ISOPERIMETRIC AND ISODIAMETRIC FUNCTIONS OF FINITE PRESENTATIONS

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Abstract. We survey current work relating to isoperimetric functions and isodiametric functions of finite presentations.

§1. Introduction and Definitions

Isoperimetric functions are classical in differential geometry, but their use in group theory derives from Gromov's seminal article [Gr] and his characterization of word hyperbolic groups by a linear isoperimetric inequality. Isodiametric functions were introduced in our article [G1] in an attempt to provide a group theoretic framework for a result of Casson's (see Theorem 3.6 below). It turned out subsequently that the notion had been considered earlier under a different name [FHL]. We have learned since that the differential geometers also have their isodiametric functions and they mean something different by them. However the analogy is too suggestive to abandon this terminology and we shall retain it here. Up to an appropriate equivalence relation (Proposition 1.1 below), isoperimetric and isodiametric functions are quasiisometry invariants of finitely presented groups. Hence these functions are examples of *geometric properties*, in the terminology of [Gh].

If $\mathcal{P} = \langle x_1, x_2, \dots, x_p \mid R_1, R_2, \dots R_q\rangle$ is a finite presentation, we shall denote by $G = G(\mathcal{P})$ the associated group; here $G = F/N$, where F is the free group freely generated by the generators $x_1, \dots, x_p$ and N is the normal closure of the relators. If w is an element of F (which we may identify with a reduced word in the free basis), we write $\ell(w)$ for the length of the word w and $\bar{w}$ for the element of G represented by w. We shall use freely the terminology of van Kampen diagrams [LS, p. 235ff] in the sequel.

We write $\mathrm{Area}_{\mathcal{P}}(w)$ for the minimum number of faces (i.e 2-cells) in a van Kampen diagram with boundary label w. Equivalently, $\mathrm{Area}_{\mathcal{P}}(w)$ is the minimum number of relators or inverses of relators occurring in all expressions of w as a product (in F) of their conjugates. The function $f : \mathbb{N} \to \mathbb{N}$ is an *isoperimetric function* for $\mathcal{P}$ if, for all n and all words w with $\ell(w) \le n$ and $\bar{w} = 1$, we have $\mathrm{Area}_{\mathcal{P}}(w) \le f(n)$. The minimum such isoperimetric function is called the *Dehn function* of $\mathcal{P}$.

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If $\mathcal{D}$ is a van Kampen diagram with boundary label w, we choose the base point v_0 in the boundary of $\mathcal{D}$ corresponding to where one starts reading the boundary label w and one defines

$$\mathrm{Diam}_{v_0}(\mathcal{D}) = \max_{v \in \mathcal{D}^{(0)}} d_{\mathcal{D}^{(1)}}(v_0, v).$$

Here $d_{\mathcal{D}^{(1)}}$ denotes the word metric for the 1-skeleton of $\mathcal{D}$, so that every edge has length 1. The function $f : \mathbb{N} \to \mathbb{N}$ is called an *isodiametric function* for $\mathcal{P}$ if, for all n and all reduced words w with $\ell(w) \leq n$ and $\bar{w} = 1$, there exists a based van Kampen diagram $(\mathcal{D}, v_0)$ for w with $\mathrm{Diam}_{v_0}(\mathcal{D}) \leq f(n)$. A more algebraic way of formulating this is as follows. Let M denote the maximum length of a relator of $\mathcal{P}$. Let f be an isodiametric function for $\mathcal{P}$. If $\bar{w} = 1$, then one can write

$$w = \prod_{i=1}^{m} R_{j_i}^{\epsilon_i u_i},$$

where R_{j_i} is a relator, $\epsilon_i = \pm 1$, $u_i \in F$ and $\ell(u_i) \leq f(\ell(w)) + M$. Here we write $a^b = bab^{-1}$ for elements a and b in a group.

A word of caution is necessary here. A diagram of minimal area is always reduced, in the sense of [LS]. However this will not be the case in general for a diagram of minimal diameter. This complicates considerably the problem of proving that a diagram is diametrically minimal. Consequently we do not introduce a diametric analog of the Dehn function.

Next we discuss the question of change of presentation.

Proposition 1.1, [Al][Sh]**.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be finite presentations for isomorphic groups. If f is an isoperimetric function (resp. isodiametric function) for $\mathcal{P}$, then there exist positive constants A, B, C, D, and E such that $n \mapsto Af(Bn+C)+Dn+E$ is an isoperimetric (resp. isodiametric) function for $\mathcal{P}'$.*

In fact, isoperimetric (resp. isodiametric) functions transform in the same way for quasiisometric presentations, so, up to the obvious equivalence relation, these are quasiisometry invariants (for the notion of quasiisometry, see [Gh]). In particular it makes sense to speak of a finitely presented group possessing a linear, quadratic, polynomial, exponential, etc., isoperimetric (resp. isodiametric) function, or more loosely, to speak of the group satisfying the the appropriate isoperimetric (resp. isodiametric) inequality.

Example. If one takes $\mathcal{P}$ to be a presentation with no relators, then the area function is identically zero, so the Dehn function is zero. However, a presentation of a free group with defining relators will have a nonzero Dehn function. Thus the awkward constants D and E in Proposition 1 are in general necessary.

Remark. An interesting variation on the notion of isoperimetric function was suggested by Gromov. We consider words w which are boundary labels of diagrams whose domains are compact orientable surfaces of some (variable) genus. Equivalently, we may assume there are words $u_1, u_2, \ldots, u_g, v_1, v_2, \ldots, v_g$ such that

$w' = w\prod_{i=1}^{g}[u_i, v_i]$ represents 1 in the group G of our finite presentation $\mathcal{P}$; here $[u_i, v_i]$ denotes the formal commutator $u_i v_i u_i^{-1} v_i^{-1}$, where u_i^{-1} denotes the formal inverse of the word u_i (invert each letter and write them in the reverse order); write $w \sim 0$ if there exists $g \geq 0$ such that this condition is satisfied. We define $\mathrm{Area}'_{\mathcal{P}}(w) = \min_{w'} \mathrm{Area}_{\mathcal{P}}(w')$, where the minimum is taken over all words w' constructed from w in this way. Then we define

$$f'(n) = \max_{\substack{w \sim 0 \\ \ell(w) \leq n}} \mathrm{Area}'_{\mathcal{P}}(w).$$

Gromov remarks that the function f' is closer in spirit to the differential geometric notion of minimal surface spanned by a loop, where one cannot control the genus of the (orientable) surface spanned.

§2. Relation with the Word Problem

The functions introduced in §1 are important for discussing the complexity of the word problem for a finitely presented group.

