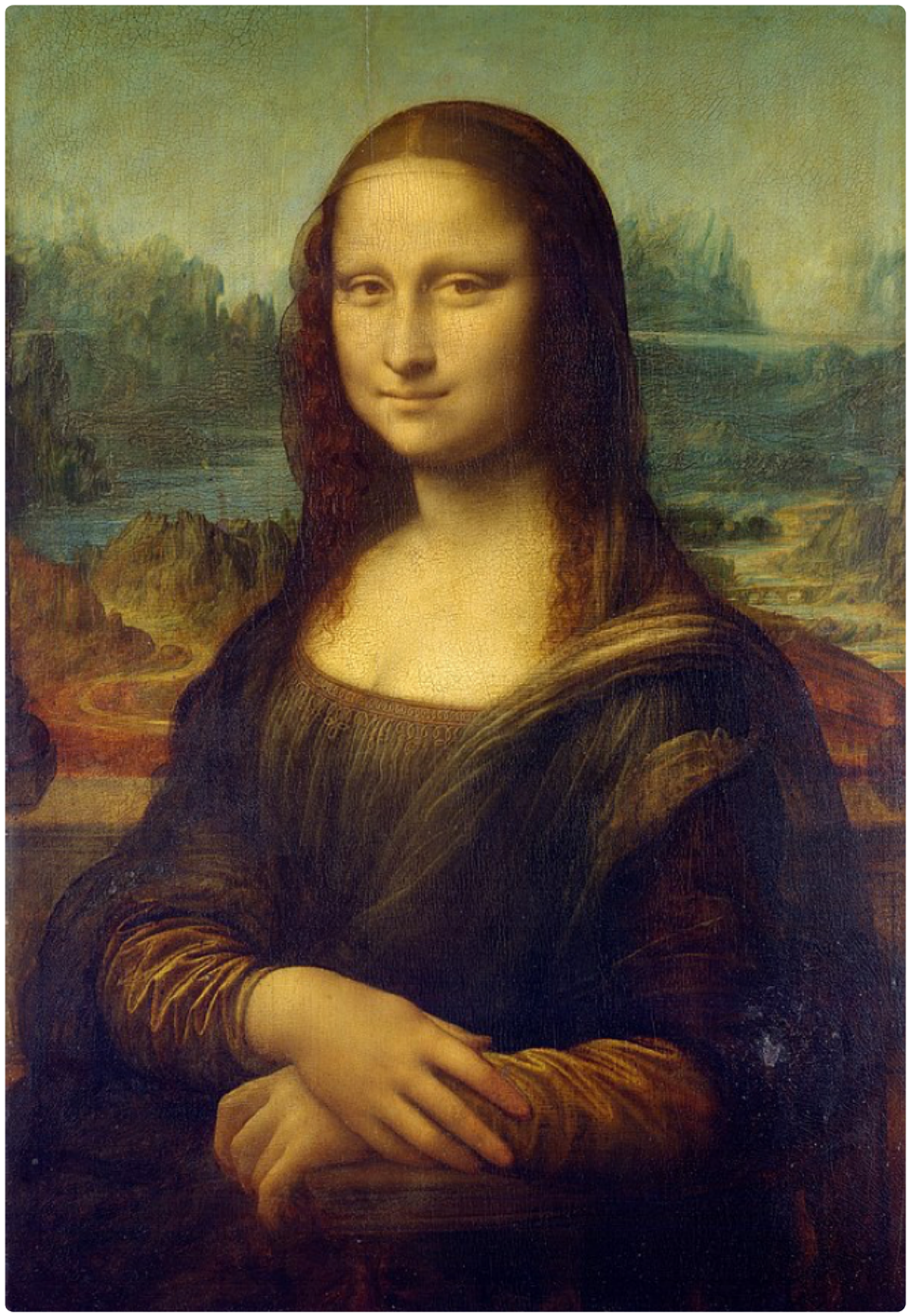
(iv) $\alpha = \sigma s_k q_i s_j \sigma'$ and $\beta = \sigma q_l s_k s_j \sigma'$, where $q_i s_j L q_l \in T$; and

(v) $\alpha = q_i s_j \sigma'$ and $\beta = q_l s_0 s_j \sigma'$, where $q_i s_j L q_l \in T$.

adding blanks at the beginning or ends of tapes.