Theorem 2.1. *The following are equivalent for a finite presentation $\mathcal{P}$.*

- 2.1.1. *$G(\mathcal{P})$ has a solvable word problem.*
- 2.1.2. *$\mathcal{P}$ has a recursive isoperimetric function (in which case, the Dehn function itself is recursive).*
- 2.1.3. *$\mathcal{P}$ has a recursive isodiametric function.*

Let us sketch the argument. For the implication (1)⇒(2), we solve the word problem for all words of length at most n, thereby obtaining for each word w satisfying $\ell(w) \leq n$ and $\bar{w} = 1$ in G some expression

$$w = \prod_{i=1}^{k(w)} R_{j_i}^{\epsilon_i u_i}$$

in the free group F. Let us define a function f by

$$f(n) = \sup_{\substack{\ell(w) \leq n \\ \bar{w} = 1}} k(w).$$

Then f is a recursive isoperimetric function for $\mathcal{P}$.

The implication (2)⇒(3) follows from the following elementary result.

Lemma 2.2. *If f is an isoperimetric function for $\mathcal{P}$, then $n \mapsto Mf(n) + n$ is an isodiametric function for $\mathcal{P}$, where M is the maximum length of a relator.*

Proof. Let w be a word of length n representing 1 in $G = G(\mathcal{P})$ and let $\mathcal{D}$ be a van Kampen diagram of minimal area for w. If V and F denote the number of vertices and faces of $\mathcal{D}$, we observe that the length of the longest edge path in $\mathcal{D}^{(1)}$ which does not contain a circuit is at most $V-1 \leq MF+n$. From this it follows that $\mathrm{Diam}_{v_0}(\mathcal{D}) \leq Mf(n)+n$, and $n \mapsto Mf(n)+n$ is an isodiametric function for $\mathcal{P}$.

The implication (3)⇒(1) proceeds as follows. Let f be an isodiametric function for $\mathcal{P}$. Suppose $\ell(w)=n$ and $\bar{w}=1$, with $G=G(\mathcal{P})$. Then there is a description

$$w = \prod_{i=1}^{m} R_{j_i}^{\epsilon_i u_i},$$

with $\ell(u_i) \leq f(n)+M$, with M as above. But the set $S_m = \{R^u \mid R \text{ a relator}, \ \ell(u) \leq m\}$ is finite and generates a finitely generated subgroup $N_m < N < F$. Since the problem of deciding whether or not a word lies in a given finitely generated subgroup of the free group F is effectively solvable, we first calculate $m = f(n)+M$ and then apply this algorithm to decide whether or not $w \in N_m$. This solves the word problem for G.

A particularly attractive geometric way of deciding whether or not $w \in N_m$ as above has been given by Stallings [St1]. His algorithm amounts to using an immersion of finite graphs as a finite state automaton. Note that the automaton depends on the word w being tested.

Remark. It is somewhat mysterious that one has to proceed from Theorem 2.1.3 to 2.1.2 via 2.1.1, thereby involving the complications of general recursive functions. A more satisfying situation is to have a formula for an isoperimetric function in terms of an isodiametric function. One conjecture, which does not contradict any known example, is that there should be an isoperimetric function of the form $n \mapsto a^{f(n)+n}$, for a constant a (Stallings raised this question in the special case when f is linear). In this connection, D. E. Cohen has recently shown [C] that *if f is an isodiametric function for a finite presentation $\mathcal{P}$, then there are positive constants a, b so that $n \mapsto a^{b^{f(n)+n}}$ is an isoperimetric function for $\mathcal{P}$.* His proof makes use of an analysis of Nielsen's reduction process for producing a basis for a subgroup of a free group (see also [G4] for a different treatment involving Stallings' folds).

Here is a striking example, which shows that the complexity of the word problem for 1-relator groups, as measured by the growth of an isodiametric function, can be quite large.

Example. Each isodiametric and each isoperimetric function for the presentation $\mathcal{P} = \langle x, y \mid x^{x^y} = x^2 \rangle$ grows faster than every iterated exponential [G1].

Remark. Magnus showed that all 1-relator groups have a solvable word problem [LS]. However his argument gives no indication of the complexity of the algorithm. It is of interest to determine how fast the Dehn function of a 1-relator presentation can grow. We have shown (unpublished) that Ackermann's function f_ω is (up to the equivalence relation of Proposition 1.1) an isoperimetric function for every 1-relator presentation. Here one defines functions $f_\alpha : \mathbb{N} \to \mathbb{N}$ for ordinals $\alpha \leq \omega$ (where ω is the first infinite ordinal) inductively by $f_1(s) = 2s$, $f_{n+1}(s) = f_n^{(s)}(s)$, where $f_n^{(s)}$ denotes the s-fold iterate of f_n, and $f_\omega(s) = f_s(s)$.

The central tool in proving these upper bounds for the Dehn functions of 1-relator presentations is the rewrite function for a pair (G, H), where H is a finitely generated subgroup of the finitely generated group G. We suppose A, B are finite sets of generators for G, H, respectively, and we let $|g|_{G,A}$ denote the distance of $g \in G$ from the identity in the word metric, and similarly define $|h|_{H,B}$ for $h \in H$. We let

$$f_{G,H}(n) = \max_{\substack{h \in H \\ |h|_{G,A} \leq n}} |h|_{H,B},$$

and we call $f_{G,H}$ the *rewrite function.* We calculate $f_{G,H}$ inductively for a 1-relator presentation and Magnus subgroup and then apply the result to calculate an isoperimetric function. The rewrite function bears the same relation to the generalized word problem that the Dehn function bears to the word problem.

It is an open question whether for each n one can find a 1-relator presentation whose Dehn function grows at least as fast as f_n. The Dehn function for the presentation $x^{x^y} = x^2$ grows at least as fast as f_3, but this is the fastest growth we have actually proved can be realized for 1-relator presentations [G2].

§3. Examples and Applications

By Proposition 1.1, the simplest invariant condition on isoperimetric functions is that they be linear. In this case there is a satisfactory characterization. If we have a given finite set of generators A for a finitely presented group G, then for sufficiently large N, the presentation $\mathcal{P}_N$, with generators A and relators consisting of all relations among the generators of length at most N, will be a presentation of G.

Theorem 3.1. *The following are equivalent for a finitely presented group G.*

3.1.1. *G has a linear isoperimetric function.*
3.1.2. *G is word hyperbolic.*
3.1.3. *There exists a finite presentation for G which satisfies Dehn's algorithm.*
3.1.4. *If A is a finite set of generators for G, then for all sufficiently large N, the presentation $\mathcal{P}_N$ for G satisfies Dehn's algorithm.*

The unexplained terms in the theorem are as follows. Let A be a finite set of generators for G and let Γ be the associated Cayley graph, equipped with the word metric. The group G is called word hyperbolic if Γ is δ-hyperbolic for some $\delta \geq 0$; here Γ is called δ-hyperbolic if every geodesic triangle Δ in it satisfies Rips's condition R_δ: every point on one side of Δ is at distance at most δ from the union of the other two sides. The finite presentation $\mathcal{P}$ for G is said to satisfy Dehn's algorithm if, given any nonempty word w with $\bar{w} = 1$, there is a relator R of $\mathcal{P}$ such that w contains greater than $\frac{1}{2}$ of the word R as a contiguous subword.

The proof of Theorem 3.1 is very attractively presented in [ABC].

The next step beyond linear is subquadratic isoperimetric functions. In this case, Gromov asserts that a finite presentation with a subquadratic isoperimetric function also possesses a linear isoperimetric function [Gr, 2.3.F]. A. Yu. Ol'shanskii recently found an elementary proof of this important result [Ol].

In order to explain how quadratic isoperimetric functions arise, it is necessary to introduce new notions.

Definition 3.2. Let G be a finitely generated group with finite set A of semigroup generators and associated Cayley graph Γ. One has the evaluation mapping $A^\star \to G$, $w \mapsto \bar{w}$, where $A^\star$ is the free monoid on A. Such a word w can be viewed as a path $w(t),\ t \geq 0$, parametrized by arc length for $t \leq \ell(w)$, starting at the base point 1 (where G is identified equivariantly with the vertex set of Γ), moving over an edge in unit time, until it reaches its end point $\bar{w}$ at time $\ell(w)$; from then on, $w(t)$ remains constant at the vertex $\bar{w}$. A *combing* is a section $\sigma : G \to A^\star$ of the evaluation mapping such that there exists a constant $k > 0$ such that

$$\text{(3.2.1)} \qquad \forall g \in G\ \forall a \in A\ \forall t \geq 0 \qquad \text{one has} \qquad |\sigma(ga)(t) - \sigma(g)(t)| \leq k;$$

here $|x - y|$ denotes the distance from x to y in Γ. The condition (3.2.1) is called the *k-fellow traveller condition.* The finitely generated group G is called combable if it admits a combing. In addition we say that σ is *linearly bounded* if there are constants $C, D > 0$ such that $\ell(\sigma(g)) \leq C|g| + D$ for all $g \in G$, where $|g| := |1 - g|$.

The combing σ is called an *automatic structure* if the subset $\sigma(G) \subset A^\star$ is a regular language; that is, $\sigma(G)$ is the precise language recognized by a finite state automaton. It is a result of [ECHLPT] that an automatic structure σ is linearly bounded.

Remark. The definition of combing adopted in [ECHLPT] is more restrictive than the one we have adopted, following [Gh, p. 26] [Sh]: the former definition implies linear boundedness. It is known that both combability and the existence of a linearly bounded combing are quasiisometry invariant conditions [Sh]. It is unknown whether the existence of an automatic structure is quasiisometry invariant, although one of the results of [ECHLPT] is an algorithm which enables one to translate an automatic structure from one finite set of semigroup generators to another.

Theorem 3.3. *If G is a finitely generated group with a linearly bounded combing, then G is finitely presented and admits a quadratic isoperimetric function.*

We give the proof, which is due to Thurston, of this important result; the reader may consult [ECHLPT], where the result is proved in conjunction with higher dimensional isoperimetric inequalities.

Suppose that A is a finite set of semigroup generators for G and that $\sigma : G \to A^\star$ is a linearly bounded combing. Let σ satisfy the k-fellow traveller condition. Let $w \in A^\star$ be such that $\bar{w} = 1$, so w represents a closed path based at the vertex 1 in the Cayley graph Γ and let $\ell(w) = n$. We shall construct a finite presentation $\mathcal{P}$ for G and a van Kampen diagram for w in $\mathcal{P}$. Let $\sigma_i = \sigma(w(i))$ for integers $0 \leq i \leq n$, so σ_i is a path in Γ from 1 to $w(i)$. Observe that

$$|\sigma_i(j) - \sigma_{i+1}(j)| \leq k,$$

for each integral time j. This means that we can consider the vertices $\sigma_i(j)$,$\sigma_i(j+1)$, $\sigma_{i+1}(j+1)$, and $\sigma_{i+1}(j)$ as lying on a quadrilateral Q_{ij} whose boundary label is a relation of length at most $2k + 2$. If we take the presentation $\mathcal{P}$ to consist of A as generators and as relators, all relations among these generators of length at most $2k + 2$, then we see that w is a consequence of these relators. Consequently $\mathcal{P}$ is a finite presentation for G. Observe that we have only used the k-fellow traveller property so far and not the linear boundedness of the combing.

The quadrilaterals Q_{ij} fit together to form a van Kampen diagram $\mathcal{D}$ for w. Observe that for fixed i we can cut off j at time $\max(\ell(\sigma_i), \ell(\sigma_{i+1}))$, since both paths, σ_i and σ_{i+1}, will have reached their end points by then. But $\ell(\sigma_i) \leq Ci + D$, since $|w(i)| \leq i$. It follows that the total number of quadrilaterals in $\mathcal{D}$ is at most

$$\sum_{i=0}^{n}(Ci + D) \leq An^2 + B$$

for constants $A, B > 0$. Thus $\text{Area}_{\mathcal{P}}(w) \leq A\ell(w)^2 + B$, and the theorem is established.

Remark. Thurston asserts that the $(2n+1)$-dimensional integral Heisenberg group for $n \geq 2$ and $\text{Sl}_n(\mathbb{Z})$, for $n \geq 4$, satisfy the quadratic isoperimetric inequality [ECHLPT]; no details are available at this time. It would appear then that there was no simple characterization of groups satisfying the quadratic isoperimetric inequality. It is proved in [ECHLPT] and [G2] that the 3-dimensional integral Heisenberg group satisfies a cubic isoperimetric inequality (see also Section 5 below). Furthermore, it is shown in [ECHLPT] that an isoperimetric function for $\text{Sl}_3(\mathbb{Z})$ must grow at least exponentially. Compare also the arguments sketched in [Gr2].

Proposition 3.4. *If the group G is combable, then it satisfies the linear isodiametric inequality.*

Proof. Let $\sigma : G \to A^\star$ be a combing, where A is a finite set of semigroup generators. Suppose σ satisfies the k-fellow traveller property. As in the first part of the proof of Theorem 3.3, we obtain a finite presentation for G whose relators are all words in $w \in A^\star$ satisfying $\bar{w} = 1$ and such that $\ell(w) \leq 2k + 2$. Furthermore, we obtain a van Kampen diagram $\mathcal{D}$ for w, as in the second part of the proof, except now we have no bounds on the lengths of the paths σ_i. Nevertheless, if we consider a vertex $\sigma_i(j)$, then by holding j fixed and letting i vary, we arrive at the boundary of $\mathcal{D}$ in at most $k\ell(w)$ steps; once we arrive at the boundary, then we can follow it in at most $\ell(w)$ additional steps to arrive at the base point. Since no vertex of $\mathcal{D}$ is farther than k from a vertex of type $\sigma_i(j)$, it follows that the distance in the word metric of $\mathcal{D}^{(1)}$ from the base point to any vertex is bounded by $(k + 1)\ell(w) + k$. This establishes the linear isodiametric inequality.