If α describes the tape at a given time, the state q_i of T, and the symbol s_j being scanned, then β describes the tape, the next state of T, and the symbol being scanned after the machine's next move. The proviso in the definition of a Turing machine that no two quadruples have the same first two symbols means that there is never ambiguity about a machine's next move: if $\alpha \to \beta$ and $\alpha \to \gamma$, then $\beta = \gamma$.

Some further explanation is needed to interpret basic moves of types (iii) and (v). Tapes are finite, but when the machine comes to an end of the tape, the tape is lengthened by adjoining a blank square. Since s_0 means blank, these two rules thus correspond to the case when T is scanning either the last symbol on the tape or the first symbol.



Turing Machine versus Mona Lisa.

SEMIGROUP WORD PROBLEM

Suppose that a semigroup Γ has a presentation

$$\Gamma = (X | \alpha_j = \beta_j, j \in J).$$

If ω and ω' are positive words on X, then it is easy to see that $\omega = \omega'$ in Γ if and only if there is a finite sequence

$$\omega \equiv \omega_1 \to \omega_2 \to \cdots \to \omega_t \equiv \omega',$$

where $\omega_i \to \omega_{i+1}$ is an ***elementary operation***; that is, either $\omega_i \equiv \sigma\alpha_j\tau$ and $\omega_{i+1} \equiv \sigma\beta_j\tau$ for some j, where σ and τ are positive words on X or $\omega_{i+1} \equiv \sigma\beta_j\tau$ and $\omega_i \equiv \sigma\alpha_j\tau$.

POST'S THEOREM

Definition. If T is a Turing machine having stopping state q_0, then its ***associated semigroup*** $\Gamma(T)$ has the presentation:

$$\Gamma(T) = (q, h, s_0, s_1, \ldots, s_M, q_0, q_1, \ldots, q_N | R(T)),$$

where the relations $R(T)$ are

$$q_i s_j = q_l s_k \qquad \text{if} \quad q_i s_j s_k q_l \in T,$$

for all $\beta = 0, 1, \ldots, M$:

$$\begin{aligned}
q_i s_j s_\beta &= s_j q_l s_\beta && \text{if} \quad q_i s_j R q_l \in T,\\
q_i s_j h &= s_j q_l s_0 h && \text{if} \quad q_i s_j R q_l \in T;\\
s_\beta q_i s_j &= q_l s_\beta s_j && \text{if} \quad q_i s_j L q_l \in T,\\
h q_i s_j &= h q_l s_0 s_j && \text{if} \quad q_i s_j L q_l \in T;
\end{aligned}$$

$$\begin{aligned}
q_0 s_\beta &= q_0,\\
s_\beta q_0 h &= q_0 h,\\
h q_0 h &= q.
\end{aligned}$$

Theorem: In $T(P)$ $h q_1 u h = q$ iff u is a stopping state for the machine P.

$\Longrightarrow$ is pretty obvious.

$\Longleftarrow$ Focus on "special words $h\alpha h$" ← instaneous description

Let ω and ω' be words on $\{s_0, s_1, \ldots, s_M, q_0, q_1, \ldots, q_N\}$ with $\omega \not\equiv q$ and $\omega' \not\equiv q$. If $\omega \to \omega'$ is an elementary operation, then ω is h-special if and only if ω' is h-special.
If $\omega = h\alpha h$ is h-special, $\omega' \not\equiv q$, and $\omega \to \omega'$ is an elementary operation of one of the first five types, then $\omega' \equiv h\beta h$, where either $\alpha \to \beta$ or $\beta \to \alpha$ is a basic move of T.

The last move is to the right

$$h\alpha h \cdots \overset{\text{or}}{\rightleftarrows} h\alpha_{k-2} h \overset{\text{or}}{\rightleftarrows} h\alpha_{k-1} h \overset{\text{or}}{\rightleftarrows} h\alpha_k h \to q$$

but $\exists!$ rightward motion so backmotion is irrelevant (i.e. is undone by the forward)

$$\alpha_{i-1} \leftarrow \alpha_i \rightarrow \alpha_{i+1}$$

$\Rightarrow \alpha_{i-1} = \alpha_{i+1}$

<u>Theorem</u>: In $T(\Gamma)$ $hq_1 uh = q$ iff u is a stopping state for the machine Γ.

The rest of the work is modifying $T(\Gamma)$ and arranging for an embedding into a gp.