Remark. It is asserted in [Gh, p. 27] that $\text{Sl}_3(\mathbb{Z})$ is not combable, where the definition of combability adopted there is the same as ours. Thurston's result (Theorem 3.3) is quoted for the proof. However, this last result applies only for a linearly bounded combing, so it must be considered open whether $\text{Sl}_3(\mathbb{Z})$ is combable or not. We proved in [G3] that all combable groups satisfy an exponential isoperimetric inequality, and this is the best result known to date in this generality.

Remark. There is an analogous notion of *asynchronously combable group*, where one has a section $\sigma : G \to A^\star$ satisfying the k-*asynchronous fellow traveller property*: after a monotone reparametrization, the paths $\sigma(g)$ and $\sigma(ga)$ are k-fellow travellers, for $g \in G$ and $a \in A$. The asynchronous combing σ is called an *asynchronously automatic structure* if the language $\sigma(G) \subset A^\star$ is regular. M. Shapiro has recently proved that the definition just given is equivalent to that of [ECHLPT] for an asynchronously automatic structure on a group [Sp2]. One can show that *every asynchronously combable group is finitely presented and satisfies a linear isodiametric inequality*. Furthermore, *if G is asynchronously automatic, then it has an exponential isoperimetric function*[ECHLPT][BGSS].

Remark. In the original version of this survey, we raised the question whether the integral Heisenberg group was combable. Gromov asserted at the conference that the real Heisenberg group is combable. There are several arguments sketched in [Gr2], but our attempts to fill in the details have only succeeded in proving the weaker result that the group is asynchronously combable; so we regard this as an open question of great interest. In this connection, we mention a recent result of M. Bridson's [Bd], that the group $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ is asynchronously combable for all $\phi \in \mathrm{Gl}_n(\mathbb{Z})$.

Theorem 3.5. *The following finitely presented groups all have linear isodiametric functions.*

- 3.5.1. *Lattices in the 3-dimensional Lie group Nil.*
- 3.5.2. *Lattices in the 3-dimensional Lie group Sol.*
- 3.5.3. $\pi_1(M)$*, where M is a compact 3-manifolds for which Thurston's geometrization conjecture* [Th] *holds.*

The statements about lattices in Nil and Sol are proved in [G1]. Here is an extremely rough sketch for lattices in Sol. The problem is reduced to showing that the "Fibonacci group" $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where $\phi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, satifies a linear isodiametric inequality. This is deduced from arithmetic properties of the Fibonacci sequence.

The argument for 3.5.3 is as follows. A result of [ECHLPT] states that if no geometric piece in the Thurston decomposition is a Nil or Sol group, then the fundamental group is automatic. Since the Nil and Sol pieces in the Thurston decomposition occur only as connected summands, it follows that the fundamental group $\pi_1(M)$ of a compact 3-manifold M for which Thurston's geometrization conjecture holds is the free product of an automatic group with a finite free product of Nil and Sol groups. Since each of these free factors satisfies the linear isodiametric inequality and since the class of finitely presented groups satisfying the linear isodiametric inequality is closed under finite free products, it follows that $\pi_1(M)$ satisfies the linear isodiametric inequality.

Remark. Bridson's recent results [Bd] strengthen Theorem 3.5, showing that the fundamental group of every compact 3-manifold satisfying Thurston's geometrization conjecture is asynchronously combable. Bridson raises the question of the "logical complexity" of the language of the combing.

Remark. It is not known how wide the class of finitely presented groups satisfying a linear isodiametric inequality is. For instance, we do not know an example of a finitely presented linear group which does not satisfy a linear isodiametric inequality (the example $x^{x^y} = x^2$ given in §2 is not a linear group). Since a finitely presented linear group has a solvable word problem, Theorem 2.1 will be of no help in constructing an example.

Remark. It follows from results of [ECHLPT] that if a compact 3-manifold satisfies Thurston's geometrization conjecture, then its fundamental group has an exponential isoperimetric function. If, in addition, there are no Nil or Sol pieces, it is automatic and satisfies the quadratic isoperimetric inequality.

Question. If ϕ is an automorphism of the finitely generated free group F and if $G = F \rtimes_\phi \mathbb{Z}$ is the corresponding split extension, does G satisfy the quadratic isoperimetric inequality? The result of [BF], that G is word hyperbolic if and only if it contains no subgroup isomorphic to $\mathbb{Z}^2$, can be viewed as positive evidence. Furthermore, G is automatic if ϕ is geometric (that is, if ϕ is induced by a homeomorphism of a compact surface with nonempty boundary). For in this case, G is $\pi_1(M)$, where M is a compact Haken 3-manifold, and Thurston's geometrization conjecture is known to hold for such M [Th]. A cohomological dimension argument shows that M has no Sol or Nil pieces, whence, by the preceding Remark, G is automatic.

That this question may be delicate is suggested by our result (unpublished) that if $\phi \in \mathrm{Aut}(F(a,b,c))$ is given by $\phi(a) = a,\ \phi(b) = ba,\ \phi(c) = ca^2$, then $G = F(a,b,c) \rtimes_\phi \mathbb{Z}$ cannot act properly discontinuously and cocompactly on any geodesic metric space satisfying Gromov's condition CAT(0) (see [GH] for the CAT(0) property).

The original motivation for introducing isodiametric functions was a result proved by Casson in 1990. We shall state a weaker version of his result which falls naturally within our framework. We say that the finite presentation $\mathcal{P}$ satisfies condition $\mathrm{ID}(\alpha)$, where $\alpha > 0$, if there is an $\epsilon \geq 0$ so that $n \mapsto \alpha n + \epsilon$ is an isodiametric function for $\mathcal{P}$. This is of course just a reformulation of a linear isodiametric inequality.

Theorem 3.6, [SG]. *Let M be a closed, orientable, irreducible, aspherical 3-manifold whose fundamental group admits a finite presentation satisfying condition* $\mathrm{ID}(\alpha)$*, where $\alpha < 1$. Then the universal cover of M is homeomorphic to* $\mathbb{R}^3$.

For example, a combable group G whose combing σ is such that each word $\sigma(g)$ is geodesic has a finite presentation satisfying condition $\mathrm{ID}(\frac{1}{2})$. Every finitely presented group which possesses an almost convex Cayley graph, in the sense of Cannon [Ca], has an $\mathrm{ID}(\frac{1}{2})$ presentation [G1] (see also §4 below). Since it is known that every Nil group has at least one almost convex Cayley graph [Sp1], it follows that Nil groups have $\mathrm{ID}(\frac{1}{2})$ presentations. Another argument, proving that Nil groups have $\mathrm{ID}(\frac{3}{4})$ presentations, appears in [G1]. The difficulty with these conditions of course is that they are not invariant under change of generators.

We should also cite additional work in connection with Casson's theorem [P][Br][St2].

§4. Relation with Peak Reduction Algorithms

In this section, G will denote a finitely presented group with finite set of semigroup generators A and associated Cayley graph Γ.

Definition 4.1. Let $\mu : G \to \mathbb{N}$ be a function such that $S = \{g \in G \mid \mu(g) = 0\}$ is a finite subgroup of G with $S \subset A$. Let $\mathcal{P}$ be a finite presentation for G with generators A and such that $\mathcal{P}$ contains all cyclic conjugates of its relators and their inverses and, in addition, $\mathcal{P}$ contains the group table for the finite group S. We say that $\mathcal{P}$ admits a peak reduction algorithm with respect to the function μ if the following condition holds: if $w \in A^\star$ is such that $\mu(\bar{w}) \le \mu(\bar{w}a) > \mu(\bar{w}aa')$ for some pair of generators $a, a' \in A$, then there is a relator of $\mathcal{P}$ of the form $aa' = a_1 a_2 \dots a_k$, with $a_i \in A$, such that $\mu(\bar{w}a_1 \dots a_i) < \mu(\bar{w}a)$ for all $1 \le i \le k$.

Define a function $f_\mu : \mathbb{N} \to \mathbb{N}$ by $f_\mu(n) = \sup_{|g| \le n} \mu(g)$.

Theorem 4.1, [G1]**.** *Suppose that $\mathcal{P}$ admits a peak reduction algorithm for the function $\mu : G \to \mathbb{N}$. If M denotes the length of the longest relator of $\mathcal{P}$, then*

4.1.1. *the function $n \mapsto M f_\mu(n) + \frac{n}{2}$ is an isodiametric function for $\mathcal{P}$, and*
4.1.2. *the function $n \mapsto n \cdot M^{f_\mu(n)+1}$ is an isoperimetric function for $\mathcal{P}$.*

We shall now give some examples of peak reduction algorithms. With G, A, Γ as above, let $\mathcal{P}_N$ be the finite presentation with generators A and relators all words $w \in A^\star$ with $\bar{w} = 1$ and $\ell(w) \le N$. Note that it follows from the fact that G is finitely presented that $\mathcal{P}_N$ is a presentation of G for all N sufficiently large. We set B_n and S_n to be the set of vertices in the ball and sphere of radius n at the identity element in Γ.

We recall [Ca] that Γ is called almost convex iff for all n and for all pairs of points $x, y \in S_n$ which are joined by a path of length at most 3 in Γ there is a path in B_n joining these points of bounded length (where the bound is independent of n, x, and y).

Proposition 4.2, [G1]**.** *The Cayley graph Γ is almost convex if and only if there exists $N > 0$ such that $\mathcal{P}_N$ satisfies peak reduction for the function $\mu(g) = |g|$.*

The proof is not difficult from the definitions.

Corollary 4.3. *If G has an almost convex Cayley graph, then it has a linear isodiametric function and an exponential isoperimetric function.* □

Theorem 4.4. *The groups $\operatorname{Aut}(F)$ and $\operatorname{Out}(F)$, where F is a finitely generated free group, have exponential isodiametric functions and isoperimetric functions of the form $n \mapsto A^{B^n}$.*

This is a consequence of results of Whitehead, Higgins and Lyndon, and McCool on the automorphism group of a finitely generated free group. For instance, for the group $\mathrm{Aut}(F)$ one chooses a free basis $x_1, x_2, \ldots, x_r$ for F and one takes $\mu(\phi) = \sum_{i=1}^{r} L(\phi(x_i)) - r$. Here $L(w)$, for a word w in F, is the length of a cyclically reduced word conjugate to w. The generators for $\mathrm{Aut}(F)$ are taken to be the Whitehead automorphisms [LS]. The function f_μ is seen to grow exponentially with n. McCool's algorithm [Mc] is a peak reduction algorithm for these data, so the assertion for $\mathrm{Aut}(F)$ follows from Theorem 4.1. The argument for $\mathrm{Out}(F)$ is similar.

Remark. These results for $\mathrm{Aut}(F)$ and $\mathrm{Out}(F)$ are surely not best possible. It is an open question whether $\mathrm{Out}(F)$ is automatic; if this were true, then the quadratic isoperimetric inequality would hold. In this connection, we have shown (unpublished) that neither $\mathrm{Aut}(F)$ for $\mathrm{rank}(F) \geq 3$ nor $\mathrm{Out}(F)$ for $\mathrm{rank}(F) \geq 4$ can act properly discontinuously and cocompactly on a geodesic metric space which satisfies Gromov's condition CAT(0). The situation for $\mathrm{Out}(F)$ when $\mathrm{rank}(F) = 3$ is still open.

Theorem 4.5. $\mathrm{Sl}_3(\mathbb{Z})$ *has an exponential isodiametric function and an isoperimetric function of the form* $n \mapsto A^{B^n}$.

This follows from a result of Nielsen's [N], that the group $\mathrm{Sl}_3(\mathbb{Z})$ satisfies a peak reduction algorithm for the function μ given by $\mu(x) = (\sum x_{ij}^2) - 3$, for $x \in \mathrm{Sl}_3(\mathbb{Z})$. The generators here are the elementary transvections $E_{ij}(1)$ and the signed permutation matrices. In this case the function f_μ grows exponentially.

Remark. It follows from results of [ECHLPT] that any isoperimetric function for $\mathrm{Sl}_3(\mathbb{Z})$ must grow at least exponentially. Thus from Theorem 4.5 we deduce that the Dehn function for a finite presentation of $\mathrm{Sl}_3(\mathbb{Z})$ has somewhere between exponential and twice-iterated exponential growth. Which, if either, is it?

Question. Can Nielsen's argument for $\mathrm{Sl}_3(\mathbb{Z})$ be generalized to a peak reduction algorithm for $\mathrm{Sl}_n(\mathbb{Z})$? The answer is surely 'yes', but it seems this has never been written down (compare [Mi, §10] where a related result is established).[1] Does $\mathrm{Sl}_3(\mathbb{Z})$ have a linear isodiametric function?

§5. Lower bounds for Isoperimetric Functions

The methods of this section for establishing lower bounds for the Dehn function of a finite presentation are due to [BMS] (other methods for finding lower bounds can be found in [G2]). We shall prove that the Dehn function of the free nilpotent group on $p \geq 2$ generators of class c grows at least as fast as a polynomial of degree $c+1$. Since it is known that every finitely generated nilpotent group has a polynomial isoperimetric function [G1], it follows that arbitrary high degree polynomial growth is exhibited by these free nilpotent groups as $c \to \infty$.

Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_p \mid R_1, R_2, \ldots, R_q \rangle$ and let F be the free group freely generated by the generators $x_1, x_2, \ldots, x_p$ and let $N \triangleleft F$ be the normal closure of the relators. We let $G = G(\mathcal{P}) = F/N$ as earlier.

[1] We have in the meantime received the preprint [Ka] which contains the peak reduction lemma for the general linear groups.

Proposition 5.1. *The group $N/[F,N]$ is a finitely generated abelian group.*

Proof. The identity $R_i^u = [u, R_i]R_i \in [F,N]R_i$ shows that the cosets of the relators R_i generate the factor group $N/[F,N]$. Since $[F,N] \supset [N,N]$, this factor group is abelian, and consequently it is a finitely generated abelian group.

Definition. Let $V = \mathbb{Q} \otimes N/[F,N]$, considered as a finitely generated vector space over $\mathbb{Q}$. If $v_1, v_2, \ldots v_d$ is a basis for V, we define the ℓ_1-norm $|v|_1$ of a vector $v \in V$ with respect to this basis to be $\sum_{i=1}^d |a_i|$, where $v = \sum_{i=1}^d a_i v_i$, $a_i \in \mathbb{Q}$.

If $w \in N$, then we define $|w|_1$ to be the ℓ_1-norm of $1 \otimes [w]$, where $[w]$ is the coset $w[F,N] \in N/[F,N]$.

Theorem 5.2. *With the notations above, there is a constant $C \geq 0$ so that for all $w \in N$ we have*

$$|w|_1 \leq C\mathrm{Area}_{\mathcal{P}}(w).$$

Proof. Let $C = \max_{1 \leq i \leq q} |R_i|_1$. This is the number C of the theorem.

Suppose now that $w \in N$, so $w = \prod_{j=1}^k R_{i_j}^{\epsilon_j u_j}$, where $\epsilon_j = \pm 1$ and $u_j \in F$ and where $k = \mathrm{Area}_{\mathcal{P}}(w)$. Observe that since $R_{i_j}^{u_j} \in [F,N]R_{i_j}$, we have $[R_{i_j}^{\epsilon_j u_j}] = \epsilon_j [R_{i_j}]$ in V. From this it follows that $|w|_1 = |\sum_{j=1}^k \epsilon_j [R_{i_j}]|_1 \leq Ck \leq C\mathrm{Area}_{\mathcal{P}}(w)$. This completes the proof.

Remark. If we change the basis of V above and calculate the ℓ_1-norm with respect to the new basis, the effect is to change the constant C in Theorem 5.2.

Next we recall some facts about nilpotent groups. A central series for a group G is sequence of subgroups

$$H_n < H_{n-1} < \ldots H_0 = G$$

so that $[G, H_i] < H_{i+1}$ for all i. The group G is called nilpotent if it has such a central series with $H_n = 1$ for some n, and the minimum such number n for all central series is called the class of nilpotence. For example, a nontrivial abelian group has class 1 and the Heisenberg group has class 2. The lower central series $\{G_n, n \geq 0\}$ for any group G is defined inductively by $G_0 = G$, $G_{n+1} = [G, G_n]$. One has that $G_i < H_i$ for any central series H_i as above, so the lower central series descends at least as fast as any central series for G.

In particular we can apply these notions to the free group F freely generated by $x_1, x_2, \ldots, x_p$, where $p \geq 2$, to get the lower central series $\{F_n\}$ of the free group. The group F/F_c is called the free nilpotent group on p generators of class c. It is a standard result that the normal subgroup $F_c \triangleleft F$ is generated by all left normed commutators of length $c+1$, $\mathrm{ad}(u_1) \circ \mathrm{ad}(u_2) \circ \cdots \circ \mathrm{ad}(u_c)(u_{c+1})$, where $u_i \in F$. Here $\mathrm{ad}(u)(v) = [u, v]$. As an example, using this observation it is easy to see that the free nilpotent group on 2 generators of class 2 is the Heisenberg group.

Theorem 5.3. *The free nilpotent group on $p \geq 2$ generators of class $c \geq 1$ has the property that its Dehn function grows at least as fast as a polynomial of degree $c+1$.*

Proof. † Since the free nilpotent group on $p \geq 2$ generators and class c retracts to that on 2 generators and class c, it suffices to prove the result for 2 generators. Let $F = F(a, b)$ be the free group freely generated by a, b and let $w_n = \mathrm{ad}(a^n)^{(c)}(b^n) \in F_c$. One checks that $\ell(w_n)$ grows linearly with n. However, when $[w_n]$ is considered in $V = \mathbb{Q} \otimes F_c/[F, F_c] = \mathbb{Q} \otimes F_c/F_{c+1}$, one has by multilinearity $[w_n] = n^{c+1}[\mathrm{ad}(a)^{(c)}(b)]$. But it is known that the Engel element $\mathrm{ad}(a)^{(c)}(b)$ of the free abelian group F_c/F_{c+1} is an element of a $\mathbb{Z}$-basis [MKS, §5.7 Problem 4], so $[\mathrm{ad}(a)^{(c)}(b)] \neq 0$ in V. It follows that $|w_n|_1 = n^{c+1}|[\mathrm{ad}(a)^{(c)}(b)]|_1 \neq 0$, so $|w_n|_1$ grows like a polynomial in n of degree $c+1$. It follows from Theorem 5.2 that $\mathrm{Area}_{\mathcal{P}}(w_n)$ grows at least as fast as a polynomial in n of degree $c + 1$, where $\mathcal{P}$ is a finite presentation for F/F_c. Since $\ell(w_n)$ is linear in n, it follows that the Dehn function for F/F_c must grow at least as fast as a polynomial of degree $c + 1$. This completes the proof.

Remark. If $N < [F, F]$ above, then $N/[F, N] \cong H_2(G, \mathbb{Z})$, as one sees from Hopf's formula. In this case the vector space V is $H_2(G, \mathbb{Q})$.

Remark. Taking $p = 2$ and $c = 2$ in Theorem 5.3, we recover the result of [ECHLPT] and [G2] that the Dehn function for the 3 dimensional integral Heisenberg group grows at least as fast as a cubic polynomial. The next result shows that this result is optimal (other proofs that the 3 dimensional integral Heisenberg group has a cubic polynomial for its Dehn function are given in [ECHLPT] and [G2]).