$\mathcal{B}(T) =$

$\langle$ *generators*: $q, q_0, \ldots, q_N, s_0, \ldots, s_M, r_i, i \in I, x, t, k$ |

relations: for all $i \in I$ and all $\beta = 0, \ldots, M$,

$$xs_\beta = s_\beta x^2, \qquad \Delta_1$$

$$r_i s_\beta = s_\beta x r_i x,$$

$$r_i^{-1} F_i^{\#} q_{i_1} G_i r_i = H_i^{\#} q_{i_2} K_i, \qquad \Delta_2$$

$$tr_i = r_i t,$$

$$tx = xt, \qquad \Delta_3$$

$$kr_i = r_i k,$$

$$kx = xk,$$

$$k(q^{-1}tq) = (q^{-1}tq)k. \qquad \rangle$$

We won't do all the combinatorics

but will note the various HNN & algebras

- $B_0 = \{x \mid \emptyset\} \cong \mathbb{Z}$

- $B_1 = \{B_0, s_0 \dots s_m \mid \Delta_1\}$

  is an HNN extension with s_i all stable letters.

- $B_2 = \{B_1 * Q, r_i \mid \Delta_2\}$

  B_2 is HNN of $B_1 * Q$ with r_i as stable letters.

- $B_3 = \{B_2, t \mid \Delta_2\}$

  B_3 is HNN of B_2 with t as stable letter

- B is HNN of B_3 with stable letter k.

KEY TOOL is the Van Kampen diagram.

(See e.g. [Riley] or Office Hours with a GEOMETRIC GROUP THEORIST)

This is used for proving "Britton's lemma" which ends up reducing the group case to the semigroup.

Subgroups of finitely presented groups

By G. Higman, F.R.S.

Department of Mathematics, University of Chicago

(*Received* 20 *February* 1961)

The main theorem of this paper states that a finitely generated group can be embedded in a finitely presented group if and only if it has a recursively enumerable set of defining relations. It follows that every countable Abelian group, and every countable locally finite group can be so embedded; and that there exists a finitely presented group which simultaneously embeds all finitely presented groups. Another corollary of the theorem is the known fact that there exist finitely presented groups with recursively insoluble word problem. A by-product of the proof is a genetic characterization of the recursively enumerable subsets of a suitable effectively enumerable set.

See [Rotman] or [Stillwell]

First Application to a variational problem

Theorem: Let M be a closed Riemannian manifold with $\pi_1 M$ having unsolvable word problem then M has ∞'ly many closed contractible geodesics.

<u>Remark</u>: (a) $\Longleftrightarrow$ periodic solutions to a certain nonlinear ODE with additional geo properties.

(b) Lusternick - Fet theorem

(c) You are never entitled to contractible ones that

are local minima

of E. (Energy).

These are exactly what

we produced here.

2nd application: Novikov's theorem

<u>Theorem</u>: There is no algorithm to decide if $M^n \cong S^n$ for manifolds of dim ≥ 5.

Unknown for $n=4$ There is an algorithm for $n=3$. (Rubinstein - Thompson 1990's)

Unfortunately this will need some algebraic preliminaries.

(Co)homology of groups,
Universal Central Extensions
and Kervaire's theorem.

Definition: $H_i(\pi)$ $H^i(\pi)$.

$H^2(\pi) \longleftrightarrow$ Central extensions

$H_2(\pi) \longleftrightarrow$ Center of Universal
Central extension.

Gordon's theorem

There is no algorithm to tell if $H_2(\Gamma) = 0$.

Baumslag-Dyer-Heller

Given a finite complex X there is an associated $G(X)$ so that $H_*(G(X)) \cong H_*(X)$

By explicit construction.
And has some "obvious" modifications.

Idea of Winkor.

Let A be a group where $A = e$ or $w + A$ is nontrivial.

$A \times_w A$ is similar.

(We'll use Baumslag-Dyer-Heller approach)

$$\overset{t}{\mathbb{Z} \times A} * \overset{t}{\mathbb{Z} \times A'}$$

t commutes with A $\quad$ $w' = w$ $\quad$ w not conjugate with A'.

$t = t'$

$\mathbb{Z} \times A \times \mathbb{Z} \times \mathbb{Z}$ 1

$\times \mathbb{Z} \times \mathbb{Z} \times A'$

所以 $\mathbb{Z}$ 的 $A \times_G A'$

A 平凡

$\Rightarrow$ 所以 1、