Proposition 5.4. *The Dehn function for the 3 dimensional integral Heisenberg group H grows like a cubic polynomial.*

Proof. We have already shown that the Dehn function grows at least as fast as a cubic polynomial. We shall obtain now a cubic polynomial upper bound. A presentation for H is $\mathcal{P} = \langle x, y, t \mid x^t = xy, y^t = y, xy = yx \rangle$. Let $\mathcal{Q} = \langle x, y, t \mid x^t = xy, y^t = y \rangle$ and let $\mathcal{R} = \langle x, y \mid xy = yx \rangle$. Observe that $\mathcal{Q}$ is a presentation for the split extension $F(x, y) \rtimes_\phi \mathbb{Z}$ of the free group $F(x, y)$, where $\phi(x) = xy, \phi(y) = y$.

Let w be a word in the generators of $\mathcal{P}$ with with $\ell(w) = n$ and such that $\bar{w} = 1$ in H. We shall find a van Kampen diagram for w in two steps. First, using only the relations of $\mathcal{Q}$, we find a sequence of cyclic words $w = w_0, w_1, \ldots, w_{n-1} = w'$, where each is obtained from the preceding by at most a single t-reduction (viewing t as the stable letter in the HNN extension $F(x, y) \rtimes_\phi \mathbb{Z}$ with base group $F(x, y)$), until one runs out of t-letters. If ℓ_x, ℓ_y, ℓ_t denote respectively the number of letters $x^\pm, y^\pm, t^\pm$ in a free word, we see inductively that $\ell_x(w_i) \leq \ell_x(w)$, $\ell_y(w_i) \leq \ell_x(w_{i-1}) + \ell_y(w_{i-1})$, and $\ell_t(w_i) \leq \max(\ell_t(w_{i-1}) - 2, 0)$. It follows that $\ell_y(w_i) \leq i\ell_x(w) + \ell_y(w) \leq (i + 1)n$, and there is an annular diagram A_i in $\mathcal{Q}$ connecting w_{i-1} with w_i of area $\mathrm{Area}(A_i) \leq \ell(w_{i-1}) \leq in$. If we fit these annular diagrams together, we obtain an annular diagram D_1 in $\mathcal{Q}$ with boundary components labelled w and w' such that $\mathrm{Area}(D_1) \leq \sum_{i=1}^{n-1} in = O(n^3)$

† H. Short told me the statement of Theorem 5.3 at the Sussex conference, from which I worked out the proof given here. Baumslag, Miller, and Short wrote me subsequently that this argument was one of several they had in mind.

Since $\ell_x(w') \leq n$, $\ell_y(w') \leq n^2$, and $\ell_t(w') = 0$, we can find a disc diagram D_2 for w' in $\mathcal{R}$ with $\text{Area}(D_2) \leq n^3$. If we fit D_1 and D_2 together along their common boundary component labelled w', we obtain a disc diagram D for w with $\text{Area}(D) \leq O(n^3) + n^3 = O(n^3)$. This completes the proof of the proposition.

The same method as in Theorem 5.3 suffices to prove the following result.

Proposition 5.5. *If G is a finitely generated nilpotent group of class c given by the exact sequence $1 \to N \to F \to G \to 1$, with F finitely generated and free (so $F_c < N$), and if the canonical map $F_c \to N/[F,N]$ has infinite image, then the Dehn function for G grows at least as fast as a polynomial of degree $c+1$.* $\square$

Example. The $(2n+1)$-dimensional integral Heisenberg group $\mathfrak{H}_{2n+1}$ is given by the presentation

$$\begin{aligned}\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n, z \mid\ & [x_i, x_j] = 1,\ [y_i, y_j] = 1 \quad \text{for all } i, j, \\ & [x_k, y_l] = 1 \quad \text{for all } k \neq l, \\ & [x_m, y_m] = z \quad \text{for all } m, \\ & z \quad \text{central}\ \rangle.\end{aligned}$$

It is more convenient to use another presentation for $\mathfrak{H}_{2n+1}$ obtained by Tietze transformations from the preceding by eliminating the central generator z. This new presentation has generators $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n$ freely generating the free group F of rank $2n$. The normal subgroup of relations N is contained in $[F,F] = F_1$ for this second presentation, and since $\mathfrak{H}_{2n+1}$ is nilpotent of class 2, we have $F_2 \subset N$. We have then the next result.

Proposition 5.6. *With the notations preceding we have*

5.6.1. *if $n \geq 2$, the canonical homomorphism $F_2/F_3 \to N/[F,N] = H_2(\mathfrak{H}_{2n+1}, \mathbb{Z})$ is the zero map, whereas*

5.6.2. *if $n = 1$, then we have $F_2/F_3 \xrightarrow{\cong} H_2(\mathfrak{H}_3, \mathbb{Z})$.*

Proof. The second assertion (5.6.2) follows from earlier remarks since $\mathfrak{H}_3$ is the free nilpotent group of class 2 on 2 generators. We proceed then to the proof of (5.6.1).

The group F_2/F_3 is generated by elements $[u,[v,w]]$ where each of u, v, w is in the set $\{x_i, y_i;\ 1 \leq i \leq n\}$. Since we have $[x_i, y_j] \in N$ for $i \neq j$ and since $[x_k, x_l],\ [y_k, y_l] \in N$ for all k, l, it follows that we have $[u,[x_i, y_j]] \in [F,N]$ for $i \neq j$ and $[u,[x_k, x_l]] \in [F,N],\ [u,[y_k, y_l]] \in [F,N]$ for all k, l, where $u \in \{x_i, y_i;\ 1 \leq i \leq n\}$.

It remains to prove that $[x_j,[x_i, y_i]] \in [F,N]$ and $[y_j,[x_i, y_i]] \in [F,N]$ for all i, j. We shall prove the first assertion, since the second follows symmetrically. If $j \neq i$, this assertion is a consequence of an identity attributed variously to E. Witt or P. Hall,

$$[b,[a^{-1},c]]^a[a,[c^{-1},b]]^c[c,[b^{-1},a]]^b = 1,$$

for all elements a, b, c in a group. We remind the reader here that our convention for commutators is $[a,b] = aba^{-1}b^{-1}$ and $a^b = bab^{-1}$. If we substitute $b = x_j, a =$

$x_i^{-1}, c = y_i$ in this identity, we find that two terms of the product are in $[F, N]$, whence the third, a conjugate of $[x_j, [x_i, y_i]]$, is also in $[F, N]$.

It remains then to prove that $[x_i, [x_i, y_i]] \in [F, N]$. Since $n \geq 2$, there is an index $j \neq i$. Observe first that $[x_i x_j, y_i y_j^{-1}] \in N$, as one sees by an elementary computation in $\mathfrak{H}_{2n+1}$, so we have $[x_i, [x_i x_j, y_i y_j^{-1}]] \in [F, N]$. The commutator $[x_i x_j, y_i y_j^{-1}]$ can be expanded as the product of four commutators, making use of the relation $[a, bc] = [a, b][a, c]^b$; we obtain $[x_i x_j, y_i y_j^{-1}] = [x_j, y_i]^{x_i}[x_i, y_i][x_j, y_j^{-1}]^{y_i x_i}[x_i, y_j^{-1}]^{y_i}$. When we expand the expression $[x_i, [x_i x_j, y_i y_j^{-1}]]$ and make use of the commutator identities (and the fact that $F_2 < N$, so $F_3 < [F, N]$), we obtain that this class in $N/[F, N]$ is that of the product of four classes, those of $[x_i, [x_j, y_i]]$, $[x_i, [x_i, y_i]]$, $[x_i, [x_j, y_j^{-1}]]$, and $[x_i, [x_i, y_j^{-1}]]$. The first, third, and fourth terms have already been shown to be in $[F, N]$. Since the product lies in $[F, N]$, it follows that the second term, $[x_i, [x_i, y_i]]$, is also in $[F, N]$. This completes the proof of the Proposition.

Remark. The proposition just proved shows that Thurston's assertion, that $\mathfrak{H}_{2n+1}$ satisfies the quadratic isoperimetric inequality for $n \geq 2$, is consistent with Proposition 5.5: if the homomorphism $F_2/F_3 \to N/[F, N]$ in (5.6.1) had had infinite image, then the Dehn function for this group would have been a cubic polynomial and not quadratic.

Remark. It is proved in [G1] that a finitely generated nilpotent group has a polynomial isoperimetric function of degree 2^h, where h is the Hirsch number. The bound on the degree was improved in [Co] to $2 \cdot 3^c$, where c is the class of nilpotence. We do not know an example of a finitely generated nilpotent group where there does not exist an isoperimetric polynomial of degree $c + 1$.

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References

[ABC] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short (ed.), *Notes on word hyperbolic groups*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 3–63.

[Al] J. Alonso, *Inégalités isopérimétriques et quasi-isométries*, C. R. Acad. Sci. Paris, Série 1, **311** (1991), 761–764.

[Bd] M. Bridson, *Combing the fundamental group of a Haken 3-manifold*, preprint, Princeton Univ., April 1992.

[BF] M. Bestvina and M. Feighn, *A combination theorem for negatively curved groups*, Jour. Diff. Geom. **35** (1992), 85–101.

[BGSS] G. Baumslag, S. M. Gersten, M. Shapiro, and H. Short, *Automatic groups and amalgams*, Jour. Pure Appl. Alg **76** (1991), 229–316.

[BMS] G. Baumslag, C. F. Miller III, and H. Short, *Isoperimetric inequalities and the homology of groups*, preprint, CUNY 1992.

[Br] S. Brick, *Filtrations of universal covers and a property of groups*, preprint, Univ. of Cal., Berkeley, 1991.

[Ca] J. W. Cannon, *Almost convex groups*, Geometriae Dedicata **22** (1987), 197–210.

[C] D. E. Cohen, *Isodiametric and isoperimetric inequalities for group presentations*, Internat. J. Algebra and Comput. **1** (1991), 315–320.

[Co] G. Conner, Ph. D. thesis, Univ. of Utah, 1992.

[ECHLPT] J. W. Cannon, D. B. A. Epstein, D. Holt, S. Levy, M. Paterson, and W. P. Thurston, *Word processing in groups*, Jones and Bartlett, Boston, 1992.

[FHL] W. J. Floyd, A. H. M. Hoare, and R. C. Lyndon, *The word problem for geometrically finite groups*, Geometriae Dedicata **20** (1986), 201–207.

[G1] S. M. Gersten, *Isodiametric and isoperimetric inequalities in group extensions*, preprint, Univ. of Utah 1991.

[G2] S. M. Gersten, *Dehn functions and ℓ_1-norms of finite presentations*, to appear in Proceedings of a Workshop on Algorithmic Problems, eds. G. Baumslag and C. F. Miller III Springer-Verlag, MSRI series, 1991.

[G3] S. M. Gersten, *Bounded cohomology and combings of groups*, preprint, Univ. of Utah, 1991.

[G4] S. M. Gersten, *The double exponential theorem for isodiametric and isoperimetric functions*, Internat. J. Algebra and Comput. **1** (1991), 321–327.

[Gh] E. Ghys, *Les groupes hyperboliques*, Sem. Bourbaki, no. 722, 1989–1990.

[GH] E. Ghys and P. de la Harpe, editors, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Birkhäuser, 1990.

[GLP] M. Gromov, J. Lafontaine and P. Pansu, *Structures métriques pour les variétés riemanniennes*, Cedic/F. Nathan, Paris, 1981.

[Gr] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), MSRI series, vol. 8, 1987, Springer-Verlag.

[Gr2] M. Gromov, *Asymptotic invariants of infinite groups*, preprint, IHES 1992.

[Ka] S. Kalajdžievski, *PRL for general linear groups*, preprint, Univ. of Manitoba 1992.

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[Mc] J. McCool, *A presentation for the automorphism group of a free group of finite rank*, Jour. London Math. Soc. **8** (1974), 259–266.

[Mi] J. Milnor, *Algebraic K-Theory*, Annals of Math. Study, vol. 72, Princeton Univ. Press, 1971.

[MKS] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, J. Wiley, 1966.

[N] J. Nielsen, *Die Gruppe der Dreidimensionalen Gittertransformationen*, Jakob Nielsen, Collected Mathematical Papers, Vol. 1 (1913–1932), Birkhäuser, 1986, pp. 147–173.

[Ol] A. Yu. Ol'shanskii, *Hyperbolicity of groups with subquadratic isoperimetric inequality*, Internat. J. Algebra and Comput. **1** (1991), 281–289.

[P] V. Poénaru, *Geometry "á la Gromov" for the fundamental group of a closed 3-manifold M^3 and the simple connectivity at infinity of $\widetilde{M}^3$*, preprint Univ. de Paris-Sud, Orsay, 1990.

[Sh] H. Short, *Groups and combings*, preprint, ENS Lyon July 1990.

[Sp1] M. Shapiro, *A geometric approach to the almost convexity and growth of some nilpotent groups*, Math. Ann. (1989), 601–624.

[Sp2] M. Shapiro, *Deterministic and non-deterministic asynchronous automatic structures*, preprint, Ohio State Univ., June 14, 1991.

[St1] J. Stallings, *Topology of finite graphs*, Inv. Math. **71** (1983), 551–565.

[St2] J. Stallings, *Brick's "QSF": a group-theoretic idea related to simple connectivity at infinity of 3-manifolds*, to appear in the Proceedings of the Conference on Geometric Methods in Group Theory held at the Univ. of Sussex, July 1991.

[SG] J. Stallings and S. M. Gersten, *Casson's idea about 3-manifolds whose universal cover is $\mathbb{R}^3$*, to appear, Internat. J. Algebra and Comput.

[Th] W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

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